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George Metcalfe

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Semilinear logics

Petr Cintula

Institute of Computer Science, Academy of Sciences – Prague, Czech Republic

cintula@cs.cas.cz

Natural conditions for an implication connective[1] in an arbitrary logic (structural consequence relation) entail a uniform definability of a 'matrix preorder' in any matrix $\mathbf{A} = \langle \mathcal{A}, D \rangle$ of the given logic:

$$x \leq^{\mathbf{A}} y \quad \text{iff} \quad x \to^{\mathcal{A}} y \in D.$$

Furthermore, they entail that such a preorder satisfies antisymmetry iff the matrix is *reduced*. As any logic is complete w.r.t. the class of its reduced matrices we can (a little nonchalantly) say that logics with implication are 'logics of ordered matrices'.

Now, *semilinear* logics are those that are complete w.r.t. the class of their *linearly* ordered matrices. The term semilinear is justified by the fact that a finitary logic is semilinear iff all its relatively subdirectly irreducible matrices are linearly ordered (following the Universal Algebra tradition of calling a class of algebras 'semi$X$' whenever its subdirectly irreducible members have the property X; e.g. as in 'semisimple'). The term 'semilinear' was introduced by Olson and Raftery in [4] (for subvarieties of residuated lattices which are usually called 'representable').

In this talk I introduce the basics of the theory of semilinear logics (based on the submitted paper [1]) paying special attention to the logics resulting from classes of residuated lattices [3]. I concentrate on showing interesting consequences of abstract 'logical' results for the 'algebraic' theory of residuated lattices.

References

[1] Petr Cintula and Carles Noguera. A hierarchy of implicational (semilinear) logics: the propositional case. Submitted.

[2] Josep Maria Font, Ramon Jansana, and Don Pigozzi. A survey of Abstract Algebraic Logic. *Studia Logica*, 74(1–2, Special Issue on Abstract Algebraic Logic II):13–97, 2003.

[3] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[4] Jeffrey S. Olson and James G. Raftery. Positive sugihara monoids. *Algebra Universalis*, 57:75–99, 2007.

[1] Let us assume here that $\to$ is a fixed connective (primitive or derived) rather than (following Abstract Algebraic Logic tradition [2]) a set of (parameterized) formulae

MV-algebras and Bezout domains

László Fuchs

Department of Mathematics, Tulane University – New Orleans LA, USA

fuchs@tulane.edu

We give a representation of an MV-algebra as a sublattice of the lattice of principal ideals of a Bezout domain. This can be generalized to the non-commutative case for certain GMV-algebras. Several properties of this representation will be discussed.

Skew relation algebras

Nikolaos Galatos

Department of Mathematics, University of Denver – Denver CO, USA

ngalatos@du.edu

Joint research with **P. Jipsen**

Jonsson and Tsinakis study relation algebras and their generalizations as residuated (or conjugated) Boolean algebras. We extend their study to a setting that does not assume a Boolean algebra reduct, but just a lattice with a De Morgan involution. The condition of residuation can be reformulated to a condition resembling conjugation and two converse operations can be defined. We further study the interaction of various involutive conditions and the Euclidean laws. Finally we provide models for generalizations of relation algebras with two distinct converse operations and axiomatize the variety they form.

Periodic varieties of unital ℓ-groups and pseudo-MV-algebras

Charles Holland

Department of Mathematics, University of Colorado – Boulder CO, USA

charles.holland@colorado.edu

Unital ℓ-groups and pseudo-MV-algebras are categorically equivalent, and so their varieties (equationally defined classes) correspond with each other. An important class of these varieties, still not fully understood, consists of the ones generated by the l- group of order-preserving automorphisms of the real line which commute with translation by $+1$, but with various choices of unit. The tools to deal with this seem closely connected with symbolic dynamics. I will discuss the known results. In particular, I will show that the Abelian variety is contained within all of these. And I will present some open problems.

Representable commutative atoms in the subvariety lattice of residuated lattices

Rostislav Horčík

Institute of Computer Science, Academy of Sciences – Prague, Czech Republic

horcik@cs.cas.cz

Starting with a totally ordered Abelian group, we construct continuum many commutative residuated chains by means of conuclei and nuclei. Then we show that these chains generate atoms in the subvariety lattice of residuated lattices. This result in fact shows that the conjecture from [3, pp. 383] is not true. Moreover, using the same construction, we can also solve the open problems [1, pp. 437] on the cardinality of atoms in the subvariety lattices of representable commutative integral FL-algebras and representable commutative FL-algebras satisfying $0 \leq x$.

References

[1] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

Topological duality, canonical extensions and decidability for lattices with quasioperators

Peter Jipsen

Chapman University – Orange CA, USA

jipsen@chapman.edu

Joint research with **A. Moshier**

Lattice-ordered monoids, residuated lattices, modal lattices and De Morgan lattices are all examples of lattices with quasioperators, i.e. operations that preserve joins or send meets to joins in each argument or preserve meets or send joins to meets in each argument. In this talk we presents a duality between bounded lattices and a natural subcategory of topological spaces, and then extend this duality to bounded lattices with quasioperators. We show that the canonical extension of such algebras can be described concretely in its topological dual in a very natural way. We consider the connection to earlier dualities of Urquhart, Hartung, and Hartonas and to the canonical extension in Gehrke and Harding, as well as to Galois frames with additional relations.

Within this framework, we then discuss techniques for proving the decidability of the equational theory of various varieties of bounded lattices with quasioperators. We will also consider algorithms for constructing finite models in these varieties, and present an implementation that enumerates all models based on a given finite Galois frame.

Cyclic involutive distributive full Lambek calculus is decidable

Michał Kozak

Faculty of Mathematics and Computer Science
Adam Mickiewicz University – Poznań, Poland

mk@amu.edu.pl

We prove the decidability of the extension **CyInDFL** of **DFL** (Distributive Full Lambek Calculus), in which the involutive law is derivable with the assumption of only one negation in the language. So far, the decidability of this logic has been an open problem. We present a Gentzen-style consecution calculus which is complete with respect to the class of cyclic involutive distributive full Lambek algebras (CyInDFL). It is an extension of the system **DFL** introduced in [4]. We prove the cut elimination theorem for this system, and give a procedure for building a finite proof search tree for any consecution.

References

[1] R. Brady. Gentzenization and Decidability of some Contraction-less Relevant Logics. *Journal of Philosophical Logic*, 20:97–117, 1991.

[2] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[3] S. Giambrone. TW_+ and RW_+ are Decidable. *Journal of Philosophical Logic*, 14:235–254, 1985.

[4] M. Kozak. Distributive Full Lambek Calculus has the Finite Model Property. *Studia Logica*, 91:201–216, 2009.

Conrad frames

Jorge Martínez

University of Florida – Gainesville FL, USA

jmartine@math.ufl.edu

The question of characterizing the lattice $\mathcal{C}(G)$ of all convex ℓ-subgroups frame-theoretically remains a thorny open question. It has long been known that such frames are algebraic, that the compact elements form a sublattice, and that they possess the *disjointification property* –called "relative normality" by Tsinakis,* et. al.*, "coherent normality" by Banaschewski.

In the frames of the title disjointification is strengthened, too much, as it happens. In this talk we characterize when $\mathcal{C}(G)$ is a Conrad frame. The result is intriguing and suggestive.

Some aspects of commutative GBL-algebras and of their logic: amalgamation, interpolation and completions

Franco Montagna

Department of Mathematics, University of Siena – Italy

montagna@unisi.it

Every weakly commutative GBL-algebra embeds into a poset sum of GMV-algebras. Moreover if the GBL algebra is also commutative and bounded, then it has a subdirect embedding into a poset sum of product algebras and MV-algebras (subdirect means that the projections are surjective). In this talk we discuss some consequences of this fact: First of all, the logic of commutative and bounded GBL-algebras is constructive, in the sense that it has the disjunction property. Moreover, such logic does not have Craig's interpolation property (a counterexample will be given). As regards to amalgamation property (AP), we do not know whether the variety of commutative and bounded GBL-algebras has the AP. However, we will exhibit some subvarieties of n-potent GBL-algebras it with and without the AP. Finally, we discuss completions of commutative GBL-algebras. Some well-known completions of residuated lattices are the McNeille completion and the canonical completion. Both completions have a nice abstract characterization, but in general a variety of residuated lattices is not closed under MCNeille completions or under the canonical completions. Thus we are looking for alternative completions such that: (1) Many varieties of residuated lattices are closed under such a completion; (2) When it is impossible to get property (1), there is still a significant subclass of the variety whose elements have a completion in the variety. We present two candidates: (a) the subdirect completion, that is, decompose the algebra A into subdirectly irreducibles $A_i : i \in I$, and take the direct product of all the McNeille completions of the A_i. (b) the poset sum completion, that is, decompose the algebra A into poset sum irreducibles $A_i : i \in I$, and take the poset sum of all the McNeille completions of the A_i. Examples: (1) The variety of MTL-algebras or of IMTL-algebras is not closed under McNeille completions, but it is closed under subdirect completions. (2) An MV-algebra has a subdirect completion in the variety of MV-algebras iff it is semisimple. Otherwise, it has no completion at all. (3) A commutative and bounded GBL-algebra (or a commutative and bounded BL-algebra) has a poset sum completion iff it is a subdirect poset sum of simple MV-algebras and of product algebras whose non-zero part is simple.

Recent results on unital lattice-ordered abelian groups

Daniele Mundici

Department of Mathematics, University of Florence – Italy

mundici@math.unifi.it

We will discuss several recent results on various classes of finitely generated unital l-groups and their associated MV-algebras. We will describe the pervasive role of bases: a basis in an MV-algebra A can be regarded as an abstract unimodular triangulation of the maximal spectral space of A.

On Birkhoff's "common abstraction" problem

Francesco Paoli

University of Cagliari – Italy

paoli@unica.it

Joint research with **Constantine Tsinakis**

In his milestone textbook *Lattice theory*, Garrett Birkhoff challenged his readers to "develop a common abstraction which includes Boolean algebras and lattice ordered groups as special cases". Over the subsequent decades, several mathematicians tried their hands at this intriguing problem. In this talk we aim at reconstructing the history of these attempts, as well as at suggesting a new solution which – so we will argue – outdoes its competitors.

Consequence relations: an abstract approach

Constantine Tsinakis

Department of Mathematics, Vanderbilt University – Nashville TN, USA

constantine.tsinakis@vanderbilt.edu

Equivalences and translations between consequence relations are quite common in logic. The notion of equivalence can be defined either syntactically or in terms of the associated lattices of theories. W. Blok and D. Pigozzi proved that the two definitions coincide in the case of an algebraizable sentential deductive system. A refined treatment of this equivalence was provided by W. Blok and B. Jónsson. Other authors have extended this result to k-deductive systems and sequents. The approach proposed in this talk – based on joint work with Nick Galatos – places under a common umbrella the aforementioned results and provides a roadmap for future research in this field..