The lattice of linear Mal’cev conditions

Jakub Opršal

Charles University in Prague

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Mal’cev conditions are naturally ordered by implication. A stronger condition is *larger* then a weaker one.
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A **clone homomorphism** (or **interpretation**) from a clone $\mathcal{A}$ to a clone $\mathcal{B}$ is a map $i: \mathcal{A} \to \mathcal{B}$ mapping $n$-ary operations to $n$-operations, and preserving composition and projections.

**Interpretation** from a variety $\mathcal{V}$ to a variety $\mathcal{W}$ is a functor $I: \mathcal{W} \to \mathcal{V}$ that is commuting with forgetful functors.
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Interpretability form quasi-order. By a standard technique, we can get the corresponding partial order (we factor by equi-interpretable).

(Garcia, Taylor: The lattice of interpretability types of varieties, 1984.)
Join of two Mal’cev conditions is the condition given by conjunction of the two.

Join of two varieties $V$ and $W$ in can be described as the variety $V \lor W$ whose operations are operations of both varieties (taken as a discrete union of operations of $V$ and operations $W$), and whose identities are all identities of both varieties.

In the other words, we can describe algebras in $V \lor W$ as $(A, F \cup G)$ where $(A, F) \in V$ and $(A, G) \in W$.

Examples

▶ Mal’cev $\lor$ Jónsson terms = Pixley term,
▶ Jónsson terms $\lor$ cube term = near unanimity.
▶ Gumm terms $\lor$ SD$(\lor)$ = Jónsson terms.
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Join of two varieties $\mathcal{V}$ and $\mathcal{W}$ in can be described as the variety $\mathcal{V} \vee \mathcal{W}$ whose operations are operations of both varieties (taken as a discrete union of operations of $\mathcal{V}$ and operations $\mathcal{W}$), and whose identities are all identities of both varieties.

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- Mal’cev $\vee$ Jónsson terms $=$ Pixley term,
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Meet of two abstract clones $\mathcal{A}$ and $\mathcal{B}$ is a clone $\mathcal{A} \times \mathcal{B}$ (the product in the category of clones) that is described by

$$(\mathcal{A} \times \mathcal{B})^{[n]} = \mathcal{A}^{[n]} \times \mathcal{B}^{[n]}$$

with the obvious composition, and obvious projections.
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For varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ the meet is described as the variety $\mathcal{V}_1 \times \mathcal{V}_2$ that is defined in such a way that

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2. it has two subvarieties \( \mathcal{V}_1' \) and \( \mathcal{V}_2' \) that are equi-interpretable with \( \mathcal{V}_1 \), \( \mathcal{V}_2 \) respectively (\( \mathcal{V}_i \) satisfies \( x_1 \cdot x_2 \approx x_i \)),

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1. its signature is disjoint union of signatures of \( \mathcal{V}_1 \) and \( \mathcal{V} \) with a new binary symbol \( \cdot \),
2. it has two subvarieties \( \mathcal{V}'_1 \) and \( \mathcal{V}'_2 \) that are equi-interpretable with \( \mathcal{V}_1 \), \( \mathcal{V}_2 \) respectively (\( \mathcal{V}_i \) satisfies \( x_1 \cdot x_2 \approx x_i \)),
3. every algebra in \( \mathcal{V}_1 \times \mathcal{V}_2 \) is a product of an algebra from \( \mathcal{V}'_1 \) and an algebra from \( \mathcal{V}'_2 \).
A linear Mal’cev condition is a condition that do not include term composition, i.e., only equations of the form

\[ f(x_{i_1}, \ldots, x_{i_n}) \approx g(x_{j_1}, \ldots, x_{i_m}), \quad \text{or} \quad f(x_{i_1}, \ldots, x_{i_n}) \approx x_j \]

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group terms, lattice terms, semilattice term, congruence uniformity, congruence singularity?.

Linear Mal’cev condition forms a subposet of the lattice of all Mal’cev conditions.
But, the subposet is not a sublattice!
Proposition

*Meet of Mal’cev term and congruence distributivity is not equivalent to any linear Mal’cev condition.*
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Definition (Barto, Pinsker)

An algebra $A$ is said to be a retract of $B$ if there are two maps $a: B \to A$ and $b: A \to B$ such that $ab = 1_A$, and for every basic operation $f$ we have

$$f_A(a_1, \ldots, a_n) = af_B(b(a_1), \ldots, b(a_n)).$$
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Observation

If $A$ is a retract of $B$ then $A$ satisfies all the linear equations that $B$ does.
We will show that meet of Mal’cev and majority is not linear.
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- $\mathcal{V}_1$ be the variety with single ternary Mal’cev operation $q$, 

- $\mathcal{V}_2$ be the variety with the majority operation $m$, 

- $\mathcal{W}$ a variety equi-interpretable with $\mathcal{V}_1 \times \mathcal{V}_2$ that is defined by linear equations.

We choose algebra in $\mathcal{V}_1'$ that has no Jonsson terms, and similarly algebra in $\mathcal{V}_2'$ that has no Mal’cev term. For example $A = (\{0, 1\}, x + y + z, \text{proj}_3 1, \text{proj}_2 1)$, and $B = (\{0, 1\}, \text{proj}_3 1, (x \lor y) \land (y \lor z) \land (x \lor z), \text{proj}_2 2)$. 


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- $A = (\{0, 1\}, x + y + z, \text{proj}_1^3, \text{proj}_1^2)$, and
- $B = (\{0, 1\}, \text{proj}_1^3, (x \lor y) \land (y \lor z) \land (x \lor z), \text{proj}_2^2)$. 
Consider the interpretation of $A \times B$ in $\mathcal{W}$, and take its retract $C$ via

\[
\begin{align*}
(0,0) & \quad \rightarrow \quad 0 \\
(0,1) & \quad \rightarrow \quad 1 \\
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Finally, let $C'$ be the interpretation of $C$ in $\mathcal{V}_1 \times \mathcal{V}_2$. Then

1. Both $B' = \{0,1\}$ and $A' = \{1,2\}$ are subuniverses of $C'$, $\text{Clo } A'$ is a reduct of $\text{Clo } A$, and $\text{Clo } B'$ is a reduct of $\text{Clo } B$. 
Consider the interpretation of $A \times B$ in $\mathcal{V}$, and take its retract $C$ via

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2. $|C'| = 3$ which is a prime! So, either $C' \in \mathcal{V}_1$, or $C' \in \mathcal{V}_2$,
Meet of linear conditions (cont.)

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3. but neither is possible since $A$ has no majority term, and $B$ has no Mal’cev term!
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Lattice of linear varieties

Problems with Mal’cev conditions
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These problems can be solved by taking all linear varieties instead. (We lose Mal’cev conditions that are not strong.)
Prime elements of the lattice

Let $X$ be a given set of variables, and $A \subseteq \mathrm{Eq}(X)$. We say that variety $V$ is $A$-colorable if there is a map $c: F_V(X) \rightarrow X$ such that

1. $c(x) = x$ for all $x \in X$, and
2. for every $\alpha \in A$ whenever $f \sim \hat{\alpha} g$ then $c(f) \sim \alpha c(g)$

where $\hat{\alpha}$ denotes the congruence of the free algebra over $X$ generated by $\alpha$.

We say that Mal'cev condition $P$ satisfies coloring condition $A$ if variety $V$ satisfies $P$ if and only if $V$ is not $A$-colorable.

Many of Mal'cev conditions that are suspected to be prime satisfy some coloring condition. Namely

- congruence $n$-permutability,
- congruence modularity,
- satisfying non-trivial congruence identity,
- $n$-cube terms,
- triviality ($x \approx y$).
(Sequeira, Barto) Let $X$ be a given set of variables, and $A \subseteq \text{Eq}(X)$. We say that variety $\mathcal{V}$ is $A$-colorable if there is a map $c: F_\mathcal{V}(X) \to X$ such that

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Theorem (Sequeira; Bentz-Sequeira)

Congruence modularity, $n$-permutability, satisfying non-trivial congruence identity, and $n$-cube term are prime with respect to varieties axiomatized by linear equations.
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Theorem (O.)

A Mal’cev condition $\mathcal{M}$ satisfy coloring condition $A$ if and only if for every linear variety $\mathcal{V}$ we have that either $\mathcal{V}$ satisfies $\mathcal{M}$, or $\mathcal{V}$ is interpretable in $\text{Pol}(X, A)$.
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Proof.

Suppose that $\mathcal{V}$ is linear and $A$-colorable ($A \subseteq \text{Eq} \ X$). Then we define an interpretation $i: \mathcal{V} \rightarrow \text{Pol}(X, A)$ as

$$i(f)(x_0, \ldots, x_n) = c(f(x_0, x_1, \ldots, x_n))$$

for every basic operation $f$, and extend to terms.
Theorem (Sequeira; Bentz-Sequeira)

Congruence modularity, n-permutability, satisfying non-trivial congruence identity, and \textit{n-cube term} are prime with respect to varieties axiomatized by linear equations.

Theorem (O.)

A Mal’cev condition \( M \) satisfy coloring condition \( A \) if and only if for every linear variety \( V \) we have that either \( V \) satisfies \( M \), or \( V \) is interpretable in \( \text{Pol}(X, A) \).

Proof.

Suppose that \( V \) is linear and \( A \)-colorable \((A \subseteq \text{Eq} X)\). Then we define an interpretation \( i: V \rightarrow \text{Pol}(X, A) \) as

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for every basic operation \( f \), and extend to terms.
We say that two subsets of elements $A$, and $B$ split a lattice if for every element $x$ of the lattice we have either $a \leq x$ for some $a \in A$, or $x \leq b$ for some $b \in B$. 

Theorem (Valeriote, Willard, 2014) \n$n$-permutability and idempotent polymorphisms of two-element poset split the lattice of idempotent varieties. 

Theorem (Kiss, Kearnes, 2013) \nSatisfying a non-trivial congruence identity and the set $\{\text{Pol}(L) : L \text{ is a semilattice}\}$ split the lattice of idempotent varieties.
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**Theorem (Valeriote, Willard, 2014)**

*n-permutability and idempotent polymorphisms of two-element poset split the lattice of idempotent varieties.*

**Theorem (Kiss, Kearnes, 2013)**

*Satisfying a non-trivial congruence identity and the set \{Pol($L$) : $L$ is a semilattice} split the lattice of idempotent varieties.*
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Find a satisfactory description of linear meet.
Some open problems...

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Is CM the linear meet of Mal’cev and CD?
Some open problems...

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Is CM the linear meet of Mal’cev and CD?

Problem
Does every prime element of the linear lattice satisfy some coloring condition? (Need to consider a little generalized conditions.)

Thank you for your attention!
Some open problems...

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Is CM the linear meet of Mal’cev and CD?

Problem
Does every prime element of the linear lattice satisfy some coloring condition? (Need to consider a little generalized conditions.)

Problem
Are the Mal’cev conditions that satisfy some coloring condition prime? (Known for Mal’cev, cyclic terms, not known for everything else.)
Some open problems... 

Problem
Find a satisfactory description of linear meet.

Problem
Is CM the linear meet of Mal’cev and CD?

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Thank you for your attention!