A representation theorem for quantale algebras *

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While the study of quantale-like structures goes back up to the 1930’s (notwithstanding that the term itself was introduced in [9] in connection with certain aspects of C*-algebras), there has recently been much interest in quantales in a variety of contexts. The most important connection probably is with Girard’s linear logic (see, e.g., [3]). In particular, [1] enunciates the following slogan: Quantales are to linear logic as frames are to the intuitionistic one.

With this paper we hope to contribute to the theory of quantales. It considers the notion of quantale algebra and shows a representation theorem for such structures generalizing the following representation theorem for quantales from [13] (notice that \(\mathcal{P}(\cdot)\) stands for the power-set):

**Theorem 1.** If \(Q\) is a quantale, there is a semigroup \(S\) and a quantic nucleus \(\mathcal{P}(S) \xrightarrow{\beta} \mathcal{P}(S)\) such that \(Q \cong \mathcal{P}(S)\).

To be precise we start by introducing the category \(Q\text{-}\text{Alg}\) of algebras over a given unital commutative quantale \(Q\) as follows (for clearness sake recall the notion of quantale algebra first).

**Definition 2.** A **quantale** is a triple \((Q, \leq, \otimes)\) such that
- \((Q, \leq)\) is a join-lattice;
- \((Q, \otimes)\) is a semigroup;
- \(q \otimes (\bigvee T) = \bigvee_{t \in T} (q \otimes t)\) and \((\bigvee T) \otimes q = \bigvee_{t \in T} (t \otimes q)\) for every \(q \in Q\) and every \(T \subseteq Q\).

A quantale \(Q\) is said to be **unital** provided that there exists an element \(1 \in Q\) such that \((Q, \otimes, 1)\) is a monoid. \(Q\) is said to be **commutative** provided that \(a \otimes b = b \otimes a\) for every \(a, b \in Q\).

**Definition 3.** Let \(Q\) be a unital commutative quantale. \(Q\text{-}\text{Alg}\) is the category, the objects of which (called **\(Q\)-algebras**) are unital \(Q\)-modules \((A, \ast)\) such that

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\bullet \ A \text{ is a quantale}; \\
\bullet \ q \ast (a \otimes b) = (q \ast a) \otimes b = a \otimes (q \ast b) \text{ for every } a, b \in A \text{ and every } q \in Q.
\]

The morphisms of the category (called \(Q\)-algebra homomorphisms) are quantale homomorphisms \(A \xrightarrow{j} B\) which are also \(Q\)-module homomorphisms.

The definition of the category \(Q\text{-Alg}\) is motivated by that of the category \(K\text{-Alg}\) of algebras over a given commutative ring \(K\) with identity (see, e.g., [4]). It generalizes the categories \(\text{Quant}\) of quantales (see, e.g., [13, 14]) and \(Q\text{-Mod}\) of (left) modules over \(Q\) (see, e.g., [1, 6, 11, 14, 15]) motivated by constructions and results from modules over a ring. Notice that while the category \(Q\text{-Mod}\) is much investigated (the first lattice analogy of ring module appeared in [6] in connection with analysis of descent theory and although the authors work with commutative structures most of the results are also valid for non-commutative case; the idea of quantale module appeared in [1] as the key notion for treatment of process semantics) the category \(Q\text{-Alg}\) is not so widespread. It is our purpose to make up the deficiency.

We continue by constructing a free quantale algebra from a semigroup, i.e., show the existence of a left adjoint for the forgetful functor to the category \(\text{SGrp}\) of semigroups (this fact together with Chapter 1 of [8] imply that the category \(Q\text{-Alg}\) is monadic over the category \(\text{Set}\) of sets). The construction goes analogously to that of a monoid ring (see, e.g., [7]): If \(S\) is a semigroup, let \(Q^S\) be the set of all maps \(S \xrightarrow{\alpha} Q\). Then \(Q^S\) is a unital \(Q\)-module with the operations given point-wise. For \(\alpha, \beta \in Q^S\) define \((\alpha \otimes \beta)(c) = \bigvee_{a \cdot b = c} \alpha(a) \otimes \beta(b)\).

Then \((Q^S, \otimes)\) is a \(Q\)-algebra free over \(S\) (denoted by \(F(S)\)).

Next we introduce the notion of quantale algebra nucleus which generalizes the notions of quantic nucleus (see, e.g., [10, 13, 14]), module nucleus (see, e.g., [11, 12]) and locale nucleus (see, e.g., [5]).

**Definition 4.** Let \(A\) be a \(Q\)-algebra. A map \(A \xrightarrow{j} A\) is called a quantum algebra nucleus on \(A\) provided that for every \(a, b \in A\) and every \(q \in Q\) the following hold:

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\bullet \ a \leq b \text{ implies } j(a) \leq j(b); \\
\bullet \ a \leq j(a); \\
\bullet \ j \circ j(a) \leq j(a); \\
\bullet \ j(a) \otimes j(b) \leq j(a \otimes b); \\
\bullet \ q \ast j(a) \leq j(q \ast a).
\]

**Proposition 5.** Let \(A\) be a \(Q\)-algebra and let \(A \xrightarrow{j} A\) be a nucleus on \(A\). Set \(A_j = \{a \in A \mid j(a) = a\}\). Then \(A_j = j^{-1}(A)\) and, moreover, \(A_j\) is a \(Q\)-algebra with the following structure:
• $\bigvee_j T = j(\bigvee T)$ for every $T \subseteq A_j$;
• $a \otimes_j b = j(a \otimes b)$ for every $a, b \in A_j$;
• $q \ast_j a = j(q \ast a)$ for every $q \in Q$ and every $a \in A_j$.

With the aforesaid notion in mind we prove the following representation theorem for quantale algebras.

**Theorem 6.** If $A$ is a $Q$-algebra, there exists a semigroup $S$ and a quantale algebra nucleus $F(S) \xrightarrow{j} F(S)$ such that $A \cong F(S)_j$.

The necessary categorical background can be found in [2, 8]. It is expected from the reader to be acquainted with basic concepts of Category Theory, e.g., with that of category and functor.

**References**


