1 Introductory remarks and results

Abel considered associative functions in [1], which is arguably the first paper about semigroups [12]. He considers a real function \( f \) such that the function \( F(x, y, z) := f(z, f(x, y)) \) is invariant under any permutation of the variables \( x, y, z \). Abel calls such a function \( F \) (which is invariant under any permutation of variables) a symmetric function. This denotation implicitly refers to a connection between associativity and a kind of symmetry.

When considering binary operations on intervals of the real line, it is clear that their commutativity is readily seen from their graph. Associativity cannot be seen so easily. Figure 1 illustrates three commutative, associative binary operations on \([0, 1]\); the minimum, the product, and the so-called Łukasiewicz t-norm, respectively. One can immediately see from the graphs the commutativity of these operations but not their associativity. Following Abel we argue that associativity (together with commutativity) may be understood as a kind of symmetry in four-dimensional space. Indeed, consider the graph of a commutative, associative binary operation \( \circ \) on \( X \) which is a subset of \( X^3 \) defined by \( \{(x, y, z) \in X^3 \mid z = x \circ y\} \). Commutativity of the operation is readily seen from the graph since it is equivalent to the invariance of the graph with respect to a reflection at the plane \( \{(x, y, z) \mid x = y\} \). Unfortunately, there exists no method for seeing the associativity of \( \circ \) in a similar manner. However, consider now \( \{(x, y, z, v) \in X^4 \mid v = (x \circ y) \circ z\} \) and call it the four-dimensional graph of \( \circ \). Associativity together with commutativity is clearly equivalent to the arguments of \( (x \circ y) \circ z \) being freely interchangeable. This is equivalent to the four-dimensional graph being invariant with respect to three reflections at the hyperplanes, given by \( \{(x, y, z, v) \in X^4 \mid x = y\} \), \( \{(x, y, z, v) \in X^4 \mid y = z\} \), and \( \{(x, y, z, v) \in X^4 \mid z = x\} \), respectively.

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The first aim of this talk is to present a method for deciding associativity from the three-dimensional graph of an operation, and even from sections of thereof. Thus, a geometric characterization of associativity is presented. Even though the related visualization is possible only on compact intervals of \( \mathbb{R} \), all the results of the geometric characterization are true on partially-ordered sets too. The aim of the talk is not only to introduce a kind of visualization aspect but, in addition, to understand algebraic relationships in a geometric manner. Visualization serves as a tool.

The basic notions are the so called rotation-invariance and pseudo-inverse properties\(^1\).

The geometric characterization turns out to be fruitful in conjecturing and proving algebraic results over, this geometric description has provided the intuition for a contribution to solving a long-standing open problem in the field of associative functions \([2, 7]\). Some corresponding results, all of which use geometric arguments, will be mentioned in the second part of the talk. A few of them are listed below. It is interesting to notice that when conjecturing algebraic results based on geometric motivations, one can first heuristically prove them using geometry; then the steps of the geometric hint can be translated step by step into an algebraic proof.

A few applications

**Theorem 1 (Disconnected Rotation)** \([9]\) Let \( \mathcal{M} = (M, \leq, 1, \circ, \rightarrow) \) be a commutative, residuated groupoid on a poset with top element 1. Extend \( M \) with a disjoint copy of it, which is equipped with the dual order of \( \leq \), and such that every element of it is smaller that any element of \( M \). Denote the resulting poset by \((M^*, \leq)\). Let \( \neg \) be the mapping on \( M^* \) which maps every element of \( M \) into its pair in the disjoint copy of \( M \), and every element of \( M^* \setminus M \) into its preimage in \( M \). Extend \( \circ \) and its residuum to \( M^* \) as follows: Let

\[
x \circ y = \begin{cases} x \circ y & \text{if } x, y \in M \\ \neg (x \circ \neg y) & \text{if } x \in M \text{ and } y \in M^* \setminus M \\ \neg (\neg y \circ x) & \text{if } x \in M^* \setminus M \text{ and } y \in M \\ 1 & \text{if } x, y \in M^* \setminus M
\end{cases}
\]

\[
x \circ \neg y = \begin{cases} x \circ \neg y & \text{if } x, y \in M^+ \\ \neg (x \circ \neg y) & \text{if } x \in M^+ \text{ and } y \in M^- \\ 1 & \text{if } x \in M^- \text{ and } y \in M^+ \\ \neg y \circ \neg x & \text{if } x, y \in M^-
\end{cases}
\]

Then \( \mathcal{M}^* = (M^*, \leq, \circ, \rightarrow, \neg, 1, \neg 1) \) (called the disconnected rotation of \((M, \leq, \circ, \rightarrow)\)) is a commutative, residuated groupoid, and its \( \neg 1 \)-level function coincides with \( \neg \).

The disconnected rotation operator, as described above, preserves associativity, conjunctivity, unit element, integrality, and being lattice-ordered.

**Theorem 2** \([9]\) The disconnected rotation of the negative cone of a commutative \( \ell \)-group is an \( MV \)-algebra.

**Theorem 3** Let \((L, \circ, \rightarrow)\) be a totally-ordered, residuated, integral \( \ell \)-monoid. Assume that \( c, e \in L \) are involutive\(^2\). Then \( a = \neg^e (\neg^e c) \) is involutive as well, \( e \leq a \), and for \( x \in [a, 1] \) we have \( \neg^e x = \neg^c (\neg^e (\neg^c x)) \).

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\(^1\)Let \((M, \leq)\) be a poset and \((M, \circ, \rightarrow)\) be a commutative residuated groupoid. Let \( c \in M \). Denote \( \neg^c x = x \rightarrow \circ c \). We say that \( \circ \) is rotation-invariant with respect to \( \neg^c \) if for all \( x, y, z \in M \) we have \( x \circ y \leq \neg^c z \implies z \circ x \leq \neg^c y \). We say that \( \circ \) admits the pseudo-inverse property with respect to \( c \) if for all \( x, y, z \in M \) we have \( x \rightarrow \circ \neg^c y = \neg^c (x \circ y) \).

\(^2\)Let \((M, \circ, \leq, 1)\) be an integral-monoid. An element \( c \in M \) is called involutive if the mapping (of type \( M_{\geq c} \rightarrow M_{\geq c} \)) defined by \( x \mapsto \neg^c x \) is an involution on \( M_{\geq c} \). In other words, \( c \in M \) is called involutive if all the elements which are greater than or equal to \( c \) are \( c \)-closed.
Theorem 4 Let $\ast$ be binary operation on a bounded poset $(M, \leq, 0, 1)$ such that $(x \ast y) \ast z = 0$ if and only if $x \ast (y \ast z) = 0$. Assume that $x \to_z 0$ exists for $x \in M$, and that $x \mapsto x \to_z 0$ is an involution of $M$. Then $\ast$ is a residuated, partially-ordered (lattice-ordered, if $M$ is a lattice), integral-monoid.

Theorem 5 [9] Suppose $(M, \leq)$ is a poset and $(M, \ast, \leq, 1, 0)$ is a bounded po-groupoid with an involutive bottom element $0$. The operation $\ast$ is divisible if and only if it is strictly increasing on the restricted positive domain.

Theorem 6 Let $(M, \leq, 0)$ be a poset with bottom element $0$, and let $\ast$ be a semigroup operation on $M$. Then $(M, \ast)$ is the semigroup reduct of a classical residuated lattice\(^3\) if and only if $\ast$ is commutative and $0$ is involutive. Moreover, $(M, \ast)$ is the semigroup reduct of an MV-algebra if and only if all elements of $M$ are involutive.

Theorem 7 [7] Let $A$ and $B$ be two different left-continuous t-norms, and let $p \in [0, 1]$. Assume that there exists $u \in [0, 1]$ such that $u$ is involutive with respect to $A$, and that the $u$-level functions of $A$ and $B$ coincide. Then $p \cdot A(x, y) + (1 - p) \cdot B(x, y)$ is not associative.

The whole talk is based on [8].

References


\(^3\)A residuated lattice is called classical if it has an involutive bottom element.