Compatible functions on commutative residuated lattices

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A commutative residuated lattice, or CRL for short, is essentially a distributive lattice $A$ equipped with a commutative monoid structure $(A, \cdot, e)$ such that for every $x \in A$, $a \cdot (\_)$ has a right adjoint, which will usually be denoted by $x \rightarrow (\_)$.

Definition: Let $L$ be a CRL. A function $f : L \rightarrow L$ is compatible with a congruence $\theta$ of $L$ if $(x, y) \in \theta$ implies $(f(x), f(y)) \in \theta$.

We say that $f$ is a compatible function of $L$ provided it is compatible with all the congruences of $L$ [1].

Let $L$ be a CRL and $x \in L$. Write $x^- := x \wedge e$ and take

$$s(x, y) := (x \rightarrow y)^- \cdot (y \rightarrow x)^- \quad (1)$$

We can give a general characterization of a compatible function $f : L \rightarrow L$ for any CRL $L$ as follows,

Theorem 1: Let $L$ be a CRL and $f : L \rightarrow L$ a map. The function $f$ is compatible if and only if for every $x, y \in L$ there is a positive integer $n$ such that

$$s(x, y)^n \leq s(f(x), f(y)) \quad (2)$$
Some particular cases of this theorem are well known results (see [1] and [3]). For example, when \( L \) is a Heyting algebra or a generalized Heyting algebra, \( n \) can be taken to be 1.

Using Theorem 1 we can prove the following

**Theorem 2:** Let \( f : L \to L \) be a compatible function, \( B \) a finite subset of \( A \) and \( x \in B \). Let \( T = \{s(b,x)^{n_b} \cdot f(b), \ b \in B\} \), where \( n_b \) is the integer associated to each pair \((b, x)\) by Theorem 1. Then, \( f(x) = \sqrt{T} \).

As an interesting consequence we obtain that

**Corollary:** The variety of CRLs is locally affine complete.

Let \( L \) be a CRL and \( P(x, y) \) be a polynomial in the language of CRLs. We say that a function \( f : L \to L \) is \( P \)-definable whenever the following inequalities hold

\[
\begin{align*}
(S1) \quad & P(x, f(x))) \leq f(x) \\
(S2) \quad & f(x) \leq y \lor P(x, y)
\end{align*}
\]

**Theorem 3:** Let \( L \) be a CRL, \( P(x, y) \) a polynomial in \( L \) and \( f : L \to L \) \( P \)-definable. Then \( f \) is compatible in \( L \) provided the following inequality holds:

\[
P(x, f(y)) \cdot (x \to y) \leq P(y, f(y)).
\]

It can be shown that every \( P \)-definable function is of the form \( x \mapsto \min E_P(x) \), with \( E_P(x) = \{y \in L : P(x, y) \leq y\} \).

**Example 1:** Consider \( L = ((0,1], \wedge, \lor, \cdot, /, 1) \) the product algebra with Goguen implication. The polynomial \( P_n(x, y) := x/y^n \) induces the \( P \)-definable function \( S_n(x) = \sqrt[n]{x} \). The operations \( S_n \) are compatible because condition (3) trivially holds for them.

Contrary to what is expected for Heyting algebras [2], the variety of CRLs enriched with \( S_1 \) is not affine complete, as the following example shows.
Example 2: Let $E$ be the subfield of the reals whose elements are the numbers constructible over the rationals. Take $A = (0,1) \cap E$ with its inherited operations. Since each element of $A$ has a square root in $A$, it follows that the successor $S(x) = \sqrt{x}$ is defined for every $x \in A$.

On the other hand, consider $1/2 \in A$. $\sqrt{1/2}$ is a root of the irreducible polynomial $2x^3 - 1 \in \mathbb{Q}(i)[x]$, and so its characteristic polynomial is of degree 3, which is not a power of 2. Thus, $\sqrt{1/2} \notin A$. Consider now the set $E_2(1/2) = \{y \in A : 1/2 \leq y\}$. Since $\mathbb{Q} \subseteq E$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, $A$ is dense in $(0,1]$. Thus, $E_2(1/2)$ has no minimum. So we have proved that $S_2$ can not be defined on $A$.

From Example 2 we conclude that $S_2$ is not given by a term in the category of CRLs with $S_1$. One can now ask whether in the the variety of CRLs with an operation $S_2$, $S_1$ becomes a term or not. The answer is no, as the next example shows.

Example 3: Consider the ordered abelian group (and hence CRL) $(\mathbb{Q}, +, 0)$. Write $\mathbb{P} := \{p/2q \in \mathbb{Q} : p, q \in \mathbb{Z} - \{0\}, (p, q) = (p, 2) = 1\}$. It is clear that $\mathbb{S} := \mathbb{Q} - \mathbb{P}$ is an ordered subgroup of $\mathbb{Q}$.

Take $E_1(1) := \{y \in \mathbb{S} : 1 \leq 2y\}$. If we consider the sequence $\{\frac{n}{2^{m-1}} : n \geq 1\}$ which is contained in $\mathbb{S}$ and converges from above to $1/2$, we conclude that $E_1(1)$ does not have a minimum in $\mathbb{S}$. Hence $S_1$ is not defined on $\mathbb{S}$.

On the other hand, for every $x \in \mathbb{S}$, $E_2(x) = \{y \in \mathbb{S} : 1 \leq 3y\}$ has always a minimum in $\mathbb{S}$: $S_2(x) = x/3$.

References

