Boolean Skeleton and Pierce representation of Bounded BCK-algebras

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Introduction

BCK-algebras were introduced by K. Iseki in [6] in order to give an algebraic framework for Meredith's implicational logic BCK ("BCK logic"). Bounded BCK-algebras were also introduced by Iseki in [7] as BCK-algebras with an additional constant which is interpreted as the lower bound. In fact, they are the algebraic counterpart of the BCK-logic plus a negation satisfying the Duns Scoto law. Since this can be expressed by means of a simple axiom, Bounded BCK-logic is algebraizable (in the sense of Blok Pigozzi [2]) and its equivalent algebraic semantics is the class of Bounded BCK-algebras.

On the other hand, distributive congruence varieties have the Boolean factor property, i.e., the factor congruences of its members form a Boolean algebra, and hence they can be represented as weak Boolean product of algebras in the variety, i.e., isomorphic to the global sections of a Boolean sheaf, called Pierce sheaf in [1].

The class of all bounded BCK-algebras is not a variety, but it is a quasivariety relatively congruence distributive Then our main purpose is to study the representability of Bounded BCK-algebras as a weak Boolean product of BCK-algebras, in similar form as the given in [1, 4] for varieties. It is clear that in our case we need to relativize the used notions.

We recall that an i-filter of a bounded BCK-algebra \( B \) is a subset \( F \) of \( B \) such that

(I1) \( \top \in F \)

and

(I2) \( a, a \rightarrow b \in F \) implies \( b \in F \).

Then the correspondence \( \theta \mapsto \top/\theta \) gives an order isomorphism from \( \text{Con}_{BCK}(B) \) onto the family \( F_i(B) \) of all implicative filters of \( B \), both ordered by inclusion. Its inverse is given by \( F \mapsto \theta_F = \{ (a, b) \in B \times B : a \rightarrow b, b \rightarrow a \in F \} \).

Our first main task is to show that in any bounded BCK-algebra \( A \), its BCK-factor congruences, or factor BCK-congruences with a BCK-congruence as a companion factor, form a Boolean algebra. That is, \( A \) has a relative "Boolean BCK-factor property". To see this we show that BCK-factors can be identify with elements of the algebra, called factor elements, whose set

\[
B_F(A) = \{ a \in A : \langle a \rangle \cap \langle \neg a \rangle = \{ \top \} \text{ and } \theta_{\langle a \rangle} \circ \theta_{\langle \neg a \rangle} = \nabla A \}.
\]

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is the universe of a subalgebra $B_F(A)$, called Boolean skeleton, of a suitable Boolean algebra $B_C(A)$ called Boolean center, which is a subalgebra of $A$. The elements in $B_C(A) = \{a \in A : (a) \cap (\neg a) = \{\top\}\}$, called Boolean elements, are the members of $A$ whose generated implicative filters is complemented in the distributive lattice of i-filters, and its complement is the implicative filter generated by its negation.

1 Boolean and Pierce representation of bounded BCK-algebras.

We recall that an algebra $A$ is representable as a weak Boolean product of a family $(A_x : x \in X)$ when

Br(1) There exists a subdirect embedding $\alpha$ from $A$ into $\prod_{x \in X} A_x$.

Br(2) There is a Boolean topological space carried on $X$, i.e., compact Hausdorff with the family of clopen subsets as a open basis, such that:

- Br(2a) for any $a, b \in A$, $[\alpha a = \alpha b] = \{x \in X : \alpha a(x) = \alpha b(x)\}$ is open subset of $X$.
- Br(2b) If $N$ is a clopen subset of $X$, then for any $a, b \in A$,

$$aa|_N \cup ab|_{X-N} \in \alpha(A)$$

By requiring in condition Br(2a) that $[\alpha a = \alpha b]$ be clopen we say that $A$ is representable as a Boolean product. As it is explained in [5], weak Boolean products (Boolean products) are the global sections of sheaves (Hausdorff sheaves) of algebras over Boolean spaces. The algebras $A_x$ are called stalks.

**Theorem 1.1** Any bounded BCK-algebra admits a weak Boolean representation of bounded BCK-algebras for each subalgebra of $B_F(A)$ over its associated Boolean space. Moreover every representation as a weak Boolean product of bounded BCK-algebras is equivalent to one of the above representations.

It follows from previous result that the weak Boolean representation given by $B_F(A)$ is the finest representation and it is called weak Pierce BCK-representation. In this case, when the representation is as Boolean product we call it Pierce BCK-representation.

Next we analyze these representations provided that its stalks satisfy some conditions. For instance, if we impose each stalk to be directly BCK-indecomposable we get what we call a good weak Pierce BCK-representation.

**Theorem 1.2** Let $Q$ be a relative subvariety of $bBCK$. Then $Q$ is well weak Pierce representable if and only if for any $A \in Q$, $B_F(A) = B_C(A) = \{a \in A : a \rightarrow \neg a = \neg a$ and $\neg a \rightarrow a = a\}$

We also characterize classes of BCK-algebras such that in its (weak) Pierce BCK-representations the stalks are finitely subdirectly BCK-irreducible or simple BCK-algebras.

**Theorem 1.3** For each bounded BCK-algebra $A$, the following are equivalent:
(i) \( \mathcal{A} \) is representable as a weak Boolean product of finitely subdirectly b\( \mathbb{BCK} \)-irreducible,

(ii) For each \( p \in \text{Prim}(B_F(\mathcal{A})) \), \( \langle p \rangle \in \text{Prim}(\mathcal{A}) \)

(iii) \( m\text{Prim}(\mathcal{A}) = \{ \langle p \rangle : p \in \text{Prim}(B_F(\mathcal{A})) \} \).

where \( \text{Prim}(\mathcal{A}) \) denotes the set of all proper prime \( i \)-filters of \( \mathcal{A} \) and \( m\text{Prim}(\mathcal{A}) \) the set of all the minimal elements in \( (\text{Prim}(\mathcal{A}), \subseteq) \)

**Theorem 1.4** Let \( Q \) be a relative subvariety of \( b\mathbb{BCK} \), then the following conditions are equivalent:

(i) All algebras in \( Q \) are hyperarchimedean i.e. weak Pierce BCK-representable with simple algebras as stalks.

(ii) There is \( n \leq 1 \) such that the equation

\[
(EM_n) (x \rightarrow y) \rightarrow (((x^n \rightarrow \perp) \rightarrow y) \rightarrow y) \approx \top
\]

hold in \( Q \), and for any \( \mathcal{A} \in Q \), \( B_F(\mathcal{A}) = B_C(\mathcal{A}) = \{ a \in A : a \rightarrow \neg a = \neg a \) and \( \neg a \rightarrow a = a \} \).

Moreover, if the above are true, then \( Q \) is a variety.

Through all the work we also include several examples of bounded BCK-algebras and relative subvarieties of bounded BCK-algebras, to illustrate our definitions and the independence of some notions.

**References**


