1 Introduction

A relation algebra is an algebra $\mathfrak{A} = \langle A, +, -, ;, 1' \rangle$ of type $(2, 1, 2, 1, 0)$, such that $\langle A, +, - \rangle$ is a boolean algebra, $\langle A, ;, 1' \rangle$ is an involuted monoid, and which satisfies the cycle law

$$(x;y) \cdot z \neq 0 \iff (z;\bar{y}) \cdot x \neq 0$$

For atoms $a, b, c$, if $(a;b) \cdot c \neq 0$ then $abc$ is called a mandatory cycle in $\mathfrak{A}$. Any cycle that is not mandatory is forbidden. $\mathfrak{A}$ is said to be representable if it is isomorphic to a subalgebra of $\langle \text{Sb}(E), \cup, -, \circ, -1, \text{Id} \rangle$, where $\text{Sb}(E)$ is all subrelations on some equivalence relation $E$ and all operations are the ordinary set-theoretic operations.

An atom $r$ in a relation algebra is said to be flexible if all cycles involving $r$ (but not involving $1'$) are mandatory. It is not difficult to show that every finite integral relation algebra with a flexible atom is representable over an infinite set. (An algebra is integral if $1'$ is an atom.) In [2], Maddux asked whether every finite integral relation algebra with a flexible atom was representable over a finite set. In [1], it was shown that finite representations are possible in case all atoms are symmetric ($\bar{a} = a$) all non-identity atoms are flexible. This proof was probabilistic in nature. In this paper, we prove that finite representations are possible in case all atoms are symmetric, and there is exactly one flexible atom.

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which participates in every mandatory cycle. Probabilistic methods alone were too messy for this special case; however, a combination of direct construction and random methods proved fruitful.

2 Constructing large representations

First we will consider the algebra \( A \) with three symmetric atoms \( 1', r, \) and \( b \), with cycles \( rrr, rrb, \) and \( rbb \) (and all permutations thereof). We construct arbitrarily large representations of this algebra. To do so, we need an edge-colored complete graph on two colors, red and blue, such that the following holds:

Given distinct vertices \( x \) and \( y \), if edge \( xy \) is colored red, then there exist vertices \( z_1, z_2, z_3, \) and \( z_4 \) such that the following obtain:

(i) edges \( xz_1 \) and \( yz_1 \) are both red;
(ii) edge \( xz_2 \) is blue and edge \( yz_2 \) is red;
(iii) edge \( xz_3 \) is red and edge \( yz_3 \) is blue; and
(iv) edges \( xz_4 \) and \( yz_4 \) are both blue,

and if edge \( xy \) is colored blue, then there exist vertices \( w_1, w_2, \) and \( w_3 \) such that the following obtain:

(v) edges \( xw_1 \) and \( yw_1 \) are both red;
(vi) edge \( xw_2 \) is blue and edge \( yw_2 \) is red; and
(vii) edge \( xw_3 \) is red and edge \( yw_3 \) is blue.

We construct such a representation in the following manner. Let \( [n] \) denote the integer set \( \{1, 2, ..., n\} \). Let \( \binom{[3k - 4]}{k} \) denote all \( k \)-subsets of \( [3k - 4] \). Let \( G \) be a complete graph with vertex set \( \binom{[3k - 4]}{k} \). Color the edges \( xy \) of \( G \) as follows:

- color \( xy \) **blue** if \( 0 \leq |x \cap y| \leq 1 \)
- color \( xy \) **red** if \( 2 \leq |x \cap y| \leq k - 1 \)

It is not hard to check that \( G \) satisfies the above conditions\(^2\). Then a map on the atoms of \( A \) can be defined as follows:

- \( 1' \mapsto \text{identity on } \binom{[3k - 4]}{k} \)

\(^2\)This is a essentially a Johnson scheme, but here the graph-theoretic language is more convenient.
\[ r \mapsto \{(x, y) : xy \text{ is colored red}\}\]
\[ b \mapsto \{(x, y) : xy \text{ is colored blue}\}\]

This map on the atoms extends to a representation of \( \mathfrak{A} \).

## 3 Recoloring \( G \) to get a new representation

We seek a finite representation of the algebra \( \mathfrak{B} \) with atoms \( r, b_1, \ldots, b_n \), with mandatory cycles \( rrr, rrb_i \), and \( rb_ib_j \) for all \( i, j \in [n] \). We can get a representation of \( \mathfrak{B} \) from a sufficiently large representation of \( \mathfrak{A} \) by reassigning edges colored blue to \emph{shades of blue} uniformly at random. The key property of our chosen representation of \( \mathfrak{A} \) that allows us to do this is that every edge has its “needs” met very many times—that is, the points \( z \) and \( w \) that “witness” the triangles in which a given edge must participate exist in large numbers. There are at least \( (k - 2)^2 \) witnesses to each need in (i)–(vii) above, and this allows us to reassign blue edges to shades of blue—one shade for each atom \( b_1, \ldots, b_n \)—and yet have conditions (i)–(vii) satisfied with increasing probability as \( k \) increases.

More precisely, given the coloring above on \( G \), recolor the blue edges of \( G \) with \( n \) shades of blue, uniformly at random. Now we have a candidate for a representation of \( \mathfrak{B} \). We need to check that conditions (i)–(vii) hold (noting, of course, that we now have \( n \) shades of blue instead of just one). The probability that an edge does not have one of its “needs” met is a function of \( n \) and \( k \); this probability goes to zero as \( k \to \infty \).

## References
