A commutative residuated lattice (briefly, CRL) is an algebra $\langle A; \cdot, \to, \wedge, \lor, e \rangle$ such that $\langle A; \cdot, e \rangle$ is a commutative monoid, $\langle A; \wedge, \lor \rangle$ is a lattice, $\to$ is a binary operation, and for all $a, b, c \in A$,

$$c \leq a \to b \iff a \cdot c \leq b$$

where $\leq$ is the lattice order. A unary operation $\neg$ on a CRL $A$ is called an involution provided that for all $a, b \in A$,

$$a \to (\neg b) = b \to (\neg a) \quad \text{and} \quad \neg\neg a = a.$$ 

In this case, the algebra $\langle A; \cdot, \to, \wedge, \lor, \neg, e \rangle$ is called an involutive CRL. We say a CRL (involutive or otherwise) is idempotent if it satisfies $x \cdot x \approx x$, linear if its lattice reduct is linearly ordered, and semilinear if it is a subdirect product of linear algebras. It is known that the class of all CRLs is an arithmetical variety, which has the congruence extension property (CEP). Every involutive CRL has the same congruences as its CRL-reduct, essentially because it satisfies $\neg x \approx x \to (\neg e)$.

By a Sugihara monoid we mean an involutive CRL which is idempotent and whose lattice reduct is distributive. The class $\text{SM}$ of Sugihara monoids is clearly a variety. As shown by Dunn in [1], every $n$–generated subdirectly irreducible member of $\text{SM}$ is a residuated chain with at most $2n + 2$ elements. Thus $\text{SM}$ is locally finite and semilinear. $\text{SM}$ is generated by a single algebra (which is based on the chain of non-zero integers). Because this algebra satisfies a quasi-equation not satisfied by $\text{SM}$, the quasivariety generated by this algebra is properly contained by $\text{SM}$.

We call a variety $V$ structurally complete if every proper subquasivariety of $V$ generates a proper subvariety of $V$. Thus $\text{SM}$ is not structurally complete. By a positive Sugihara monoid we mean any $\{\neg\}$–free subreduct of a Sugihara monoid. $\text{PSM}$ will denote the class of all positive Sugihara monoids, which happens to be a variety. Of course, $\text{PSM}$ is locally finite because $\text{SM}$ is. We are interested in whether $\text{PSM}$ is structurally complete. The question for some related varieties is answered in [5]. The larger class of idempotent, semilinear CRLs is not structurally complete, although the class of its $\{e\}$–free subreducts is. In fact, this class of subreducts is primitive in the sense that each of its subquasivarieties is already a variety. The methods used in [5] do not apply to $\text{PSM}$, so we need a new approach to obtain the main result, that $\text{PSM}$ is not only structurally complete, but primitive as well.

We mention here that the properties under discussion have a significant interpretation in logic. A logical consequence relation $\vdash$ is said to be structurally
complete if each of its proper extensions contains some new theorems (as opposed to having only new derivable rules). If a variety $V$ is the equivalent algebraic semantics for $\vdash$ (in the sense of [3]), then $V$ is structurally complete iff $\vdash$ is. Sugihara monoids constitute the equivalent algebraic semantics of the logical system $RM^t$. This system is derived from the relevance logic “$R$–mingle” (axiomatized in [1]) by adding the sentential constant $t$ to the language, as well as the axioms $\vdash t$ and $\vdash t \rightarrow (x \rightarrow x)$. Thus, the fact that $SM$ is not structurally complete reflects the same fact (known for some time) regarding $RM^t$. Because $PSM$ algebraizes the full negation-free fragment of $RM^t$, the primitivity of $PSM$ implies that this fragment is structurally complete, as are all of its extensions.

We investigate the structure of directly indecomposable members of $PSM$, first by showing:

**Theorem 1.** A finite, idempotent semilinear CRL is directly indecomposable if and only if its lattice reduct has a unique co-atom.

Given a finite CRL $A$, one may define a CRL $A^{\perp \top}$ by adding new greatest and least elements ($\top$ and $\bot$, respectively), and defining $\bot \cdot a = a$ for any $a \in A \cup \{\bot, \top\}$, and $\top \cdot a = \top$ whenever $a \neq \bot$. The resulting structure is residuated, and it is a directly indecomposable CRL. In the case of $PSM$, we can prove a converse:

**Theorem 2.** Every directly indecomposable member of $PSM$ of size greater than 2 is of the form $A^{\perp \top}$ for some $A \in PSM$.

This useful fact makes proof by induction on the size of algebras in $PSM$ possible. For example, we can show:

**Proposition 3.** The lattice reduct of every finite positive Sugihara monoid is self-dual.

This follows from the fact that (the lattice reduct of) $A^{\perp \top}$ is self-dual if $A$ is, and that a product of two self-dual algebras is also self-dual.

To show that $PSM$ is primitive, we will use the following characterization of primitivity, which is a variation of results in [4] and [2].

**Theorem 4.** A variety $V$ is primitive iff whenever $B$ is a subdirectly irreducible homomorphic image of a finitely generated algebra $A \in V$ then $B$ may be embedded into some ultrapower of $A$.

So as a locally finite variety, $PSM$ is primitive iff every subdirectly irreducible homomorphic image of a finite algebra $A$ may be embedded back into $A$. (This specialization to locally finite varieties appears in [4], and with a direct proof in [6].) We can show that $PSM$ actually has a stronger property. An algebra $A$ is projective over a class of algebras (in the algebraic sense) provided that it can be embedded into any algebra in the class which has $A$ as a homomorphic image, and in such a way that the composition of the two mappings is the identity on $A$. We can show
Theorem 5. Every finite subdirectly irreducible positive Sugihara monoid is projective in PSM (in the algebraic sense).

The following is an immediate consequence.

Theorem 6. PSM is primitive, that is, every subquasivariety of PSM is actually a variety.

And so we have:

Theorem 7. The full negation-free fragment of RM^t is structurally complete, as are all of its extensions.

References


