An abstract clone is a multi-based algebra

\[ C := \left( (C^{(n)})_{n \geq 1}, (S_m^n)_{m,n \geq 1}, (e_i^n)_{1 \leq i \leq n}, (m, n \text{ integers}) \right) \]

with \( n + 1 \)-ary operations \( S_m^n \) and nullary operations \( e_i^n \), which is an element of a variety of multi-based algebras defined by the following identities

\[ \text{(C1)} \quad S_m^n (z, S_m^n(x_1, \ldots , x_n), \ldots , S_m^n(y_p, x_1, \ldots , x_n)) \approx S_m^n(S_p^n(z, y_1, \ldots , y_p), x_1, \ldots , x_n), m, n, p \geq 1, \]

\[ \text{(C2)} \quad S_m^n(e_i^n, x_1, \ldots , x_n) \approx x_i, n \geq 1, i \leq n, \]

\[ \text{(C3)} \quad S_n^n(y, e_1^n, \ldots , e_n^n) \approx y, n \geq 1. \]

Here \( S_m^n, S_n^n, S_p^n, S_i^n \) and \( e_i^n, 1 \leq i \leq n \) are symbols for the corresponding operations.

Let \( a, b \in C^{(n)} \). Then \( a + b := S_m^n(a, b, \ldots , b) \) defines a binary associative operation + on \( C^{(n)} \) and one obtains a semigroup \( (C^{(n)}; +) \). We study two concrete versions of this semigroup, the semigroup of all \( n \)-ary operations and the semigroup of all \( n \)-ary cooperations defined on a finite non-empty set \( A \).
Let $O^n(A)$ be the set of all $n$-ary operations defined on the finite set $A$. If $|A| = k \geq 2$, the elements of $O^n(A)$ are called $n$-ary functions of $k$-valued logic. If $f \in O^n(A)$ and $g_1, \ldots, g_n \in O^m(A)$, we define the superposition operation $S^n_{m,A}$ by $S^n_{m,A}(f, g_1, \ldots, g_n)(a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$ for all $a_1, \ldots, a_m \in A$. If $n = m$ we will write for short $S^n_{n,A}$ instead of $S^n_{n,n,A}$. Together with the projections $e^n_{i,A}: A^n \to A, i \leq n$, defined by $e^n_{i,A}(a_1, \ldots, a_n) := a_i$ for all $a_1, \ldots, a_n \in A$ we get the multi-based algebra

$$(O^n(A))_{n \geq 1}, (S^n_{m,A})_{m,n \geq 1}, (e^n_{i,A})_{1 \leq i \leq n}$$

which is a model of the axioms (C1), (C2), (C3). We study the properties of the semigroup $(O^n(A); +)$ and its subsemigroups and are especially interested in the following problems:

1. Determine the order of all elements of $(O^n(A); +)$!

For $f \in O^n(A)$, $A = \{0, 1\}$, there are only three possibilities: $2f = f, 3f = f$, or $3f = 2f$. In the general case, i.e. for arbitrary finite sets $A$ with $|A| > 2$ we characterize all idempotent elements of $(O^n(A); +)$.

2. Determine all regular elements of $((O^n(A); +)$!)

For $A = \{0, 1\}$ an operation $f \in O^n(A)$ is regular iff $2f = f$ or $3f = f$. This is not true for $|A| > 2$. For $A = \{0, 1\}$ the set of all regular elements forms an orthodox semigroup.

3. Characterize all subsemigroups of $(O^n(A); +)$ satisfying certain identities!

We characterize all right-zero semigroups, left-zero semigroups, semilattices, rectangular bands, normal bands and regular bands contained in $(O^n(A); +)$.

4. Characterize Green’s relations!

We characterize Green’s relations $\mathcal{L}$ and $\mathcal{R}$ on $(O^n(A); +)$.

As a second example of a class of semigroups derivable from abstract clones we consider semigroups of cooperations.

Let $A$ be a non-empty set. For each $n \geq 1$ we denote the $n$-th copower of $A$, that is the union of $n$ disjoint copies of $A$ by $A^{\circ n}$. Specifically, $A^{\circ n} := \{1, \ldots, n\} \times A$, and an element $(i, a)$ corresponds to the element $a$ in the $i$-th copy of $A$. An
$n$-ary cooperation on $A$ is then a mapping $f : A \to A^\leq n$. Each $n$-ary cooperation is uniquely determined by a pair $(f_1, f_2)$ of mappings $f_1 : A \to \{1, \ldots, n\}$ and $f_2 : A \to A$.

Let $cO^n(A)$ be the set of all $n$-ary cooperations defined on $A$. If $f \in cO^n(A)$ and $g_1, \ldots, g_n \in cO^m_A$, then we define a $k$-ary cooperation $f[g_1, \ldots, g_n] : A \to A^\leq m$ by

$$a \mapsto ((g_{f_1(a)})_1(f_2(a)), (g_{f_1(a)})_2(f_2(a)))$$

for all $a \in A$. The cooperation $f[g_1, \ldots, g_n]$ is called the composition of $f$ and $g_1, \ldots, g_n$. Instead of $f[g_1, \ldots, g_n]$ we will also write $comp^n_m(f, g_1, \ldots, g_n)$. The injections $i_{i,A}^n$ are special cooperations which are defined by $i_{i,A}^n : A \to A^\leq n$ with $a \mapsto (i, a)$ for $1 \leq i \leq n$. Then we get a multi-based algebra $((cO^n(A))_{n \geq 1}; (comp^n_m)_{m,n \geq 1}, (i_{i,A}^n)_{1 \leq i \leq n})$. This multi-based algebra is an abstract clone, i.e. satisfies the clone axioms $(C1), (C2), (C3)$. Using the operation $comp^n_m$ we define a binary operation $+$ on $cO^n(A), n \geq 1$ and by $(C1)$ we obtain the semigroup $(cO^n(A); +)$ of all $n$-ary cooperations defined on $A$.

We study the properties of the semigroup $(cO^n(A); +)$ and its subsemigroups and similar to the case of operations we are especially interested in the following problems:

1. Determine the order of all elements of $(cO^n(A); +)$ !

As for operations also for cooperations on $A = \{0, 1\}$ there are only three possibilities: $2f = f, 3f = f$, or $3f = 2f$.

2. Determine all idempotent and all regular elements of $(cO^n(A); +)$ !

Regular and idempotent elements of $(cO^n(A); +)$ can be characterized as follows:

(i) $f \in cO^n(A)$ is idempotent iff $a \in f^{-1}((i,a))$ for all $(i,a) \in Imf$.

(ii) $f \in cO^n(A)$ is regular iff for any two elements $(i,a), (j,b)$ of $Imf$ there follows $a = b \Rightarrow i = j$. For $A = \{0, 1\}$ the set of all regular elements forms an orthodox subsemigroup of $(cO^n(A); +)$ and the set of all idempotent elements forms a regular band. For $|A| > 2$ this is not longer true.
3. Characterize all subsemigroups of $(cO^n(A); +)$ satisfying certain identities!

We characterize all right-zero semigroups, left-zero semigroups, semilattices, rectangular bands and normal bands contained in $(cO^n(A); +)$.

4. Characterize Green’s relations!

We characterize Green’s relations $\mathcal{L}$ and $\mathcal{R}$ on $(cO^n(A); +)$.

**Conclusion**: The results allow to compare the properties of the semigroups $(O^n(A); +)$ and $(cO^n(A); +)$ for $|A| = 2$ with the case that $|A| > 2$ and of operations with cooperations.

**References**

