A discriminator variety of linear Heyting algebras with operators arising in quantum computation

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1 Introduction

Unlike classical computation, quantum computation [7] allows one to encode two atomic information bits in parallel. Here, in fact, the appropriate counterpart of a classical bit is the qubit, defined as a unit vector in $\mathbb{C}^2$, i.e. $|\psi\rangle = a|0\rangle + b|1\rangle$, where $a, b$ are complex numbers s.t. $|a|^2 + |b|^2 = 1$, and $\{|0\rangle, |1\rangle\}$ is the canonical base. Supposing that, in analogy with the classical case, $|0\rangle$ and $|1\rangle$ represent maximal and precise pieces of information, the superposition state $|\psi\rangle$ corresponds to an uncertain information: as dictated by the Born rule, $|a|^2$ yields the probability of the information described by the pure state $|0\rangle$, while $|b|^2$ yields the probability of the information described by the pure state $|1\rangle$. A system of $n$ qubits, also called a $n$-quregister, is represented by a unit vector in the $n$-fold tensor product Hilbert space $\otimes^n \mathbb{C}^2$. Qubits and quregisters, therefore, encode possibly uncertain, yet maximal information. Non-maximal information pieces are matched, on a mathematical level, by qumixes, i.e. density operators in $\mathbb{C}^2$ or in appropriate tensor products $\otimes^n \mathbb{C}^2$ of $\mathbb{C}^2$.

Similarly to the classical case, we can introduce and study the behaviour of a number of quantum logical gates operating on such information units. These gates are mathematically represented by unitary operators on the appropriate Hilbert spaces.

Hereafter we mention a crucial example in the framework of quregisters: for any $n \geq 1$, the square root of the negation on $\otimes^n \mathbb{C}^2$ is the unitary operator $\sqrt{\text{Not}}^{(n)}$ such that, for every element $|a_1, ..., a_n\rangle$ of the computational basis $B^{(n)}, \sqrt{\text{Not}}^{(n)}(|a_1, ..., a_n\rangle) = |a_1, ..., a_{n-1}\rangle \otimes \frac{1}{2} ((1+i)|a_n\rangle + (1-i)|1-a_n\rangle)$, where $i$ is the imaginary unit. The counterpart of $\sqrt{\text{Not}}^{(n)}$ in the framework

\footnote{By $B^{(n)}$ we denote the set $\{|a_1, ..., a_n\rangle : a_i \in \{0, 1\}\}$, which is an orthonormal basis for the space $\otimes^n \mathbb{C}^2$.}
of qumixes is referred to as $\sqrt{\text{NOT}}^{(n)}$ and is defined, for any qumix $\rho$, by the following formula $\sqrt{\text{NOT}}^{(n)} \rho = \sqrt{\text{NOT}}^{(n)} \rho \sqrt{\text{NOT}}^{(n)*}$, where $\sigma^*$ denotes the adjoint of the operator $\sigma$. In this way, one ends up defining an array of quantum computational logics ([1], [3], [4]) which may differ from one another along two degrees of freedom: the language - i.e. the number and type of quantum logical gates they include; the consequence relation - which may be either weak or strong, as the following definition points out.

**Definition 1** Let $\rho, \sigma$ be density operators whose respective probabilities\(^2\) are denoted by $p(\rho), p(\sigma)$. $\sigma$ is a weak consequence of $\rho$ iff $p(\rho) \leq p(\sigma)$; $\sigma$ is a strong consequence of $\rho$ iff $p(\rho) \leq p(\sqrt{\text{NOT}} \rho) \leq p(\sqrt{\text{NOT}} \sigma)$.

It can be shown [2] that - from a logical viewpoint - it is unnecessary to consider information quantities in Hilbert spaces other than $\mathbb{C}^2$; in fact, the algebra whose universe is the set of all qumixes of $\mathbb{C}^2$ and whose operations correspond, in an appropriate sense, to the main quantum logical gates generates the same (weak or strong) consequence relation as the algebra over the set of all qumixes of arbitrary $n$-fold tensor products of $\mathbb{C}^2$. In order to investigate the algebraic counterpart of the strong consequence relation in a logical system associated to quantum logical gates we introduced quasi MV-algebras and $\sqrt{7}$ quasi MV-algebras [6], [5]. Successively we investigated an expansion of lattice ordered $\sqrt{7}$ QMV algebras whereby: i) lattice meet and join are promoted to the rank of fundamental operations; ii) lattice meet is residuated by a Gödel-like implication. We called these structures *linear Heyting quantum computational algebras* (shortly, LHQC-algebras). In this talk we outline a representation and a standard completeness theorem for this structure. Finally we show that LHQC-algebras form a discriminator variety.

**References**


\(^2\)This notion of probability for a density operator $\rho$ is defined as the trace of the compound operator $P_i^{(n)} \rho$, where $P_i^{(n)}$ is a projection operator which, in a suitable sense, represents the "truth-property" [4].
