Mereotopology and Boolean contact algebras

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The origins of mereotopology go back to the works of Leśniewski [4] on mereology and, on the other hand, the works of de Laguna [1], Tarski [7], and Whitehead [9] who used regions instead of points as the basic entity of geometry. In this “pointless geometry”, points are now second order definable as sets of regions, similar to the representation of Boolean algebras, where elements can be recovered as ultrafilters. Whitehead’s addition to the mereological structures of Leśniewski (which were, basically, complete Boolean algebras $B$ without a smallest element) was a “connection” (or “contact”) relation $C$ among nonempty regions, which, in its simplest form is a reflexive and symmetric relation satisfying an additional extensionality axiom. Historically, standard (models for) mereotopological structures were collections of regular open (or regular closed) sets of topological spaces $\langle X, \tau \rangle$ with the standard (Whiteheadian) contact among regions defined by

$uCv \iff \text{cl}(u) \cap \text{cl}(v) \neq \emptyset.$

(1)

The primary example is the collection of all nonempty regular open sets of the Euclidean plane.

A Boolean contact algebra (BCA) is a structure $\langle B, C \rangle$, where $B$ is a Boolean algebra, and $C$ a binary relation on $B$ that satisfies the following:

$C_0. \ (\forall x)0(-C)x$

$C_1. \ (\forall x)[x \neq 0 \RightarrowxCx]$

$C_2. \ (\forall x)(\forall y)[xCy \Rightarrow yCx]$

$C_3. \ (\forall x)(\forall y)(\forall z)[xCy \land y \leq z \Rightarrow xCz].$

$C_4. \ (\forall x)(\forall y)(\forall z)[xCy + z \Rightarrow (xCy \lor xCz)].$

Additionally, we will consider the following properties:

$C_5. \ (\forall x)(\forall y)[C(x) = C(y) \Rightarrow x = y].$ (The extensionality axiom).

$C_6. \ (\forall x)(\forall y)[(\forall z)[xCz \lor yCz^*] \Rightarrow xCy].$ (The interpolation axiom).

$C_7. \ (\forall x)[(x \neq 0 \land x \neq 1) \Rightarrow xCx^*]$ (The connection axiom).

$C$ is called an extensional contact relation (ECR) if it satisfies $C_0 - C_5$, and $C$ is called connected if it satisfies $C_7$. $C_6$ is one of the main pillars of proximity structures [5].

A (standard) representation of a BCA $\langle B, C \rangle$ is an isomorphism $h$ from $\langle B, C \rangle$ onto a subalgebra of the algebra $\text{RegOp}(X)$ of regular closed sets of some topological space $\langle X, \tau \rangle$, where the contact relation $C_7$ on $\text{RegOp}(X)$ is defined by (1). A

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topological space \( (X, \tau) \) is weakly regular, if it is semiregular (i.e. it has a basis of regular opens sets) and for each non-empty \( u \in \tau \) there is some non-empty \( v \in \tau \) such that \( \text{cl}(v) \subseteq u \). Weak regularity may be called a “pointless version” of regularity, and each regular space is weakly regular. \( (X, \tau) \) is called \( \kappa \)-normal, if any two disjoint regular closed sets can be separated by disjoint open sets [6].

In reverse order, the following representation results have been found:

**Theorem 1.** Suppose that \( \langle B, C \rangle \) is a BCA.

1. Each BCA has a standard representation in a compact semiregular \( T_0 \) space. [2]
2. If \( \langle B, C \rangle \) satisfies \( C_5 \), then \( \langle B, C \rangle \) has a standard representation in a weakly regular \( T_1 \) space \( X \). Furthermore, \( X \) is connected if and only if \( C \models C_7 \), and \( X \) is \( \kappa \)-normal if and only if \( C \models C_6 \). [3]
3. If \( \langle B, C \rangle \) satisfies \( C_5 \land C_7 \), then \( \langle B, C \rangle \) has a standard representation in a connected compact Hausdorff space. [8]

On every Boolean algebra with at least four elements there are are contact relations \( C, C', C'' \) such that \( C \models C_5 \), \( C' \models C_6 \), and \( C'' \models C_7 \). It is not clear, however, whether one can define on any atomless Boolean algebra a contact relation \( C \) that satisfies \( C_5 \) and one of \( C_6 \) or \( C_7 \). For a large class of BAs this is possible:

**Theorem 2.**
1. Suppose that \( A, B \) are atomless, \( B \leq_{\text{reg}} A \), and that \( C \) is a contact relation on \( B \) that satisfies \( C_5 \) and \( C_7 \). Then, there is such a contact relation on \( A \).
2. If \( A \) is an infinite free BA, then there is a contact relation on \( A \) that satisfies \( C_5 \) and \( C_7 \).

**Corollary 3.** If \( A \) is atomless and has an infinite free regular subalgebra, then there is a contact relation on \( A \) that satisfies \( C_5 \) and \( C_7 \).

In view of the representation results above, this implies a representation theorem for BAs:

**Corollary 4.** If \( A \) is atomless and has an infinite free regular subalgebra, then \( A \) is isomorphic to a dense subalgebra of the regular–closed algebra of a connected weakly regular \( T_1 \) space.

Turning to the collection \( \mathcal{C} \) of contact relations on a fixed BA \( A \), we can show the following:

**Theorem 5.**
1. Let \( \text{Rel}^{rs}(\text{Ult}(A))(\text{Ult}(A)) \) be the set of all reflexive and symmetric relations on \( \text{Ult}(A) \) that are close in the product topology of the Stone topology of \( \text{Ult}(A) \). The mapping \( r : \mathcal{C} \rightarrow \text{Rel}^{rs}(\text{Ult}(A))(\text{Ult}(A)) \) defined by \( r(C) = \{ (F, G) \in \text{Ult}(A)^2 : F \times G \subseteq C \} \) is bijective and order preserving.
2. \( \mathcal{C} \) is a complete atomistic co–Heyting algebra.

We have some results on the fine structure of \( \mathcal{C} \); for \( i \in \{ 5, 6, 7 \} \), let \( \mathcal{C}_i \) be the collection of all contact relations on \( A \) that satisfy \( C_i \).

**Theorem 6.**
1. \( \mathcal{C}_5 \) is an ideal of \( \mathcal{C} \).
2. \( C \in \mathcal{C}_6 \iff \text{R}_C \) is an equivalence relation.
3. For each \( C \in \mathcal{C}_7 \) there is a minimal \( C' \in \mathcal{C}_7 \) such that \( C' \subseteq C \). If \( B \) is finite, then \( C \in \mathcal{C}_7 \) if and only if \( \text{R}_C \) is a connected graph and \( \text{dom}(\text{R}_C \setminus 1^1) = \text{Ult}(B) \).
References


