Study of canonical extensions of BL-algebras

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Canonical extensions of lattice ordered algebras provide an algebraic
formulation of what is otherwise treated via topological duality or relational
methods. They were firstly introduced by Jónsson and Tarski for Boolean
algebras with operators (see [8] and [9]) and generalized for distributive
lattices with different operations in [5], [4] and [3].

If \( A = (A, \{ f_i, i \in I \}) \) is a distributive lattice with operations, the canon-
ical extension \( A^\sigma \) of the lattice \( (A, \wedge, \vee) \) is a doubly algebraic distributive
lattice that contains \( A \) as separating and compact sublattice. The main
problem is to extend the extra operations \( \{ f_i, i \in I \} \) to \( A^\sigma \) and check if this
new structure is an algebra in the same class of \( A \). There are two different
ways to extend an operation \( f \): one is the canonical extension \( f^\sigma \) and the
other is the dual canonical extension \( f^\pi \) (see [5]). This gives us two possible
candidates for the canonical extension of \( A \), namely the canonical exten-
sion \( A^\sigma \) and the dual canonical \( A^\pi \). A class of algebras is called \( \sigma \)-canonical
or \( \pi \)-canonical if it is closed under canonical or dual canonical extensions
respectively.

BL-algebras were introduced by Hájek (see [7]) as the algebraic counter-
part of basic logic. Since BL-algebras can be viewed as distributive lattices
with additional operations, one can analyze the canonicity of different sub-
varieties of these algebras.

BL-algebras form a variety \( BL \) that contains as important subvarieties
the variety of MV-algebras, the variety of Product algebras and the variety
of Gödel algebras (linear Heyting algebras). The study of canonicity in these

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proper subvarieties of BL-algebras has already been accomplished (see [5], [6] and [2]). Some of these results are summarized in the next lemma:

Lemma 1 (see [2]). The subvarieties $\mathcal{MV} \subseteq \mathcal{BL}$ of MV-algebras and $\mathcal{PL} \subseteq \mathcal{BL}$ of Product algebras are neither $\sigma$-canonical nor $\pi$-canonical. The subvariety $\mathcal{G} \subseteq \mathcal{BL}$ of Gödel algebras is $\pi$-canonical but not $\sigma$-canonical.

The non $\sigma$-canonicity and non $\pi$-canonicity of the whole variety of BL-algebras are also proved in [2]. In [5, Corollary 4.6] it is proved that if $\mathcal{V}$ is a variety of distributive lattices with operations that is finitely generated, then $\mathcal{V}$ is $\sigma$-canonical and $\pi$-canonical. In this presentation we will give some results about $\sigma$ and $\pi$-canonicity of subvarieties of $\mathcal{BL}$. Concerning $\sigma$-canonicity we obtain the next result:

Theorem 2 Let $\mathcal{V}$ be a variety of BL-algebras. Then $\mathcal{V}$ is $\sigma$-canonical iff $\mathcal{V}$ is finitely generated by finite algebras.

We next present some of the ideas we use to prove this fact.

Recall that every subvariety of BL-algebras is generated by its totally ordered members. Each totally ordered BL-algebra (BL-chain from now on) can be uniquely decomposed into the ordinal sum of a totally ordered family of non trivial totally ordered Wajsberg hoops (see [1]). The decomposition of BL-chains into totally ordered Wajsberg hoops together with the fact that chains generate subvarieties of $\mathcal{BL}$ give us the clue that in order to study canonicity of BL-algebras one has to study how do canonical and dual canonical extensions of totally ordered Wajsberg hoops look like and how do they behave with respect to the operation of ordinal sum. One of the important results we prove in this direction is the following:

Theorem 3 Let $W_1, W_2$ be totally ordered hoops and $\oplus$ denotes the operation of ordinal sum. Then

$$(W_1 \oplus W_2)^\sigma \cong W_1^\sigma \oplus W_2^\sigma \quad \text{and} \quad (W_1 \oplus W_2)^\pi \cong W_1^\pi \oplus W_2^\pi.$$ 

Although the canonical extensions are not preserved when the ordinal sum involves an arbitrary non finite set of summands the previous result is enough to our aim.

Next we investigate the canonical (and dual canonical) extensions of totally ordered Wajsberg hoops. Since each totally ordered Wajsberg hoop is either cancellative (in case it is not bounded) or it is the zero free reduct of a totally ordered MV-chain, the following two results together with a careful investigation about possible subvarieties of BL-algebras are enough to obtain the non $\sigma$-canonicity of every non finitely subvariety generated of BL-algebras:
Result 1) ([2]) If \( C \) denotes the Chang MV-algebra and \( C^\sigma \) and \( C^\pi \) denote the canonical and the dual canonical extensions of \( C \), then neither \( C^\sigma \) nor \( C^\pi \) are hoops.

Result 2) Let \( W \) be the unique totally ordered cancellative Wajsberg hoop defined in the real interval \([0,1]\). Let \( W^\sigma \) and \( W^\pi \) be its canonical and dual canonical extensions respectively. Then neither \( W^\sigma \) nor \( W^\pi \) are hoops.

Concerning dual canonicity we still have non \( \pi \)-canonicity for a large number of subvarieties of BL-algebras. But using some results of [1] and [5] we prove that there are (at least) countably many different subvarieties of BL-algebras that are \( \pi \)-canonical. The results are:

Theorem 4 Let \( V \) be a subvariety of BL-algebras and let \( B \in V \) be a chain. If in the decomposition of \( B \) as ordinal sum of non trivial Wajsberg hoops there is a non finite Wajsberg hoop, then \( V \) is non \( \pi \)-canonical.

Theorem 5 Let \( L_n \) be the \( n \)-elements MV-chain and let \( A \) be a non finite Gödel chain. The subvariety of BL-algebras generated by \( L_n \oplus A \) is \( \pi \)-canonical.

References


