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Abstract We introduce and present results about a class of quantales, the topological relational quantales, that can be associated with tuples \((X, R)\) such that \(X\) is a topological space and \(R\) is a lower-semicontinuous equivalence relation on \(X\) (see Definition 1 below). Our motivating case studies and examples of topological relational quantales include concrete quantales such as the relational quantales of [11] and the quantale \(Pen\) introduced in [12] (see Examples 1–5 below). In this setting, we focus in particular on the question of the representability of quantales into quantales of binary relations on a set, which has already been studied by some authors in the literature: for instance, in [1] it is shown that every quantale \(Q\) can be order-embedded into a quantale of relations in such a way that the noncommutative product is represented as relational composition, but this method does not extend to involutive quantales and the join of \(Q\) is not in general represented as the union. In [10], it is shown that any involutive quantale is embeddable into a quantale of join-semilattice endomorphisms, which give back the quantales of relations when the sup lattice is a powerset.

We present sufficient conditions for the representability of unital involutive quantales (see [13] for reference) into quantales of relations, so that joins are represented as unions, non-commutative products as compositions of relations, and involutions as taking inverses. One of the most significant conditions for our representation theorem is that the quantale \(Q\) is join-generated by its functional elements, i.e. those \(a \in Q\) s.t. \(a^* \cdot a \leq e\). This condition is analogous to that given by Jónsson and Tarski in [9] for representability of relation algebras, and indeed our methodology is similar in that is based on notions derived from the theory of canonical extensions. This connection with the theory of canonical extensions is an element of novelty, for, as far as we know, the theorems of representation for quantales that appear in the literature rely on techniques inspired only to functional analysis and the theory of \(C^*\)-algebras. On the other hand, it is worth remarking that the theory of canonical extensions such as it is presented in [3, 6, 7, 8] cannot be straightforwardly applied to the context of quantales, because the infinite joins would be destroyed when passing to \(Q^\sigma\). So we work in a setting of meet-dense extensions of quantales that preserve all joins, where we consider only the \(\pi\)-extended operations. Indeed, we assume that \(Q\) can be embedded into a \(\mathcal{P}(X)\) so that \(\bigvee_Q\) is the union in \(\mathcal{P}(X)\) and \(Q\) is meet-dense in \(\mathcal{P}(X)^1\). A special class of representable quantales is then formed by those topological relational quantales that are join-generated by \(\{\text{graph}(f) \subseteq R \mid f : A \to X\text{ is a continuous map and } A \subseteq X\text{ is open in } X\}\).

Let \(X\) be a topological space and \(R \subseteq X \times X\) an equivalence relation which is lower-

\(^1\text{If } Q\text{ is also a frame, namely if the finite meet induced by the complete join-semilattice order distributes over arbitrary joins, then this condition is equivalent to saying that } Q, \text{ seen as a frame, is spatial and corresponds to a } T1 \text{ topology.}
semi-continuous (l.s.c. for short) i.e. such that, for any open set \( A \subseteq X \), the set \( R^{-1}[A] = \{ x \in X \mid \langle x, y \rangle \in R \text{ for some } y \in A \} \) is open. Moreover, \( R \) the selection property holds for \( R \) (cf. [5]) if \( R \) is the union of graphs \( \Gamma_f = \{ \langle x, y \rangle \mid x \in X, \ y = f(x) \} \) of continuous maps \( f : A \to X \) such that \( A \) is open in \( X \). Notice that if the selection property holds for \( R \), then \( R \) is l.s.c., but the converse is not in general true.

**Definition 1.** A unital involutive quantale \( Q \) is a topological relational quantale if there exists a topological space \( X \) and an l.s.c. equivalence relation \( R \) on \( X \) s.t. \( Q \) is isomorphic to a subquantale of \( \mathcal{P}(R) \) that is join-generated by the finite compositions of graphs \( \Gamma_f \subseteq R \) and inverses of graphs \( \Gamma_f^{-1} \subseteq R \) of a collection of continuous maps \( f : A \to X \) with open domain. For every \( X \) and \( R \) as above, the full topological relational quantale \( \mathcal{F}(R) \) is the subquantale of \( \mathcal{P}(R) \) that is join-generated by the finite compositions of graphs \( \Gamma_f \subseteq R \) and inverses of graphs \( \Gamma_f^{-1} \subseteq R \) of all the continuous maps \( f : A \to X \) with open domain.

By definition, \( \mathcal{F}(R) \) is indeed a unital involutive sub-quantale of \( \mathcal{P}(R) \), moreover the top element of \( \mathcal{F}(R) \) is \( R \) if and only if the selection property holds for \( R \). Let us give some examples to illustrate the concepts just introduced.

**Example 1.** If \( X \) has the discrete topology, then the selection property holds for every equivalence relation \( R \subseteq X \times X \) (hence in particular it is l.s.c.), since the singletons \( \{ \langle x, y \rangle \} \) are special cases of graphs of continuous functions with open domain. So \( \mathcal{F}(R) = \mathcal{P}(R) \) and we recover the classical relational quantales of [11] as special cases of full topological relational quantales.

**Example 2.** If \( X \) is a Stone space and \( R \) a l.s.c. equivalence relation s.t. \( R[x] \) is closed for every \( x \in X \), then, as a consequence of the general theory of continuous selections [4], \( R \) is union of graphs \( \Gamma_f \), with \( f : A \to X \) continuous and \( A \) is open. Then the top element of \( \mathcal{F}(R) \) is \( R \).

**Example 3.** If \( X \) is a Stone space and \( R = \bigcup R_i \) is the union of equivalence relations \( R_i \subseteq X \times X \) each of which satisfies the conditions in Example 2, then again the top element of \( \mathcal{F}(R) \) is \( R \).

**Example 4.** Let \( X \subset 2^{\mathbb{N}_0} \) be the set of Penrose sequences, i.e. the sequences \( x = (x_k)_{k \in \mathbb{N}} \) s.t. \( x_k = 1 \) implies \( x_{k+1} = 0 \). \( X \) is a closed subset of \( 2^{\mathbb{N}_0} \) for the Tychonoff topology, hence it is a Stone space. One defines the equivalence relation \( R \) on \( X \) by \( xRy \) iff there exists \( n \geq 0 \) such that \( x_k = y_k \) for every \( k > n \). The equivalence classes of \( R \) classify the isomorphism classes of Penrose tilings of the plane. In [12], Mulvey and Resende defined a quantale \( \text{Pen} \), by generators and relations, with the property that its algebraically irreducible relational representations are in one-to-one correspondence with the equivalence classes of \( R \). This quantale admits a canonical relational representation \( \text{Pen}^{\equiv} \) through which any relational representation of \( \text{Pen} \) factors. The quantale \( \text{Pen}^{\equiv} \) is defined by Mulvey and Resende as the sub-quantale of \( \mathcal{P}(R) \) generated by the relations

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\begin{align*}
l_n &= \{ \langle x, y \rangle \in R \mid y_n = 0 \text{ and } x_k = y_k \text{ for any } k > n \} \\
s_n &= \{ \langle x, y \rangle \in R \mid y_n = 1 \text{ and } x_k = y_k \text{ for any } k > n \}
\end{align*}
\]

together with their inverses \( l_n^{-1} \) and \( s_n^{-1} \). We showed the following result, which does not follows straightforwardly from the definition of Mulvey and Resende:
Proposition 2. \( \overline{\text{Pen}} \) coincides with the full topological relational quantale \( \mathcal{F}(R) \), and it is join-generated by graphs of local homeomorphisms \( f : A \to B \), with \( A, B \) clopen.

An important property of \( R \), which we use in the proof of the above result, is that \( R = \bigcup R_n \) where \( R_n = \{ (x, y) \in R \mid x_k = y_k \text{ for every } k > n \} \). Each \( R_n \) is an equivalence relation and it is closed in \( X \times X \), hence \( R \) fits the requirements of Example 3. Moreover one can prove that \( \overline{\text{Pen}} \) is a frame, its finite meets coinciding with finite intersections. Hence \( \overline{\text{Pen}} \) is a topology on \( R \) that can be shown to be strictly finer than the inherited product topology. More precisely \( \overline{\text{Pen}} \) is the limit topology on \( R \) induced by the chain of open inclusions \( R_0 \subset R_1 \subset \cdots \subset R_n \subset \cdots \). This was indeed our guiding example in this work, and we hope that our interpretation of \( \overline{\text{Pen}} \) as a full topological relational quantale will help to elucidate its relation with the \( C^* \)-algebra that Connes associates with Penrose tilings [2].

Example 5. Let \( X \) be a locally connected Hausdorff space and \( G \) be a discrete group acting freely on \( X \) by homeomorphisms. Then one defines the equivalence relation \( xRy \) iff \( \exists g \in G \) such that \( y = g(x) \). Observe that a continuous function with open and connected domain \( f : A \to X \) has graph \( \Gamma_f \subset R \) if and only if there exists \( g \in G \) such that \( f(x) = g(x) \) for any \( x \in A \). In this case the intersection of two such graphs \( \Gamma_g \) and \( \Gamma_{g'} \) is either empty or is another graph of this type. Then it follows that \( \mathcal{F}(R) \) is a frame, with finite meet given by finite intersection. Also in this case, as in the previous example, \( \mathcal{F}(R) \) is join-generated by graphs of homeomorphisms \( f : A \to B \) with \( A, B \) open.

Problem 3. Find conditions on an equivalence relation \( R \subseteq X \times X \), at least for \( X \) a Stone space, such that \( R \) is the union of graphs of local homeomorphisms.

Problem 4. Is it possible to describe axiomatically the topological relational quantales?

Motivated by this last question, we give some sufficient conditions for the representability as relational quantales of unital involutive quantales generated by functional elements.

Definition 5. Let \( \mathcal{Q} = (Q, \lor, \cdot, *, e) \) be a unital involutive quantale [13]. \( \mathcal{Q} \) is SGF if the following conditions hold:

1. \( Q \) is strongly Gelfand (abbreviated as SG), i.e. \( c \leq c \cdot c^* \cdot c \) for every \( c \in Q \).

2. \( Q \) is join-generated by its functional elements, i.e. for every \( c \in Q \), \( c = \lor f_i \) with \( f_i^* \cdot f_i \leq e \) for each \( i \).

3. \( Q \) is embedded in \( \mathcal{P}(X) \) as a meet-dense join-semilattice, i.e. \( \lor l_i = \bigcup l_i \) for every family \( l_i \in Q \) and \( A = \bigcap \{ l \in Q \mid A \subseteq l \} \) for every \( A \in \mathcal{P}(X) \).

Notice that a join-semilattice \( Q \) can be always embedded in \( \mathcal{P}(X) \) with a join-preserving map (take for example \( X = Q \) and define the embedding as \( Q \ni c \mapsto Q \setminus \{ c \} = \{ q \in Q \mid q \not\geq c \} \)). The non-trivial condition is the meet-denseness of the embedding. Notice also that in any SG quantale one has \( f = f \cdot f^* \cdot f \) for any functional element \( f \).
For SGF quantales $Q$, one can extend the quantale operations to $\mathcal{P}(X)$ as follows: for every $A, B \in \mathcal{P}(X)$, $A \cdot B := \bigcap \{ c \cdot q \mid c, q \in Q, A \subset c, B \subset q \}$ and $A^* := \bigcap \{ c^* \mid A \subset c, c \in Q \}$ (which corresponds to taking the $\pi$-extensions).

**Lemma 6.** $\langle \mathcal{P}(X), \cup, \cdot, *, e \rangle$ is a unital involutive quantale, and it is generated by its functional elements.

By this lemma, we can prove the following representation theorem under the hypothesis that $Q$ is atomic, without loss of generality:

**Proposition 7.** Let $Q$ be an SGF quantale. Then the map $Q \rightarrow \mathcal{P}(Q \times Q)$ defined by $x \mapsto F(x) = \{(a, b) \mid a, b $ atoms, $ a \leq x \cdot b \}$ is a strictly unital embedding of unital involutive quantales.

The embedding defined in the proposition above does not map atoms of $Q$ into atoms of $\mathcal{P}(Q \times Q)$, but we can show an analogous result to theorem 4.30 in [9] under the analogous additional condition: let $Q$ be an atomic SG quantale and let us assume that the following property holds for $Q$:

4. for every atom $b \in Q$, $b \cdot 1 \cdot b^* \leq e$.

This condition implies that every atom of $Q$ is a functional element, so $Q$ is an SGF quantale. In this case we can construct an embedding of involutive unital quantales $Q \rightarrow \mathcal{P}(X \times X)$ by which any atom is mapped to some singleton $\{\langle x, y \rangle \} \in \mathcal{P}(X \times X)$. If $R$ is the image of the top element $1 \in Q$ in $\mathcal{P}(X \times X)$ then $R$ is an equivalence relation and $Q$ will be identified with $\mathcal{P}(R)$.

**Proposition 8.** Let $Q$ be an atomic SGF quantale for which property 4 holds. Let $X = \{ a \cdot a^* \mid a \in L $ atom $ \}$. Then the map $L \rightarrow \mathcal{P}(X \times X)$ defined by $x \mapsto r_x = \{(a \cdot a^*, a^* \cdot a) \mid a $ atom and $ a \leq x \}$ is an embedding of unital involutive quantales.

**References**


