1. Introduction

Just over sixty years ago, Alfred Tarski proved:

**Theorem 1.1.** *Every relation algebra can be embedded in a complete and atomic relation algebra.*

This theorem is an acorn from which a mighty oak has grown. I have used this metaphor once before, about a theorem by another mathematician. See [J95], p. 354. The two instances are remarkably similar. In each case the proof of a moderately interesting theorem, when viewed from the right perspective, was seen to exemplify a previously unformulated principle with broad applicability. Perhaps we should pay more attention to this phenomenon. It might teach us to look behind the details of a proof and identify fundamental ideas that make it work. This talk will be about the theory that evolved from Tarski’s result, but it will also illustrate the potential rewards for careful reading, and the price we may pay for not being thorough enough. When I first read Tarski’s proof, I realized that his method could be applied to other algebraic structures beside relation algebras. This led to the theory of canonical extensions of Boolean algebras with operators. Almost half a century later, Mai Gehrke and I extended this to bounded distributive lattices with operators, but we soon realized that a still more general notion of canonical extensions was needed. A patchwork of definitions was considered, but eventually Gehrke proposed one that includes all of them as special cases. This concept, which applies to
arbitrary bounded distributive lattice expansions, i.e. to arbitrary algebras having a bounded distributive lattice as a reduct, was introduced and investigated in [GJ04]. In spite of its extreme generality, it has turned out to be quite manageable and powerful. This is the concept that I should have come up with when I read Tarski’s proof over half a century earlier!

2. Tarski’s proof

A relation algebra $A = (A_0, \circ, \sim, 1')$ consists of a Boolean algebra (BA) $A_0$ and operations $\circ, \sim$ and $1'$ of arities 2, 1 and 0, respectively. The operations $\circ$ and $\sim$ are required to distribute over joins. There are some additional axioms, but these need not concern us here. Tarski began his proof by associating with any BA $B$ a complete and atomic extension. The extension that he used is now called the canonical extension of $B$, and is written $B^\sigma$. It is characterized up to equivalence by the condition that there exists a commutative diagram

$$
\begin{array}{ccc}
B & \cong & S \\
\downarrow & & \\
B^\sigma & \cong & \wp(X)
\end{array}
$$

where $X = (X, \tau)$ is the Stone space of $B$ and $S$ is the set field of all clopen subsets of $X$. Tarski also characterized the extension $B^\sigma$ of $B$ intrinsically by two simple and important properties. These properties are so basic to the theory of canonical extensions that it is useful to have names for them.

Definition 2.1. Suppose $B'$ is a complete and atomic extension of a BA $B$. We say that

(i) $B$ is compact in $B'$ if, for any sets $S, T \subseteq B$ with $\land S \leq \lor T$ (in $B'$), there exist finite sets $S' \subseteq S$ and $T' \subseteq T$ with $\land S' \leq \lor T'$.

(ii) $B$ is separating in $B'$ if, for any distinct atoms $p$ and $q$ of $B'$ there exists $a \in B$ with $p \leq a$ and $q \leq a^-$.

Theorem 2.2. For any BA $B$, the extension $B^\sigma$ of $B$ is characterized up to equivalence by the properties that $B^\sigma$ is complete and atomic, and that $B$ is compact and separating in $B^\sigma$. 
We shall also refer to Tarski’s extension of a relation algebra $A = (A_0, \circ, \sim, 1')$ as its canonical extension, for in the current terminology, that is what it is. Of course he took $A_\sigma$ to be the BA reduct of this algebra, and then he completed the construction by extending the auxiliary operations. Actually there is no choice how this should be done, for in a complete relation algebra, the relative multiplication preserves joins of arbitrary sets of elements and all meets of down-directed sets, while conversion preserves all joins and all meets. Using the fact that every atom of $A^\sigma$ is the meet of all the members of $A$ above it, and that every member of $A^\sigma$ is the join of all the atom below it, one can therefore give explicit formulas for the extensions of the auxiliary operations. This is what Tarski did, and he then completed the proof by showing that the algebra he had constructed was in fact a relation algebra and an extension $A$.

The proof of Tarski’s theorem is not particularly difficult, but this is partly due to his elegant characterization of the canonical extension of a BA. This characterization continues to play a major role in the theory of canonical extensions. Thanks to it, we rarely have to go back to the details of the construction of canonical extensions. We can, however, use topological concepts in disguise. The algebraic version of compactness is an important example of this. As another example, we refer to the elements of a BA $B$ as clopen elements of its canonical extension $B^\sigma$, and to the meets and joins of subsets of $B$ as closed elements and open elements of $B^\sigma$, because these are the elements that correspond to clopen subsets, closed subsets and open subsets of the Stone space of $B$. We denote by $K(A^\sigma)$ and $O(A^\sigma)$, or simply by $K$ and $O$, the sets consisting of all closed elements, and of all open elements of $A^\sigma$.

3. Generalizing the construction

To what classes of algebras should we try to apply Tarski’s ideas? The papers [JT51, 52], [GJ94] and [GJ04] offer three suggestions:

- Boolean algebras with operators [BAOs],
- Bounded distributive lattices with operators [DLOs],
Bounded distributive latticed expansions [DLEs].

A operator is an operation that preserves joins in each of its argument, an a DLE is an algebra having a DL as a reduct.

The canonical extension of a BA was already been defined. The corresponding concept for DLs is defined analogously, except that now the Stone representation is replaced by the Priestley representation. Thus the canonical extension $B^\sigma$ of a DL $B$ is characterized up to equivalence by the condition that there exists a commutative diagram

$$
\begin{array}{ccc}
B & \subseteq & (B\delta)^\delta \\
\downarrow & & \downarrow \\
B^\sigma & \subseteq & (B\delta)^\circ^*
\end{array}
$$

where the two pairs of maps

$$
B \longmapsto B\delta, \quad S \longmapsto S\delta
$$

and

$$
C \longmapsto C^*, \quad P \longmapsto P^*
$$

represent the duality between DLs and Priestley spaces, and the duality between doubly algebraic DLs and posets, while $\circ^*$ is the forgetful functor that sends each Priestley space $S = (S, \tau, \leq)$ into its poset reduct $S^\circ = (S, \leq)$. As in the Boolean case, there is an intrinsic characterization.

**Theorem 3.1.** The canonical extension $B^\sigma$ of a DL $B$ is characterized up to equivalence by the properties that $B^\sigma$ is a doubly algebraic, distributive lattice, and that $B$ is a compact and separating sublattice of $B^\sigma$.

In [JT 51, 52] and [GJ94], canonical extensions of BA and DL maps are defined essentially as in Tarski’s proof of his theorem. The only significant difference is that in the papers closed elements are used where Tarski used atoms. This modified version can reasonable be applied to isotone maps, but for more general maps it does not even yield an extension of the original map. A different definition was therefore needed. In retrospect, Gehrke’s idea seems embarrassingly simple: Use the fact that a DL is dense in its canonical extension. To make more precise,
we need a topology on $A^\sigma$. Actually there are six topologies on $A^\sigma$ that play a role in the study of canonical extensions, although only one of them is involved in the definition of the concept.

**Definition 3.2.** Consider a DL $A$. We denote by $\sigma(A^\sigma), \sigma^\uparrow(A^\sigma)$ and $\sigma^\downarrow(A^\sigma)$, or simply by $\sigma, \sigma^\uparrow$ and $\sigma^\downarrow$, the topologies on $A^\sigma$ having as bases, respectively, all sets of the form $\uparrow p \cap \downarrow u, \uparrow p$ and $\downarrow u$ with $p \in K$ and $u \in O$.

The other three topologies can be defined on an arbitrary doubly algebraic DL $C$. Denote by $J^\infty(C)$ and $M^\infty(C)$ the sets of all members of $C$ that are, respectively, strictly join irreducible and strictly meet irreducible, and by $J^\infty_\omega(C)$ and $M^\infty_\omega(C)$ the sets consisting, respectively, of all joins of finite subsets of $J^\infty(C)$, and the set of all meets of finite subset of $M^\infty(C)$.

**Definition 3.3.** Consider a DL $A$. We denote by $\iota(C), \iota^\uparrow(C)$ and $\iota^\downarrow(C)$, or simply by $\iota, \iota^\uparrow$ and $\iota^\downarrow$, the topologies on $A^\sigma$ having as bases, respectively, all sets of the form $\uparrow p \cap \downarrow u, \uparrow p$ and $\downarrow u$, with $p \in J^\infty_\omega(C)$ bad $u \in M^\infty_\omega(C)$.

The topology $\sigma$ has two important properties.

(i) $A$ is dense in $A^\sigma$,
(ii) The members of $A$ are precisely the isolated points of $A^\sigma$.

We are now ready to formulate the basic definition.

**Definition 3.4.** For any DLE $A = (A_0, \omega^A, \omega \in \Omega)$, we let

$$A^\sigma = (A^\sigma_0, (\omega^A)^\sigma, \omega \in \Omega),$$

$$A^\pi = (A^\pi_0, (\omega^A)^\pi, \omega \in \Omega).$$

The DLEs $A^\sigma$ and $A^\pi$ are referred to as the canonical extension of $A$ and the dual canonical extension of $A$, respectively.

An explanation of the last definition is in order. If $\omega^A$ is an operation with arity $n$, then $\omega^A : A^n \rightarrow A$, hence $(\omega^A)^\sigma : (A^n)^\sigma \rightarrow A^\sigma$. Of course we want $\omega^A^\sigma : (A^\sigma)^h \rightarrow A^\sigma$. The algebras $(A^n)^\sigma$ and $(A^\sigma)^n$ are isomorphic, but to justify the definition we must assume that they
are equal A more accurate definition of $\omega^A$ would be to say that it is equal to $(\omega^A)^h \sigma^{-1}$, where $h : (A^n)^\sigma \preceq (A^\sigma)^n$ takes each member of $A^n$ into itself.

4. About the topologies

Here are some facts about the six topologies.

**Theorem 4.1.** For any DL map $f : A \to B$, $f^\sigma$ is the largest $(\sigma, \iota^\dagger)$-continuous extension of $f$, and $f^\pi$ is the smallest $(\sigma, \iota^\dagger)$-continuous extension of $f$.

**Theorem 4.2.** For any DL map $f : A \to B$, the following implications hold:

(i) $f$ is isotone $\Rightarrow f^\sigma$ is $(\sigma^\dagger, \iota^\dagger)$-continuous;
(ii) $f$ is an operator $\Rightarrow f^\sigma$ is $(\iota^\dagger, \iota^\dagger)$-continuous;
(iii) $f$ is join preserving $\Rightarrow f^\sigma$ is $(\sigma^\dagger, \iota^\dagger)$-continuous;
(iv) $f$ is join and meet preserving $\Rightarrow f^\sigma$ is $(\sigma, \sigma)$-continuous.

**Definition 4.3.** A DL map $f$ is said to be smooth if $f^\sigma = f^\pi$.

**Theorem 4.4.** If the DL map $f$ is smooth, then the map $f^\sigma = f^\pi$ is $(\sigma, \iota)$-continuous. Conversely, if $f$ has a $(\sigma, \iota)$-continuous extension $g$, then $g$ is unique, and $g = f^\sigma = f^\pi$, so $f$ is smooth.

5. Preservation

The tools are now in place, but how well do they work? Only a few results will be given. Two basic questions will be considered, preservation of homomorphisms and canonicity.

**Definition 5.1.** (i) A class $\mathcal{K}$ of DLEs is said to have (PH) if, for every $\mathcal{K}$-homomorphism $h : A \to B$, the map $h^\alpha : A^\sigma \to B^\sigma$ is a homomorphism.

**Theorem 5.2.** Suppose $\mathcal{K}$ is a class of DLs that is closed under ultraproduct. If $\mathcal{K}$ has (PH), then so does the variety generated by $\mathcal{K}$.

As an immediate corollary, every finitely generated variety of DLEs has (PH), but we can do much better than that!
Definition 5.3. An operation $f : A^n \rightarrow A$ is said to be *monotone* if, for each $i < n$, the operations obtained by fixing all its arguments except the $i$–th one are either all isotone or all antitone, a DLE $A$ is said to be *monotone* if all the operations $\omega^A$ are monotone, and a class of DLEs is said to be monotone if all its members are monotone.

Theorem 5.4. *Every variety of DLEs that is generated by a class of monotone algebras has (PH).*

Definition 5.5. A class $\mathcal{K}$ of algebras is said to be *canonical* if $\mathcal{K}$ has (PH) and is closed under canonical extensions.

Theorem 5.6. *Suppose $\mathcal{V}$ is a variety of DLEs with (PH) and $\mathcal{K}$ is a class that generates $\mathcal{V}$. If $A^\sigma \in \mathcal{V}$ for all $A \in \text{Pu}(\mathcal{K})$, then $\mathcal{V}$ is canonical.*

Corollary 5.7. *Suppose $\mathcal{K}$ is an elementary class of DLEs with (PH). If $\mathcal{K}$ is canonical then so is the variety generated by $\mathcal{K}$."

References


