Positive scalar curvature and a new index theory for noncompact manifolds

Stanley Chang
Department of Mathematics
Wellesley College

NCGOA Spring Institute 2013
Vanderbilt University
May 6, 2013
Low-dimensional results

Theorem (CWY 2010)

*The only noncompact contractible 3-manifold with positive scalar curvature is $\mathbb{R}^3$.***
Some high-dimensional results

Joint work with Shmuel Weinberger and Guoliang Yu.

Theorem (CWY 2013)

There are (contractible) noncompact manifolds with uncountably many positive scalar curvature components.

Theorem (CWY 2013)

There are (contractible) manifolds $M$ with a positively curved exhaustion but which itself cannot carry a positive scalar curvature metric.
Scalar Curvature

Given a Riemannian metric $g$ on a manifold $M$ of dimension $n$, we can define a function $\kappa_g : M \to \mathbb{R}$ measuring the scalar curvature of the manifold $M$ at each point.

**Definition**

If $g$ is a Riemannian metric on $M$ of dimension $n$, the scalar curvature is a smooth function $\kappa_g : M \to \mathbb{R}$ obtained from the curvature tensor by contracting twice.

$$\frac{\text{vol}_g B_r(M, p)}{\text{vol}_g B_r(\mathbb{R}^n, 0)} = 1 - \frac{\kappa_g(p)}{6(n + 2)} r^2 + \cdots .$$
Let $M$ be an $n$-dimensional manifold endowed with a Riemannian metric $g$, and let $p \in M$. Suppose that $\kappa(p) \neq 0$; i.e. $M$ is not flat at $p$. Then there is an $\varepsilon > 0$ such that, for all $r \in (0, \varepsilon)$, one of the following is true:

1. $\operatorname{vol}_g B_r(M, p) < \operatorname{vol}_g B_r(\mathbb{R}^n, 0)$;
2. $\operatorname{vol}_g B_r(M, p) > \operatorname{vol}_g B_r(\mathbb{R}^n, 0)$.

In these cases, we say that $M$ is (1) positively curved, (2) negatively curved at $p$. 
**The curvature problem**

**Relative Baum-Connes**

**The Dirac operator**

**Manifolds with exhaustion**

**Space of metrics**

**Contractible case**

---

**Introduction**

The curvature problem

Relative Baum-Connes

The Dirac operator

Manifolds with exhaustion

Space of metrics

Contractible case

---

**Trichotomy Theorem**

**Theorem (Kazdan-Warner)**

Let $M^n$ be a closed differentiable manifold of dimension $n$. Then $M$ belongs to exactly one of the following three classes:

1. **those admitting some Riemannian metric** $g$ **for which** $\kappa_g > 0$ (**positive manifolds**);

2. **those admitting no Riemannian metric** $h$ **with** $\kappa_h > 0$, **but admitting a metric** $g$ **with** $\kappa_g \equiv 0$;

3. **those admitting no Riemannian metric** $h$ **with** $\kappa_h \geq 0$, **but admitting a metric** $g$ **with** $\kappa_g < 0$. 

---

**Introduction**

The curvature problem

Relative Baum-Connes

The Dirac operator

Manifolds with exhaustion

Space of metrics

Contractible case
Dirac operator methods

**Theorem (Lichnerowicz)**

*If M is a closed spin manifold of dimension 4k, endowed with the Atiyah-Singer Dirac bundle S. If D is the Dirac operator on this bundle and \( \nabla \) is the standard Levi-Civit\`a connection, then

\[
D^2 = \nabla^* \nabla + \frac{\kappa}{4}.
\]

**Corollary**

*If \( \kappa > 0 \), then the index of D, given by

\[
\text{ind}(D) \equiv \dim \ker(D) - \dim \text{coker}(D),
\]

must vanish. Note: The Atiyah-Singer index theorem says that \( \text{ind}(D) \) is equal to the topological invariant \( \hat{A}(M) \).*
Indicial Receptacles

For particular types of manifolds $M$, we can define an Dirac-like index that lies in the following groups.

<table>
<thead>
<tr>
<th>Author</th>
<th>Group</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atiyah</td>
<td>$\mathbb{Z}$</td>
<td>1963</td>
</tr>
<tr>
<td>Hitchin</td>
<td>$KO^{-\ast}(pt)$</td>
<td>1974</td>
</tr>
<tr>
<td>Gromov-Lawson</td>
<td>$KO_{\ast}(B\pi)$</td>
<td>1980</td>
</tr>
<tr>
<td>Gromov-Lawson-Rosenberg</td>
<td>$KO_{\ast}(C_{r}^{\ast}\pi)$</td>
<td>1986</td>
</tr>
<tr>
<td>Roe</td>
<td>$K_{\ast}(C^{\ast}(M))$</td>
<td>1995</td>
</tr>
</tbody>
</table>

**General idea:** If $M$ can be endowed with a positive scalar curvature metric, then the index vanishes.
The Gromov-Lawson-Rosenberg Conjecture

**Conjecture:** If $M = K(\pi, 1)$ is a closed aspherical manifold, then it is not positive.

**Conjecture:** (GLR) Suppose that $M$ is a connected closed spin manifold of dimension $n \geq 5$. Then $M$ is positive iff a particular Dirac index $\hat{\alpha}(M)$ vanishes in $KO_*(C_r^*\pi)$. 
### Summary of results

Necessary vanishing condition to positive scalar curvature in compact manifolds of dimension $\geq 5$.

<table>
<thead>
<tr>
<th></th>
<th>spin</th>
<th>nonspin</th>
</tr>
</thead>
<tbody>
<tr>
<td>simply connected</td>
<td>Hitchin invariant in $KO^{-\ast}(pt)$</td>
<td>none</td>
</tr>
<tr>
<td>not simply connected</td>
<td>Dirac index in $K_\ast(C_r^\ast \pi)$  (False: $\mathbb{Z}^4 \times \mathbb{Z}_3$)</td>
<td>Universal class in $H_n(B_\pi)$  (False: $\mathbb{Z}^4 \times \mathbb{Z}_3$)</td>
</tr>
</tbody>
</table>
Let $X$ be a locally compact metric space and let $H$ be an $X$-module. Denote by $C^*(X, H)$ the $C^*$-algebra on $H$ generated by the locally compact, finite propagation operators on $X$.

If $A$ is a $C^*$-algebra, denote by $\mathcal{I}A$ the $C^*$-algebra of bounded uniformly continuous functions $f: [1, \infty) \to A$.

If $\mathcal{A}$ is the collection of all $f \in \mathcal{IC}^*(X, H)$ such that $f(t)$ is a finite propagation operator and $\text{prop}(f(t)) \to 0$ as $t \to \infty$. The closure of $\mathcal{A}$ in $\mathcal{IC}^*(X, H)$ is called the localization algebra $C^*_L(X)$. 
Definition of relative $C^*$-algebra

Let $Y \subseteq X$ be compact metric spaces. Consider the inclusion $i: C^*_L(Y) \to C^*_L(X)$. Define

$$C_i = \{(a, f): f \in C_0([0, 1), C^*_L(X)), a \in C^*_L(Y), f(0) = i(a)\}.$$ 

Define the relative $KO$-homology group of $(X, Y)$ to be $KO_*(X, Y) \equiv KO_*(SC_i)$. 

Definition
Definition of reduced and maximal $C^*$-algebra

Let $\mathbb{C}[G]$ be the group ring of a discrete group $G$. It has two completions to a $C^*$-algebra:

The reduced group $C^*$-algebra $C^*_r(G)$ is obtained by completing $\mathbb{C}[G]$ in the operator norm for its regular representation on $\ell^2(G)$.

The maximal group $C^*$-algebra $C^*_\text{max}(G)$ or just $C^*(G)$, is defined by the following universal property: any $\ast$-homomorphism $f : \mathbb{C}[G] \to B(H)$ factors through the inclusion $\mathbb{C}[G] \hookrightarrow C^*_\text{max}(G)$.

If $G$ is amenable then $C^*_r(G) = C^*_\text{max}(G)$. 
Definition of reduced relative \( C^* \)-algebra

Let \( \phi_j : C^*_{\text{max}}(\pi_1(Y)) \to C^*_{\text{max}}(\pi_1(X)) \) be the map induced by the homomorphism \( j : \pi_1(Y) \to \pi_1(X) \).

Consider the mapping cone \( C^* \)-algebra of \( \phi_j \) given by

\[
C_{\phi_j,\text{max}} = \{(a, f) : f \in C_0([0, 1), C^*_{\text{max}}(\pi_1(X))), a \in C^*_{\text{max}}(\pi_1(Y)), f(0) = \phi_j(a)\}.
\]

Define \( C^*_{\text{max}}(\pi_1(X), \pi_1(Y)) \) to be the suspension \( SC_{\phi_j,\text{max}} \) of \( C_{\phi_j,\text{max}} \). We call it the \textit{maximal relative group} \( C^* \)-algebra of \( (\pi_1(X), \pi_1(Y)) \).

If in fact the homomorphism \( j \) is an injection, we can likewise define a \textit{reduced relative} \( C^* \)-algebra \( C^*_{\text{red}}(\pi_1(X), \pi_1(Y)) \).
The relative Baum-Connes maps

There is a map

$$\mu_{\text{max}} : KO_*(X, Y) \to KO_*(C^*_{\text{max}}(\pi_1(X), \pi_1(Y)))$$

is called the *maximal relative Baum-Connes map*. A reduced relative Baum-Connes map

$$\mu_{\text{red}} : KO_*(X, Y) \to KO_*(C^*_{\text{red}}(\pi_1(X), \pi_1(Y)))$$

can be similarly constructed if the homomorphism $j$ from $\pi_1(Y)$ to $\pi_1(X)$ is injective.
Conjecture: Let $Y \subseteq X$ and suppose that $X$ and $Y$ are both aspherical compact spaces.

1. (Relative Novikov conjecture) The maximal relative Baum-Connes map

$$\mu_{\text{max}} : KO_* (X, Y) \to KO_* (C_{\text{max}}^* (\pi_1 (X), \pi_1 (Y)))$$

is an injection.

2. (Relative Baum-Connes conjecture) If $j : \pi_1 (Y) \to \pi_1 (X)$ is an injection, then the reduced relative Baum-Connes map

$$\mu_{\text{red}} : KO_* (X, Y) \to KO_* (C_{\text{red}}^* (\pi_1 (X), \pi_1 (Y)))$$

is an isomorphism.
Proof of theorem

Theorem

Suppose that $Y \subseteq X$ are aspherical compact spaces such that $\pi_1(Y)$ and $\pi_1(X)$ are $K$-amenable and satisfy the Baum-Connes conjecture.

1. Then $\mu_{\max}$ is an isomorphism.
2. Assume also that $\pi_1(Y) \to \pi_1(X)$ is an injection. Then $\mu_{\text{red}}$ is an isomorphism.
Commutative diagram

\[
\begin{align*}
KO_{n+1}(Y) & \longrightarrow KO_{n+1}(C^*(\pi_1(Y))) \\
KO_{n+1}(X) & \longrightarrow KO_{n+1}(C^*(\pi_1(X))) \\
KO_{n+1}(X, Y) & \longrightarrow KO_{n+1}(C^*(\pi_1(X), \pi_1(Y))) \\
KO_{n}(Y) & \longrightarrow KO_{n}(C^*(\pi_1(Y))) \\
KO_{n}(X) & \longrightarrow KO_{n}(C^*(\pi_1(X)))
\end{align*}
\]
Relative $KO\text{-}homology$ class or $D$

$\partial M = N$

$W = N \times [0, \infty)$

$T_n = N \times [\frac{n}{2}, n]$

$Z_n = M \cup_N (N \times [0, n])$
Let $D$ be the Dirac operator on $Z = M \cup_N W$. Let $F_D = \frac{D}{\sqrt{1+D^2}}$ and let $[F]$ be its homology class in $KO_0(Z)$. We can define $\text{ind}_L([F])$ as a local index in $KO_0(C_L^*(Z))$ defined by Yu. Define $C_L^*(W, Z)$ to be the closed two-sided ideal of $C_L^*(Z)$ generated by $C_L^*(W)$. The inclusion map $C_L^*(W) \to C_L^*(W, Z)$ induces an isomorphism in $KO$-theory. Let $\pi : C_L^*(Z) \to C_L^*(Z)/C_L^*(W, Z)$ be the natural quotient map and let

$$q_D = \pi_* (\text{ind}_L([F_D])) \in KO_*(C_L^*(Z)/C_L^*(W, Z)) \cong KO_*(M, N).$$

We call this class the relative $KO$-homology class of $D$. 
Vanishing Theorem

**Theorem**

Suppose that \((M, \partial M)\) is a spin compact manifold with boundary endowed with a metric of positive scalar curvature that is collared at the boundary. Let

\[ \mu : KO_\ast(M, \partial M) \to KO_\ast(C^\ast(\pi_1(M), \pi_1(\partial M))) \]

be the relative Baum-Connes map previously defined. If \(q_D\) is the class constructed above, then \(\mu(q_D) = 0\).
The Gromov-Lawson-Rosenberg conjecture states that a closed spin manifold $M^n$ with $n \geq 5$ has a metric of positive scalar curvature iff its Dirac index vanishes in $KO_*(C^*_r(\pi))$, where $\pi = \pi_1(M)$. We formulate now a relative version of this conjecture.

Conjecture: (Relative Gromov-Lawson-Rosenberg) Let $(N, \partial N)$ be a compact spin manifold with boundary. Suppose that the map

$$\mu: KO_n(N, \partial N) \to KO_n(C^*(\pi_1(N), \pi_1(\partial N)))$$

satisfies $\mu(q_D) = 0$, where $q_D$ is the relative $KO$-homology class of $D$. Then there is a metric of positive scalar curvature on $N$ that is collared near $\partial N$. 
Admissible exhaustion

Definition

Let $Y$ be a noncompact space. Assume that $Y$ is a CW complex. Let $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ be a sequence of connected compact subsets of $Y$. We say that \{ $Y_i$ \} is an admissible exhaustion if

1. $Y = \bigcup_{i=1}^{\infty} Y_i$;
2. for each $j > i$, there is a compact subset $Y_{i,j} \subseteq Y$ such that $Y_j = Y_{i,j} \cup Y_i$ and $Y_{i,j} \cap Y_i = \partial Y_i$. 
Admissible exhaustion

Let $Y_i$ be an admissible exhaustion of $Y$. Define $D_i^*$ to be the $C^*$-algebraic inductive limit given by

$$\lim_{j \to \infty, j > i} \mathcal{C}_{max}^*(\pi_1(Y_j), \pi_1(Y_{i,j})) \otimes K.$$

There is a natural homomorphism $\rho_{i+1}: D_{i+1}^* \to D_i^*$ induced by group homomorphisms given by inclusions of the corresponding spaces. Let

$$A(Y) \equiv \left\{ (a_1, a_2, \ldots) \in \prod_{i=1}^{\infty} D_i^* : \rho_{i+1}(a_{i+1}) = a_i, \sup_{i} \|a_i\| < \infty \right\}$$

There is an index map $\sigma: KO_*(Y) \to KO_*(A(Y))$. 

Theorem

Let \( Y \) be a noncompact space with an admissible exhaustion \( \{Y_i\} \). Let \( M \) be a noncompact manifold. Assume that there is a proper map \( f: M \to Y \) with an admissible exhaustion \( \{M_i\} \) of \( M \) such that

1. each \( M_i \) is a compact manifold with boundary \( \partial M_i \),
2. \( f^{-1}(Y_i) = M_i \) and
3. \( f^{-1}(\partial Y_i) = \partial M_i \).

Suppose that \( M \) is spin and let \( D_M \) be the Dirac operator on \( M \). If \( M \) admits a metric of uniform positive scalar curvature, then the index \( \sigma(f_*[D_M]) \) of \( D_M \) is zero in \( KO_*(A(Y)) \), where \( f_*: KO_*(M) \to KO_*(Y) \) is the homomorphism induced by \( f \).
The notion of $\lim^{1}$

If $\{G_i\}$ is an inverse sequence of abelian groups indexed by the positive integers together with a coherent family of maps $f_{j,i} : G_j \to G_i$ for all $j \geq i$, then $\lim^{1} G_i$ is categorically defined to be the first derived functor of $\lim$.

Eilenberg-Moore also provides a description in the following. If $\Psi : \prod G_i \to \prod G_i$ is defined by $\Psi(g_i) = (g_i - f_{i+1,i}(g_i))$, then $\lim^{1} G_i$ is defined by $\lim^{1} G_i \equiv \text{coker}(\Psi)$.

Gray proves that, if each $G_i$ is countable, then $\lim^{1} G_i$ is either zero or uncountable.
The notion of \( \lim^1 \)

An example of an inverse system with a nontrivial \( \lim^1 \) term is

\[
S = \left\{ \mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{3} \ldots \right\}
\]

in which case we have the uncountable group \( \lim^1 S = \widehat{\mathbb{Z}_3}/\mathbb{Z} \).

Consider the composite mapping cylinder \( B_S \) of the infinite composite

\[
S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \ldots
\]

where each map takes \( z \in S^1 \) to \( z^3 \). This \( B_S \) is a fiber bundle over a circle whose fiber is the unique 4-regular tree with a single root.
Consider the space $B_S$ as a union of level sets and denote by $S^1_i$ the level set right below the level of the $i$-th circle. Let $A_n$ be the annular region between successive circles. Let $Y_j = \bigcup_{i=1}^{j} A_i$. Consider the sequence
\[
0 \rightarrow \lim^1 KO_{n+1}(Y_j, \partial Y_j) \rightarrow KO^lf(B_S) \rightarrow \lim KO_n(Y_j, \partial Y_j) \rightarrow 0.
\]

**Proposition**

The group $\lim^1 KO_{n+1}(Y_j, \partial Y_j)$ is nontrivial.
Theorem

Let \([c] \in KO_n^f(B_S)\), where \(B_S\) is endowed with an exhaustion by compact sets \(\{Y_i\}\) as above. There is \((M, f) \in \Omega^{\text{spin}}_n(B_S)\) such that

1. \(f_*[D_M] = [c];\)
2. the inverse images \((M_i, \partial M_i) = f^{-1}(Y_i, \partial Y_i)\) are compact manifolds with boundary such that the induced maps
   \(\pi_1(M_i) \rightarrow \pi_1(Y_i)\) and \(\pi_1(\partial M_i) \rightarrow \pi_1(\partial Y_i)\) are all isomorphisms.
Main noncontractible theorem

Theorem

Let $\xi$ be a nonzero class $\lim^{1}KO_{n+1}(Y_j, \partial Y_j)$ and consider $\xi$ also as an element of $KO^lf_n (BS)$. Let $M$ be as given in the above theorem with the exhaustion $(M_i, \partial M_i)$. Then each $M_i$ has a metric of positive scalar curvature which is collared at the boundary, but $M$ itself does not have a metric of uniformly positive scalar curvature.
Recall the notation $D_i^* = C_{\text{max}}^*(\pi_1(Y_i), \pi_1(\partial Y_i)) \otimes \mathcal{K}$. We have a commutative diagram

$$
egin{array}{cccccc}
0 & \rightarrow & \lim^1 KO_{n+1}(M_i, \partial M_i) & \rightarrow & KO_n^{lf}(M) & \rightarrow & \lim KO_n(M_i, \partial M_i) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \lim^1 KO_{n+1}(Y_i, \partial Y_i) & \rightarrow & KO_n^{lf}(BS) & \rightarrow & \lim KO_n(Y_i, \partial Y_i) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \lim^1 KO_{n+1}(D_i^*) & \rightarrow & KO_n(A(BS)) & \rightarrow & \lim KO_n(D_i^*) & \rightarrow & 0 
\end{array}
$$
Let $M$ be a spin manifold.

1. $\pi_0(\text{Riem}^+(M^n))$ is nontrivial if $n \equiv 0, 1 \mod 8$. (Hitchin)
2. $\pi_1(\text{Riem}^+(M^n))$ is nontrivial if $n \equiv -1, 0 \mod 8$.
3. The moduli space $\text{Riem}^+(M)/\text{Diff}_{x_0}(M)$ often has higher homotopy. (Botvinnik-Hanke-Schick-Walsh)

**Theorem**

There is a noncompact manifold $M$ for which $\pi_0(\text{Riem}^+(M))$ is uncountably large.
Proof

1. Find manifold $N$ that has metrics $\alpha$ and $\alpha'$ of positive scalar curvature which are in different components of $\text{Riem}^+(N)$.

2. Form the infinite connected sum $M = N \# N \# \cdots$ with the obvious exhaustion $M_i = \#_i N - D^n$.

3. There are uncountably many positive scalar curvature metrics, and the relative index notion of Xie-Yu show that the metrics are all in different connected components.
Proof

1. Find manifold $N$ that has metrics $\alpha$ and $\alpha'$ of positive scalar curvature which are in different components of $\text{Riem}^+(N)$.

2. Form the infinite connected sum $M = N \# N \# \cdots$ with the obvious exhaustion $M_i = \#_i N - D^n$.

3. There are uncountably many positive scalar curvature metrics, and the relative index notion of Xie-Yu show that the metrics are all in different connected components.

4. To construct a contractible case, we resort to homology sphere that can maintain a metric of positive scalar curvature, and reflect it across its highest dimensional simplices to produce a Davis-like manifold.
Introduction
The curvature problem
Relative Baum-Connes
The Dirac operator
Manifolds with exhaustion
Space of metrics
Contractible case

The main theorem

Theorem

There is a contractible manifold \( M \) with a nested exhaustion by compact manifolds \( M_i \) with boundary such that each \( M_i \) has a metric of positive scalar curvature which is collared at the boundary, but \( M \) itself does not have a metric of uniformly positive scalar curvature.
Outline of proof

1. Form the infinite mapping cylinder $\mathcal{E}$ of maps $f_i: S^n \to S^n$ of degree 3.
Outline of proof

1. Form the infinite mapping cylinder $\mathcal{E}$ of maps $f_i: S^n \to S^n$ of degree 3.

2. Apply Baumslag-Dyer-Heller to this complex to obtain spaces $X_\mathcal{E} \to Y_\mathcal{E} \to \mathcal{E}$ of the forms $\{BH_i; BG_i\}$ and $\{BH_i^+; BG_i^+\}$. 
Outline of proof

1. Form the infinite mapping cylinder $\mathcal{E}$ of maps $f_i: S^n \rightarrow S^n$ of degree 3.

2. Apply Baumslag-Dyer-Heller to this complex to obtain spaces $X_{\mathcal{E}} \rightarrow Y_{\mathcal{E}} \rightarrow \mathcal{E}$ of the forms $\{BH_i; BG_i\}$ and $\{BH_i^+; BG_i^+\}$.

3. Find an exhaustion $\{P_i\}$ such that $\lim^1 KO^\lf_{*} (P_i, BG_{i+1}^+) \text{ is nontrivial}$. Pick a nonzero class $\xi$. Consider it as an element of $KO^\lf_n (Y_{\mathcal{E}})$.
Outline of proof

1. Form the infinite mapping cylinder $\mathcal{E}$ of maps $f_i: S^n \to S^n$ of degree 3.

2. Apply Baumslag-Dyer-Heller to this complex to obtain spaces $X_\mathcal{E} \to Y_\mathcal{E} \to \mathcal{E}$ of the forms $\{BH_i; BG_i\}$ and $\{BH_i^+; BG_i^+\}$.

3. Find an exhaustion $\{P_i\}$ such that $\lim^1 KO_{*}^f (P_i, BG_{i+1}^+)$ is nontrivial. Pick a nonzero class $\xi$. Consider it as an element of $KO^f_n(Y_\mathcal{E})$.

4. Use surjectivity of the proper $KO$-theory Hurewicz map $h_{*}^{pr} : [\mathbb{R}^n, Y_\mathcal{E}]_{pr} \to KO^f_n(Y_\mathcal{E})$ to find a map $f_{\xi} : \mathbb{R}^n \to Y_\mathcal{E}$ such that $h_{*}^{pr}(f_{\xi}) = \xi$. 
Outline of proof

1. Use a proper version of the Hausmann-Vogel theorem to pull $f_\xi$ to an element $(M, j) \in \Omega^{sys}_n(\mathcal{X}_\mathcal{E})$. This manifold is acyclic, since it is constructed out of homology annuli. Also $(M_i, \partial M_i)$ is null-cobordant in the sense of pairs over its fundamental group.
Outline of proof

1. Use a proper version of the Hausmann-Vogel theorem to pull $f_\xi$ to an element $(M, j) \in \Omega^n_{\text{sys}}(X_\mathcal{E})$. This manifold is acyclic, since it is constructed out of homology annuli. Also $(M_i, \partial M_i)$ is null-cobordant in the sense of pairs over its fundamental group.

2. The image of the Dirac operator under $KO^f_n(M) \to KO_n(A(X_\mathcal{E}))$ is nonzero but is zero in $\varprojlim KO_n(M_i, \partial M_i)$.

\[
\begin{array}{cccccc}
0 & \xrightarrow{\varprojlim} & KO_{n+1}(M_i, \partial M_i) & \xrightarrow{f} & KO^f_n(M) & \xrightarrow{\varprojlim} & KO_n(M_i, \partial M_i) & \xrightarrow{\varprojlim} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{\varprojlim} & KO_{n+1}(Q_i, BG_{i+1}) & \xrightarrow{f} & KO^f_n(X_\mathcal{E}) & \xrightarrow{\varprojlim} & KO_n(Q_i, BG_{i+1}) & \xrightarrow{\varprojlim} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{\varprojlim} & KO_{n+1}(D_i^*) & \xrightarrow{f} & KO_n(A(X_\mathcal{E})) & \xrightarrow{\varprojlim} & KO_n(D_i^*) & \xrightarrow{\varprojlim} & 0
\end{array}
\]
The proof of the main theorem

Let $n \geq 5$ and let $X_i$ be a copy of the $n$-sphere for all $i \geq 1$. Let $f_i: X_{i+1} \to X_i$ be a map of degree 3. For example, the maps $f_i$ can be taken to be the $(n-1)$-fold suspension of the degree 3 circle map $S^1 \to S^1$ given by $z \mapsto z^3$. Let $I = [0, 1]$ denote the unit interval and $\mathcal{E}$ be the infinitely iterated mapping cylinder given by

$$\mathcal{E} = (X_1 \times I) \cup_{f_2} (X_2 \times I) \cup_{f_3} (X_3 \times I) \cup_{f_4} \cdots$$

where each $X_{i+1} \times I$ is attached to $X_i \times I$ by identifying points of $X_{i+1} \times \{0\}$ with points of $X_i \times \{1\}$ via the map $f_{i+1}$. Note that, by the nature of the $f_i$, if

$$\mathcal{E}_i = (X_1 \times I) \cup_{f_2} (X_2 \times I) \cup_{f_3} \cdots \cup_{f_i} (X_i \times [0, 1/2])$$

is the truncation of $\mathcal{E}$ midway through the $i$-th cylinder, then the derived limit $\varprojlim KO_n(\mathcal{E}_i, \partial \mathcal{E}_i)$ is nontrivial.
The proof of the main theorem

Without loss of generality, one may assume by the simplicial approximation theorem that the spheres $X_i$ are triangulated in such a way that the $f_i$ are all simplicial. Extend these triangulations to a triangulation to the entirety of $\mathcal{E}$. In each cylinder $X_i \times I$, let $Y_i = X_i \times \{1/2\}$. Let $Z_1 = X_1 \times [0, 1/2]$ and let $Z_i$ be the closed subset of $\mathcal{E}$ between $Y_{i-1}$ and $Y_i$ for all $i \geq 2$. 

\[
\begin{align*}
\mathcal{E} &= X_1 \times I \quad X_2 \times I \quad X_3 \times I \\
Y_1 &\quad Y_2 &\quad Y_3 \\
\{Z_1\} &\quad \{Z_2\} &\quad \{Z_3\} &\ldots
\end{align*}
\]
The proof of the main theorem

Notice that $Y_i$ is a subspace of $Z_i$ and $Z_{i+1}$ for all $i$. The space $\mathcal{E}$ can be visualized as a system of inclusions

\[ Z_1 \leftarrow Y_1 \rightarrow Z_2 \leftarrow Y_2 \rightarrow Z_3 \leftarrow Y_3 \]

with $\mathcal{E} = Z_1 \cup Y_1 Z_2 \cup Y_2 Z_3 \cup Y_3 \cdots$. 
The Quillen plus construction

The plus construction is a method for simplifying the fundamental group of a space without changing its homology and cohomology groups.

Given a perfect normal subgroup of the fundamental group of a connected CW complex $X$, we attach two-cells along loops in whose images in the fundamental group generate the subgroup. This operation generally changes the homology of the space, but these changes can be reversed by the addition of three-cells.
Baumslag-Dyer-Heller

Theorem (Baumslag-Dyer-Heller, 1980) Let $F$ be a finite simplicial complex. There is a finitely presented group $\pi$ such that $B\pi^+$ is homotopy equivalent to $F$. This construction is functorial with respect to simplicial inclusion. In other words, if $E \subseteq F$ is an inclusion of finite simplicial complexes, then there are groups $\rho \hookrightarrow \pi$ such that

1. $B\rho^+$ is homotopy equivalent to $E$ and
2. $B\pi^+$ is homotopy equivalent to $F$.

In addition the maps $B\rho \to E$ and $B\pi \to F$ are homology equivalences with arbitrary coefficients (since $B\rho \to B\rho^+$ and $B\pi \to B\pi^+$ are homology equivalences); it also induces an isomorphism at the level of KO-theory.
The proof of the main theorem

We apply Baumslag-Dyer-Heller to the entire complex $\mathcal{E}$ (one piece at a time) to achieve groups $G_i$ and $H_i$ such that

1. $G_i$ is a subgroup of $H_i$ and $H_{i+1}$ for all $i$;
2. there are homotopy equivalences $BG_i^+ \rightarrow Y_i$ and $BH_i^+ \rightarrow Z_i$ for all $i$;
3. $BG_i^+$ is a subset of both $BH_i^+$ and $BH_{i+1}^+$ for all $i$.

We can in addition assume that

4. the groups $G_i$ and $H_i$ are all superperfect;
5. the image of each map $G_i \rightarrow H_i$ generates $H_i$ normally.
The proof of the main theorem

The Baumslag-Dyer-Heller theorem then gives a complex

\[ Y_\mathcal{E} = BH_1^+ \cup_{BG_2} BH_2^+ \cup_{BG_3} BH_3^+ \cup_{BG_4} \cdots \]

with a map \( Y_\mathcal{E} \to \mathcal{E} \):

\[ BH_1^+ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
The proof of the main theorem

We now consider an appropriate exhaustion on the complex \( Y_\mathcal{E} \). For all \( i \geq 1 \), let \( P_i = BH_1^+ \cup BG_2^+ BH_2^+ \cup BG_3^+ \cdots \cup BG_i^+ BH_i^+ \). Then \( \{P_i\} \) is a nested compact exhaustion of \( Y_\mathcal{E} \). Let \( Q_i = BH_1 \cup BG_2 BH_2 \cup BG_3 \cdots \cup BG_i BH_i \) be the corresponding exhaustion of \( X_\mathcal{E} \). Clearly \( g^{-1}(P_i) = Q_i \) and \( g^{-1}(BG_i^+) = BG_{i+1} \). Since the plus construction is a homology isomorphism, even at the level of \( KO \)-theory, the vertical arrows of the diagram below are all isomorphisms:
The proof of the main theorem

Since $\lim_{i} KO_{\ast}^lf(P_{i}, BG_{i+1}^{+})$ is nontrivial by assumption, pick a nonzero element $\xi \in \lim_{i} KO_{\ast}^lf(P_{i}, BG_{i+1}^{+})$. Regard it also as a nonzero element in $KO_{\ast}^lf(X_{\mathcal{E}})$. 
The proof of the main theorem

Definition

Let $X$ be a space and $n$ a positive integer. Let $u_n$ be a chosen generator of $KO_n(S^n) \cong \mathbb{Z}$. Let $f : S^n \to X$ be an element of $\pi_n(X)$. Let $f_* : KO_n(S^n) \to KO_n(X)$ be the induced map. There is then a well-defined map $h_* : \pi_n(X) \to KO_n(X)$ given by $h_*[f] = f_*(u_n)$, called the $KO$-theory Hurewicz map.

Let $h_{pr}^* : [\mathbb{R}^n, Y_\mathcal{E}]_{pr} \to KO_n^I(Y_\mathcal{E})$ be the proper $KO$-theory Hurewicz map that generalizes the usual $KO$-theory Hurewicz map. This map is surjective so that there is $f_\xi : \mathbb{R}^n \to Y_\mathcal{E}$ such that $h_{pr}^*(f_\xi) = \xi$. 


Hausmann-Vogel

Definition

Let $X$ be a space. The $i$-th homology sphere bordism $\Omega^HV_i(X)$ of $X$ is the set of pairs $(M, f)$ where $M$ is an oriented homology $i$-sphere and $f: M \to X$ is a pointed continuous map.

The pairs $(M_1, f_1)$ and $(M_2, f_2)$ are equivalent in $\Omega^HV_i(X)$ if there is a cobordism $(W, M_1, M_2)$ with an extension $F: W \to X$ of both $f_1$ and $f_2$ with $H_*(W, M_1; \mathbb{Z})$ and $H_2(W, M_2; \mathbb{Z})$ trivial.
If $X$ is the inductive limit of spaces $X_i$ such that $\pi_1(X_i)$ is a finitely presented group acting trivially on $\pi_2(X_i)$ and $\Omega^HV_n(X)$ is the $n$-th homology sphere bordism of $X$, then there is an isomorphism $\Omega^HV_n(X) \to \pi_n(X^+) = [S^n, X^+]$ for $n \geq 5$.

This result of Hausmann-Vogel can be generalized to the proper setting with respect to the map $g : X_\mathcal{E} \to Y_\mathcal{E}$ of complexes.
The proof of the main theorem

Define $\Omega_{n}^{\text{sys}}(X_{E})$ to be the bordism group of proper maps $j: M \to X_{E}$ from a manifold $M$ to $X_{E}$ such that two maps $j_{1}: M_{1} \to X_{E}$ and $j_{2}: M_{2} \to X_{E}$ are identified if and only if there is a cobordism $W$ of $M_{1}$ and $M_{2}$ with a map $k: M \to X_{E}$ restricting to $j_{1}$ and $j_{2}$ on the boundary such that $k^{-1}(BG_{i})$ is a homology $h$-cobordism between $j_{1}^{-1}(BG_{i})$ and $j_{2}^{-1}(BG_{i})$ and $k^{-1}(BH_{i})$ is a homology $h$-cobordism between $j_{1}^{-1}(BH_{i})$ and $j_{2}^{-1}(BH_{i})$ for all $i$. The generalization of the usual Hausmann-Vogel theorem states that there is a bijective correspondence between $[\mathbb{R}^{n}, Y_{E}]_{pr}$ and $\Omega_{n}^{\text{sys}}(X_{E})$. 
The proof of the main theorem

The proper map $f_\xi$, considered as an element in $[\mathbb{R}^n, Y_\mathcal{E}]_{pr}$, is then associated to a map $j: M \to X_\mathcal{E}$ in $\Omega^\text{sys}_n(X_\mathcal{E})$ fitting into the following diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{j} & X_\mathcal{E} \\
\downarrow & & \downarrow g \\
\mathbb{R}^n & \xrightarrow{f_\xi} & Y_\mathcal{E}
\end{array}
$$

Here the left-hand vertical map $M \to \mathbb{R}^n$ is a degree one map obtained by applying plus construction on each $j^{-1}(BH_i)$ and $j^{-1}(BG_i)$. 
The proof of the main theorem

The element \((M, j)\) can be chosen in \(\Omega^\text{sys}_n(X_\mathcal{E})\) to enjoy the following conditions:

1. if \(D_M\) denotes the Dirac operator on \(M\), then \(j_*[D_M] = \xi\) in \(KO^\text{lf}_n(X_\mathcal{E})\);
2. the inverse images \(M_i = j^{-1}(Q_i)\) and \(j^{-1}(\partial M_i) = BG_{i+1}\) are compact manifolds such that the collection \(\{M_i\}\) forms an exhaustion of \(M\) and the maps \(\pi_1(M_i) \to \pi_1(Q_i)\) and \(\pi_1(\partial M_i) \to \pi_1(BG_{i+1})\) are all isomorphisms;
3. the image of \(\xi\) in \(\varprojlim KO_n(Q_i, BG_{i+1})\), which is zero by exactness, can be lifted compatibly to the zero element in the spin bordism groups given by \(\Omega^\text{spin}_n(B\pi_1(M_i), B\pi_1(\partial M_i))\); i.e. each \((M_i, \partial M_i)\) is nullbordant in the sense of pairs over their fundamental groups.
The proof of the main theorem

Let $D_i^*$ and $A(X_{\mathcal{E}})$ be the $C^*$-algebras as described before. We have the following commutative diagram, in which all vertical arrows from the second row to the third row are isomorphisms.

\[ \begin{array}{ccccccccc}
0 & \longrightarrow & \lim^1 KO_{n+1}(M_i, \partial M_i) & \longrightarrow & KO^l_n(M) & \longrightarrow & \lim KO_n(M_i, \partial M_i) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim^1 KO_{n+1}(Q_i, BG_{i+1}) & \longrightarrow & KO^l_n(X_{\mathcal{E}}) & \longrightarrow & \lim KO_n(Q_i, BG_{i+1}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim^1 KO_{n+1}(D_i^*) & \longrightarrow & KO_n(A(X_{\mathcal{E}})) & \longrightarrow & \lim KO_n(D_i^*) & \longrightarrow & 0 
\end{array} \]
The proof of the main theorem

There is a relative version of Gromov-Lawson surgery, whose proof is essentially identical to the non-relative case, that applies to manifolds with boundary and builds metrics that are products near the boundary.

In addition, every nullcobordant \((M_i, \partial M_i)\) can be obtained from the \(n\)-ball \(\mathbb{D}^n\) using surgeries of codimension at least 3 and that \(\mathbb{D}^n\) itself has a metric of positive scalar curvature that is collared at the boundary.
Then the relative Gromov-Lawson result can be applied to build a metric of positive scalar curvature on $M_i$ that is collared near $\partial M_i$. However $M$ itself does not have a complete metric of uniformly positive scalar curvature because the image of the Dirac operator under the map $KO_n^l(M) \to KO_n(A(X_\varepsilon))$ is nonzero.
Construction of relative Baum-Connes map

Let $\psi_{max}$ be the composition of the maps

$$C^*_\text{max}(\bar{Y})\pi_1(Y) \cong C^*_\text{max}(\pi_1(Y)) \otimes \mathcal{K} \to C^*_\text{max}(\pi_1(X)) \otimes \mathcal{K} \cong C^*_\text{max}(\bar{X})\pi_1(X).$$

Let

$$C_{\psi, max} = \{(a, f) : f \in C_0([0, 1), C^*_L(\bar{X})\pi_1(X), a \in C^*_L(\bar{Y})\pi_1(Y), f(0) = \psi_{max}(a)\}.$$

Let the map $\chi_{s, max} : C_i \to C_{\psi, max}$ be given by

$$(b, f) \mapsto (\phi_{s, max}(b), \phi_{s, max}(f)),$$

where

$$(\phi_{s, max}(f))(u) \equiv \phi_{s, max}(f(u)) \text{ for all } u \in [0, 1).$$

The induced map

$$\mu_{max} \equiv (\chi_{s, max})_* : KO_*(X, Y) \cong KO_*(SC_{\psi, max}) \xrightarrow{e_*} KO_*(C^*_\text{max}(\pi_1(X), \pi_1(Y)))$$

is called the maximal relative Baum-Connes map. Here $e$ is the evaluation homomorphism and $e_*$ is the induced map on $KO$-theory.
By the definition of $K$-amenability of Cuntz the natural homomorphisms

$$C^*_{\text{max}}(\pi_1(X)) \to C^*_r(\pi_1(X))$$

$$C^*_{\text{max}}(\pi_1(Y)) \to C^*_r(\pi_1(Y))$$

induce $KK$-equivalences. If $\pi_1(X)$ and $\pi_1(Y)$ are $K$-amenable and satisfy the Baum-Connes conjecture, and if $\pi_1(Y)$ injects into $\pi_1(X)$, then the $K$-theory of the reduced relative group $C^*$-algebra and the maximal relative group $C^*$-algebra coincide. For simplicity, in this case we use $C^*(\pi_1(X), \pi_1(Y))$ to denote both the reduced and maximal relative group $C^*$-algebra. The proof of the theorem follows from the following commutative diagram, the five-lemma.
The proof of the main theorem

This last property can easily be obtained by making use of a trick from Weinberger that replaces an inclusion of groups $B \subseteq A$ which is surjective on first homology by a square

\[
\begin{array}{ccc}
B & \longrightarrow & A \\
\downarrow & & \downarrow \\
B' & \longrightarrow & A'
\end{array}
\]

where all arrows are inclusions, and the vertical arrows are homology isomorphisms, and the bottom horizontal arrow is a homology isomorphism.
The proof of the main theorem

The manifold $M$ is acyclic, since it is constructed out of homology annuli via Hausmann’s results. To show that it is in fact contractible, we now only need to show that it is simply connected. The manifold $M$ has the fundamental group properties of $X$, aside from the innermost core, where we insert a contractible manifold bounded by homology sphere. This manifold is now contractible using the normal generation of each $G_i$ in $H_i$ by an inductive application of van Kampen’s theorem. The fundamental group of the boundary of this $r$-ball is trivialized in the ball itself, and it normally generates the annulus of which it is the inner boundary, and therefore kills the next ball.