Discrete integrable systems and cluster algebras

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Abstract

We discuss discrete time non-linear evolution equations arising from the study of quantum integrable spin chains in physics, and show how they fit in the cluster algebra structure of Fomin and Zelevinsky. The latter is a sort of dynamical system on formal coordinate patches that are transformed according to rational mutation rules, guaranteeing the Laurent property that each mutated coordinate is a Laurent polynomial of the variables in any other coordinate patch. We shall focus on the main open positivity conjecture of cluster algebra, stating that these Laurent polynomials have only non-negative integer coefficients.

The combinatorial content of our evolution equations is revealed by their exact solutions, which make use of their discrete integrable structure to express them as partition functions for Viennot's heaps of pieces, or alternatively for positively weighted paths on target graphs. Cluster algebra mutations are implemented by local transformations in various guises: local continued fraction rearrangements, matrix representations local moves a la Yang-Baxter, etc.

This generalizes nicely to a non-commutative setting, by use of the notion of quasi-determinants introduced by Gelfand and Retakh, and allows to prove some Kontsevich non-commutative cluster algebra positivity conjecture in rank 2. In higher rank, we obtain a non-commutative version of the discrete Hirota equation.

(Lectures based on work in collaboration with R. Kedem).