Hopf cyclic cohomology and transverse characteristic classes

Lecture 2: Hopf cyclic complexes

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Section 1: Hopf cyclic cohomology with coefficients in SAYD modules
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Hopf cyclic cohomology and SAYD modules

- MPI and SAYD and their examples
- Hopf cyclic cohomology
- Lie algebra cohomology
- Examples
Hopf Cyclic Cohomology

- Hopf cyclic cohomology invented by Connes and Moscovici as a computational tool for computing the index cocycle appears in the local index formula. Roughly speaking, they used the Hopf algebra of general transverse symmetry to compute their desire part of the cyclic cohomology, differential cyclic cohomology, of the crossed product algebra representing the transverse manifold of a foliation.

- The ingredient for Hopf cyclic cohomology is a Hopf algebra endowed with a modular pair in involution.

- Based on different needs we generalize Hopf cyclic cohomology in different directions:
  - Allowing noncommutative symmetries, i.e., Hopf (co)module (co)algebra
  - Allowing higher dimensional coefficients, i.e., SAYD modules
  - Allowing Hopf algebras and (co)algebras with several objects, i.e., $\times$-Hopf algebras
Let $\mathcal{H}$ be a Hopf algebra. By definition, a character $\delta : \mathcal{H} \to \mathbb{C}$ is an algebra map. A group-like $\sigma \in \mathcal{H}$ is the dual object of the character, i.e., $\Delta(\sigma) = \sigma \otimes \sigma$. The pair $(\delta, \sigma)$ are called modular pair in involution if

$$\delta(\sigma) = 1, \text{ and } S_{\delta}^{2} = Ad_{\sigma}.$$ 

Here $Ad_{\sigma}(h) = \sigma h \sigma^{-1}$ and $S_{\delta}$ is defined by

$$S_{\delta}(h) = \delta(h_{(1)}) S(h_{(2)}).$$

A right-left stable-anti-Yetter-Drinfeld module over a Hopf algebra $\mathcal{H}$ is a right $\mathcal{H}$-module $M$, which is also a left $\mathcal{H}$-comodule and satisfying

$$\nabla(m \cdot h) = S(h_{(3)}) m_{< -1 >} h_{(1)} \otimes m_{<0>} \cdot h_{(2)}, \quad m_{<0>} \cdot m_{<-1>} = m,$$

**Lemma** Any MPI defines a one dimensional SAYD module and all one dimensional SAYD modules come this way.
Let \((\mathfrak{g}, \mathcal{F})\) be a Lie Hopf algebra.

1. Define the derivation \(\delta_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{C}, \quad \delta_{\mathfrak{g}}(X) = \text{Tr}(Ad_X)\).
2. Extend \(\delta_{\mathfrak{g}}\) to an algebra map on \(\mathcal{F} \boxtimes \mathcal{U}\)
   \(\delta := \varepsilon \boxtimes \delta_{\mathfrak{g}} : \mathcal{F} \boxtimes \mathcal{U}(\mathfrak{g}) \to \mathbb{C} \quad \text{by} \quad \delta(f \boxtimes u) = \varepsilon(f)\delta_{\mathfrak{g}}(u)\)
3. The canonical rep. functions \(f_i^j \in \mathcal{F}\) defined for a fixed basis \(X_1, \ldots, X_m\) of \(\mathfrak{g}\),
   \[
   \nabla(X_j) = \sum_{i=1}^{m=\dim \mathfrak{g}} X_i \otimes f_i^j.
   \]
   It is clear that \(f_i^j\) are independent of \(\{X_1, \ldots, X_m\}\). Set
   \[
   \sigma_{\mathcal{F}} := \sum_{\pi \in S_m} (-1)^\pi f_1^{\pi(1)} \cdots f_m^{\pi(m)}.
   \]

**Theorem** The pair \((\delta, \sigma)\) is a modular pair involution for the Hopf algebra \(\mathcal{F} \boxtimes \mathcal{U}\).
Induced module

Let $M$ be a left $g$-module and a right $\mathcal{F}$-comodule via $\nabla_M : M \to M \otimes \mathcal{F}$. We say that $M$ is an induced module if

$$\nabla_M(X \cdot m) = \sum X_{<0>} \cdot m_{<0>} \otimes X_{<1>} m_{<1>} + m_{<0>} \otimes X \triangleright m_{<1>}$$

$\mathcal{H}$ act on $M$ from left via $(f \triangleleft u) \cdot m = \epsilon(f) u \cdot m$

$\mathcal{H}$ coact on $M$ via, $\nabla_M(m) = \sum m_{<0>} \otimes m_{<1>} \triangleright \triangleright 1$

**Lemma** Any induced module is a YD-module over $\mathcal{F} \triangleright \triangleleft \mathcal{U}$.

Let $\sigma M_\delta := M$ as a vector space.

Then the following make $\sigma M_\delta$ a SAYD module over $\mathcal{F} \triangleright \triangleleft \mathcal{U}$.

$$m \triangleleft (f \triangleleft u) := \sum \epsilon(f) \delta(u_{(2)}) S(u_{(2)}) m.$$  

$$\nabla(m) := \sum \sigma S(m_{<1>}) \triangleright \triangleright 1 \otimes m_{<0>}.$$
Let $M$ be a left $g := g_1 \bowtie g_2$-module such that the restriction of the action results a locally finite $g_2$-module. Then, via the $g_1$ action on $M$, by restriction, and the coaction defined by

$$\nabla(m) = \sum m_{<0>} \otimes m_{<1>}, \quad \text{iff} \quad v \cdot m = \sum m_{<1>}(v)m_{<0>}.$$

$M$ becomes an induced $(g_1, R(g_2))$-module. Conversely, every induced $(g_1, R(g_2))$-module comes this way.

So for any representation $M$ of $g = g_1 \bowtie g_2$ we have a $\sigma M_\delta$ as a SAYD module on $R(g_2) \bowtie U(g_1)$.

Similarly for a matched pair of Lie groups $G = G_1 \bowtie G_2$, and any representation $M$ so that the action of $G_2$ is locally finite we have a SAYD module $\sigma M_\delta$ on $R(G_2) \bowtie U(g_1)$.

For a complete characterization of SAYD modules on geometric bicrossed product Hopf algebras one needs more advanced technology which is not suitable to be discussed in this lecture series.
Cyclic module of a Hopf algebra with a SAYD module

\[ C^q(H, M) := M \otimes H^\otimes q, \quad q \geq 0. \]

We recall the following operators on \( C^*(H, M) \):

- **Face operators** \( \partial_i : C^q(H, M) \rightarrow C^{q+1}(H, M), \quad 0 \leq i \leq q + 1 \)
- **Degeneracy operators** \( \sigma_j : C^q(H, M) \rightarrow C^{q-1}(H, M), \quad 0 \leq j \leq q - 1 \)
- **Cyclic operators** \( \tau : C^q(H, M) \rightarrow C^{q}(H, M), \)

by

\[
\begin{align*}
\partial_0(m \otimes h^1 \otimes \ldots \otimes h^q) &= m \otimes 1 \otimes h^1 \otimes \ldots \otimes h^q, \\
\partial_i(m \otimes h^1 \otimes \ldots \otimes h^q) &= \sum m \otimes h^1 \otimes \ldots \otimes h^{i(1)} \otimes h^{i(2)} \otimes \ldots \otimes h^q, \\
\partial_{q+1}(m \otimes h^1 \otimes \ldots \otimes h^q) &= \sum m_{<0>} \otimes h^1 \otimes \ldots \otimes h^q \otimes m_{<-1>}, \\
\sigma_j(m \otimes h^1 \otimes \ldots \otimes h^q) &= m \otimes h^1 \otimes \ldots \otimes \varepsilon(h^{j+1}) \otimes \ldots \otimes h^q, \\
\tau(m \otimes h^1 \otimes \ldots \otimes h^q) &= \sum m_{<0>} h^{1(1)} \otimes S(h^{1(2)}) \cdot (h^2 \otimes \ldots \otimes h^q \otimes m_{<-1>}),
\end{align*}
\]

where \( H \) acts on \( H^\otimes q \) diagonally.
The graded module $C^*(Hc, M)$ endowed with the above operators is then a cocyclic module. This means that $\partial_i$, $\sigma_j$, and $\tau$ satisfy the following identities:

\[
\begin{align*}
\partial_j \partial_i &= \partial_i \partial_{j-1}, & \text{if } & i < j, \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1}, & \text{if } & i \leq j, \\
\sigma_j \partial_i &= \begin{cases} 
\partial_i \sigma_{j-1}, & \text{if } i < j \\
\text{Id} & \text{if } i = j \text{ or } i = j + 1 \\
\partial_{i-1} \sigma_j, & \text{if } i > j + 1,
\end{cases}
\tau \partial_i &= \partial_{i-1} \tau, & 1 \leq i \leq q \\
\tau \partial_0 &= \partial_{q+1}, \\
\tau \sigma_i &= \sigma_{i-1} \tau, & 1 \leq i \leq q \\
\tau \sigma_0 &= \sigma_n \tau^2, \\
\tau^{q+1} &= \text{Id}.
\end{align*}
\]
One uses the face operators to define the Hochschild coboundary $b : C^q(H, M) \to C^{q+1}(H, M)$, by

$$b := \sum_{i=0}^{q+1} (-1)^i \partial_i$$

One uses the rest of the operators to define the Connes boundary operator, $B : C^q(H, M) \to C^{q-1}(H, M)$, by

$$B := \left( \sum_{i=0}^{q} (-1)^{qi} \tau^i \right) \sigma_{q-1}(1 - \tau).$$

It is shown for any cocyclic module that $b^2 = B^2 = (b + B)^2 = 0$. 
As a result, one defines the cyclic cohomology of $H$ with coefficients in SAYD module $M$, which is denoted by $HC^*(H, M)$, as the total cohomology of the bicomplex

$$C^{p,q}(\mathcal{H}, M) = \begin{cases} M \otimes \mathcal{H}^q, & \text{if } 0 \leq p \leq q, \\ 0, & \text{otherwise}. \end{cases}$$

(1)

One also defines the periodic cyclic cohomology of $\mathcal{H}$ with coefficients in $M$, which is denoted by $HP^*(\mathcal{H}, M)$, as the total cohomology of direct sum total of the following bicomplex

$$C^{p,q}(\mathcal{H}, M) = \begin{cases} M \otimes \mathcal{H}^q, & \text{if } p \leq q, \\ 0, & \text{otherwise}. \end{cases}$$

(2)

It is seen that the periodic cyclic complex and hence cohomology is $\mathbb{Z}_2$ graded.
Let \( g \) be a finite dimensional Lie algebra and \( V \) is a right \( g \)-module. The Chevalley-Eilenberg complex of the \((g, V)\) is defined by \( C^n(g, V) = \text{Hom}(\wedge^q g^*, V) \) \( q \geq 0 \) with

\[
V \xrightarrow{\partial_0} C^1(g, V) \xrightarrow{\partial_1} C^2(g, V) \xrightarrow{\partial_2} \cdots ,
\]

\[
\partial(\omega)(Y_0, \ldots, Y_q) = \sum_{i<j} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_q) + \sum_i (-1)^i \omega(Y_0, \ldots, \hat{Y}_i, \ldots Y_q) Y_i.
\]

Alternatively, for \( \{\theta^i, X_i\} \) as a dual basis pair for \( g^* \) and \( g \)

\[
\partial_0(v) = vX_i \otimes \theta^i, \quad \partial_q(v \otimes \omega) = vX_i \otimes \theta^i \wedge \omega + v \otimes \partial_{dR}(\omega).
\]

Here \( \partial_{dR} : \wedge^q g^* \to \wedge^{q+1} g^* \) is the de Rham coboundary which is a derivation of degree 1 and recalled here by

\[
\partial_{dR}(\theta^k) = \frac{1}{2} C^{k}_{i,j} \theta^i \wedge \theta^j
\]
Let $V$ be a representation of $\mathfrak{g}$. We easily see that $V$ is a SAYD module on $U(\mathfrak{g})$ provided the coaction defined trivially.

The antisymmetrization map $\alpha : V \otimes \wedge^q \mathfrak{g} \to V \otimes U(\mathfrak{g}) \otimes^q$ defined by

$$\alpha(v \otimes Y_1 \wedge \cdots \wedge Y_q) = \frac{1}{q!} \sum_{\sigma \in S_q} v \otimes Y_{\sigma(1)} \otimes \cdots \otimes Y_{\sigma(q)}$$

We see that

$$b\alpha = 0, \quad B\alpha = \alpha \partial$$

Using some standard homotopy arguments we see

$$HP^*(U(\mathfrak{g}), V) \cong \bigoplus_{i = \ast \mod 2} H_i(\mathfrak{g}, V)$$
Let \( l \) be the solvable radical of \( g \), i.e., \( l \) is the unique maximal solvable ideal of \( g \). The Levi decomposition of Lie algebras implies that \( g = s \ltimes l \), where \( s \) is a semisimple subalgebra of \( g \) called a Levi subalgebra.

\[
\mathcal{D}_{Gr} : V \otimes F^\otimes q \to C^q(g, h, V),
\]
\[
\mathcal{D}_{Gr}(v \otimes f^1 \otimes \ldots \otimes f^q)(X_1, \ldots, X_q) = \sum_{\mu \in S_q} (-1)^\mu \left. \frac{d}{dt_1} \right|_{t_1=0} \ldots \left. \frac{d}{dt_q} \right|_{t_q=0} f^1(\exp(t_1 X_{\mu(1)})) \ldots f^q(\exp(t_q X_{\mu(q)}) v).
\]

**Theorem** Let \( G \) be a Lie group, \( V \) a representative \( G \)-module, and \( g = s \ltimes l \) be a Levi decomposition. Then \( \mathcal{D}_{Gr} \) induces the isomorphism

\[
HP^*(R(G), V) \cong \bigoplus_{* \equiv i \mod 2} H^i(g, s, V).
\]
Theorem  Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{l}$. Then for any finite dimensional $\mathfrak{g}$-module $V$, the map $D_{\text{Alg}}$ induces the isomorphism

$$HP^*(R(\mathfrak{g}), V) \cong \bigoplus_{i=0}^{\infty} H^i(\mathfrak{g}, \mathfrak{s}, V).$$

Theorem  Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{l}$. Then for any finite dimensional $\mathfrak{g}$-module $V$, the map $D_{\text{Alg}}$ induces the isomorphism

$$D_{\text{Alg}} : V \otimes R(\mathfrak{g})^{\otimes q} \to (V \otimes \wedge^q \mathfrak{l}^*)^\mathfrak{s},$$

$$D_{\text{Alg}}(v \otimes f^1 \otimes \ldots \otimes f^q)(X_1, \ldots, X_q) = \sum_{\sigma \in S_q} (-1)^\sigma f^1(X_{\sigma(1)}) \ldots f^q(X_{\sigma(q)})v.$$
Hopf cyclic cohomology of Lie-Hopf algebras

- Bicomplexes associated to Lie Hopf algebras
- van Est isomorphism
Let $\mathcal{F}$ a $\mathfrak{g}$-Hopf algebra. We denote the bicrossed product Hopf algebra $\mathcal{F} \rhd\lhd U(\mathfrak{g})$ by $H$. Let the character $\delta$ and the group-like $\sigma$ be the canonical modular pair in involution. In addition, let $M$ be an induced $(\mathfrak{g}, \mathcal{F})$-module and $\sigma M_\delta$ be the associated SAYD module over $Hc$.

The Hopf algebra $\mathcal{U} := U(\mathfrak{g})$ admits the following right action on $\sigma M_\delta \otimes \mathcal{F} \otimes q$, which plays a key role in the definition of the next bicocyclic module:

$$(m \otimes \tilde{f})u = \sum \delta_\mathfrak{g}(u^{(1)})S(u^{(2)}) \cdot m \otimes S(u^{(3)}) \bullet \tilde{f},$$

$$u \bullet (f_1 \otimes \ldots \otimes f^n)$$

$$:= \sum u^{(1)}_{<0>} \triangleright f_1 \otimes u^{(1)}_{<1>} u^{(2)}_{<0>} \triangleright f_2 \otimes \ldots$$

$$\ldots \otimes u^{(1)}_{<n-1>} \ldots u^{(n-1)}_{<1>} u^{(n)} \triangleright f^n.$$
One then defines a bicocyclic module $C^{\bullet, \bullet}(\mathcal{U}, \mathcal{F}, M)$, where

$$C^{p, q}(\mathcal{U}, \mathcal{F}, s\ M_\delta) := s\ M_\delta \otimes \mathcal{U}^\otimes p \otimes \mathcal{F}^\otimes q, \quad p, q \geq 0,$$

whose horizontal part is the cocyclic module associated to $\mathcal{U}$ with coefficients in

$$\sigma\ M_\delta \otimes \mathcal{F}^\otimes q$$

The vertical part is the cocyclic module associated to $\mathcal{F}$ with coefficients in

$$\sigma\ M_\delta \otimes \mathcal{U}^\otimes p$$

One notes that, by definition, a bicocyclic module is a bigraded module whose rows and columns form cocyclic modules and any horizontal arrow commute with any vertical one.
Bicomplex associated to Lie Hopf algebra

So each row and column have their own Hochschild coboundary and Connes boundary maps. These boundaries and coboundaries are denoted by $\rightarrow B$, $\uparrow B$, $\rightarrow b$, and $\uparrow b$, which are demonstrated in the following diagram.
In the next move, we identify the standard Hopf cocyclic module $C^*(H, \sigma M_\delta)$ with the diagonal subcomplex $D^*$ of $C^{\bullet\bullet}$. This is achieved by means of the map $\Psi : D^\bullet \to C^*(H, \sigma M_\delta)$ together with its inverse $\Psi^{-1} : C^*(H, \sigma M_\delta) \to D^*$. They are explicitly defined as follows:

$$\Psi(m \otimes u^1 \otimes \ldots \otimes u^n \otimes f^1 \otimes \ldots \otimes f^n) = \sum m \otimes f^1 \triangleright u^1_{<0>} \otimes f^2 u^1_{<1>} \triangleright u^2_{<0>} \otimes \ldots \otimes f^n u^1_{<n-1>} \ldots u^{n-1}_{<1>} \triangleright u^n,$$

respectively

$$\Psi^{-1}(m \otimes f^1 \triangleright u^1 \otimes \ldots \otimes f^n \triangleright u^n) = \sum m \otimes u^1_{<0>} \otimes \ldots \otimes u^{n-1}_{<0>} \otimes u^n \otimes f^1 \otimes \otimes f^2 S(u^1_{<n-1>}) \otimes f^3 S(u^1_{<n-2>} u^2_{<n-2>}) \otimes \ldots \otimes f^n S(u^1_{<1>} \ldots u^{n-1}_{<1>}).$$
Simplification of the bicomplex

- The bicocyclic module $C^{•,•}(\mathcal{U}, \mathcal{F}, \sigma M_\delta)$ can be further reduced to the bicomplex

$$C^{•,•}(g, \mathcal{F}, \sigma M_\delta) := \sigma M_\delta \otimes \wedge^\bullet g \otimes \mathcal{F}^{\otimes \bullet},$$

via the antisymmetrization map

$$\tilde{\alpha} : \sigma M_\delta \otimes \wedge^q g \otimes \mathcal{F}^{\otimes p} \otimes \rightarrow \sigma M_\delta \otimes \mathcal{U}^{\otimes q} \otimes \mathcal{F}^{\otimes p}, \quad \tilde{\alpha}_g = \alpha \otimes \text{Id},$$

the pullback of the vertical $b$-coboundary in vanishes, while the vertical $B$-coboundary becomes $\partial_g$

- On the other hand, since $\mathcal{F}$ is commutative, the coaction

$$\nabla : g \rightarrow g \otimes \mathcal{F},$$

extends from $g$ to a unique coaction

$$\nabla_g : \wedge^p g \rightarrow \wedge^p g \otimes \mathcal{F}.$$ After tensoring with the right coaction of $\sigma M_\delta$,

$$\nabla(m \otimes X^1 \wedge \cdots \wedge X^q)$$

$$= m_{<0>} \otimes X^1_{<0>} \wedge \cdots \wedge X^q_{<0>} \otimes \sigma^{-1} m_{<1>} X^1_{<1>} \cdots X^q_{<1>}.$$
We arrive at the bicomplex $C^{\bullet,\bullet}(g, \mathcal{F}, \sigma M_\delta)$, described by the diagram

\[ \cdots \xrightarrow{\partial_g} \sigma M_\delta \otimes \wedge^2 g \xrightarrow{b_\mathcal{F}} \sigma M_\delta \otimes \wedge^2 g \otimes \mathcal{F} \xrightarrow{b_\mathcal{F}} \sigma M_\delta \otimes \wedge^2 g \otimes \mathcal{F} \otimes 2 \xrightarrow{b_\mathcal{F}} \cdots \]

\[ \cdots \xrightarrow{\partial_g} \sigma M_\delta \otimes g \xrightarrow{b_\mathcal{F}} \sigma M_\delta \otimes g \otimes \mathcal{F} \xrightarrow{b_\mathcal{F}} \sigma M_\delta \otimes g \otimes \mathcal{F} \otimes 2 \xrightarrow{b_\mathcal{F}} \cdots \]

\[ \cdots \xrightarrow{\partial_g} \sigma M_\delta \xrightarrow{b_\mathcal{F}} \sigma M_\delta \otimes \mathcal{F} \xrightarrow{b_\mathcal{F}} \sigma M_\delta \otimes \mathcal{F} \otimes 2 \xrightarrow{b_\mathcal{F}} \cdots \]
Let $C^{p,q}(\mathfrak{g}^*, \mathcal{F}, M) = M \otimes \wedge^p \otimes \mathcal{F} \otimes q$

We apply the Poincaré isomorphism $D_P : C^{p,q}(\mathfrak{g}, \mathcal{F}, \sigma M_\delta) \rightarrow C^{m-p,q}(\mathfrak{g}^*, \mathcal{F}, M)$

The result is the following bicomplex whose columns are Lie algebra cohomology complex and the row are Hopf cyclic cohomology.
One uses the facts that $\alpha = g_1 \Join g_2$, and $g_2 = h \Join l$ is a Levi decomposition to see that the relative Lie algebra cohomology $H(\alpha, h, M)$ is computed by the total complex of the following bicomplex. Here the vertical and horizontal arrows are induced by the Lie algebra cohomology coboundaries of $g_1$ and $g_2$ with values in $M \otimes \wedge \bullet g_2$ and $M \otimes \wedge \bullet g_1$. 

\[ H(g_1 \Join g_2, h, M) \]
Let \((g, g_2)\) be a matched pair of Lie algebras and \(M\) be a finite dimensional representation of \(g_1 \bowtie g_2\).

Let \(F\) be \((g_1, g_2)\)-related Hopf algebra, that is \(F\) is a \(g_1\)-Hopf algebra with a \(g_1\) respected Hopf duality from \(U(g_2)\).

One then defines the following Van Est Isomorphism

\[
\mathcal{V} : M \otimes \wedge^p g_1^* \otimes F^\otimes q \rightarrow M \otimes \wedge^p g_1^* \otimes \wedge^q g_2^*
\]

\[
\mathcal{V}(m \otimes \omega \otimes f^1 \otimes \ldots \otimes f^q)(Y_1, \ldots, Y_p, \zeta_1, \ldots, \zeta_q) = \sum_{\sigma \in S_q} (-1)^\sigma \omega(Y_1, \ldots, Y_p) f^1(\zeta_{\sigma(1)}) \ldots f^q(\zeta_{\sigma(q)})
\]
Theorem Let \((g_1, g_2)\) be a matched pair of Lie algebras and \(\mathcal{F}\) a \((g_1, g_2)\)-related Hopf algebra such as \(R(g_2)\), \(R(G_2)\), or \(P(G_2)\). Assume that \(g_2 = h \ltimes \mathfrak{l}\) is a Levi decomposition such that \(h\) is \(g_1\)-invariant and the natural action of \(h\) on \(g_1\) is given by derivations. Then for any \(\mathcal{F}\)-comodule and \(g_1\)-module \(M\), the Van Est map \(\mathcal{V}\), is a map of bicomplexes and induces an isomorphism between Hopf cyclic cohomology of \(\mathcal{F} \acts U(g_1)\) with coefficients in \(\sigma M_\delta\) and the Lie algebra cohomology of \(\alpha := g_1 \ltimes g_2\) relative to \(h\) with coefficients in the \(\alpha\)-module induced by \(M\). In other words,

\[
HP^\bullet(\mathcal{F} \acts U(g_1), \sigma M_\delta) \cong \bigoplus_{i=\bullet \mod 2} H^i(g_1 \ltimes g_2, h, M).
\]
Hopf cyclic cohomology of $\mathcal{H}_n$

- Hopf-Koszul bicomplex
- Group cohomology bicomplex
- Equivariant bicomplex
- Wedge-Equivariant bicomplex
- Hopf cyclic cohomology of $\mathcal{H}_n$ relative to $gl_n$
- Hopf cyclic cohomology of $\mathcal{H}_n$
Passage to Lie algebra cohomology

We recall that the Koszul resolution associated to the Lie algebra $\mathfrak{g}$ is the complex

$$V(\mathfrak{g}^\ast) : \quad \mathcal{U} \xrightarrow{\partial_K} \mathfrak{g}^\ast \otimes \mathcal{U} \xrightarrow{\partial_K} \wedge^2 \mathfrak{g}^\ast \otimes \mathcal{U} \xrightarrow{\partial_K} \ldots,$$

$\mathfrak{g}$ acts on $\mathcal{U}$ from the right via $u \triangleleft X = -Xu$

$\nabla_K : \mathcal{V}^{\oplus} (\mathfrak{g}^\ast) \to \mathcal{F} \otimes \mathcal{V}^{\oplus}(\mathfrak{g}^\ast)$ defined as follows:

$$\nabla_K (\omega \otimes u) = S(u_{<0>(2)}) \triangleright S(u_{<1>\omega_{<1>}}) \otimes \omega_{<0>} \otimes u_{<0>(1)}.$$

$$V^\bullet(\mathfrak{g}^\ast) \otimes \mathcal{U} \overset{\mathcal{C}}{\leftarrow} \frac{b_{\mathcal{F},K}}{B_{\mathcal{F},K}} V^\bullet(\mathfrak{g}^\ast) \otimes \mathcal{U} \mathcal{F} \overset{b_{\mathcal{F},K}}{\leftarrow} \frac{b_{\mathcal{F},K}}{B_{\mathcal{F},K}} V^\bullet(\mathfrak{g}^\ast) \otimes \mathcal{U} \mathcal{F}^\otimes 2 \overset{b_{\mathcal{F},K}}{\leftarrow} \ldots$$

Theorem

$C^\bullet_{\mathcal{U}}(\mathcal{F}, V^\bullet(\mathfrak{g}^\ast))$ is a $\mathfrak{g}$-equivariant Hopf cyclic complex of $\mathcal{F}$ with coefficients in the differential graded SAYD $V^\bullet(\mathfrak{g}^\ast)$, and the map $\kappa : \ C^\bullet_{\mathcal{U}}(\mathcal{F}, V^\bullet(\mathfrak{g}^\ast)) \to C^\bullet(\mathcal{F}, \mathfrak{g}^\ast)$ induces an isomorphism of total complexes. $\kappa(\omega \otimes u \otimes \mathcal{U} f^1 \otimes \ldots \otimes f^q) = \omega \otimes u \bullet (f^1 \otimes \ldots \otimes f^q),$
We define $C_{\text{coinv}}^{p,q}(g^*, F) := (\wedge^p g^* \otimes F \otimes q+1)^F$ by

$$\alpha \otimes \tilde{f} \in (\wedge^p g^* \otimes F \otimes q+1)^F$$

if $\alpha_{<0>} \otimes \tilde{f} \otimes S(\alpha_{<1>}) = \alpha \otimes \tilde{f}_{<0>} \otimes \tilde{f}_{<1>}.$

We define $I : \wedge^p g^* \otimes F \otimes q \to (\wedge^p g^* \otimes F \otimes q+1)^F$ by

$$I(\alpha \otimes \tilde{f}) =
= a_{<0>} \otimes f^{1}_{(1)} \otimes S(f^{1}_{(2)}) f^{2}_{(1)} \otimes \ldots \otimes S(f^{q-1}_{(2)}) f^{q}_{(1)} \otimes S(\alpha_{<1>} f^{q}_{(2)})$$

induces an isomorphism

The isomorphism $I$ turns the $\bullet$ action into the diagonal action

$$X \rhd (f^0 \otimes \ldots \otimes f^q) = \sum_{i=0}^{q} f^0 \otimes \ldots \otimes X \rhd f^i \otimes \ldots \otimes f^q.$$
One obtains the bigraded module

\[
\begin{align*}
\left( \bigwedge^2 g^* \otimes \mathcal{F} \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \left( \bigwedge^2 g^* \otimes \mathcal{F} \otimes 2 \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \left( \bigwedge^2 g^* \otimes \mathcal{F} \otimes 3 \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \ldots \\
\left( g^* \otimes \mathcal{F} \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \left( g^* \otimes \mathcal{F} \otimes 2 \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \left( g^* \otimes \mathcal{F} \otimes 3 \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \ldots \\
\left( C \otimes \mathcal{F} \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \left( C \otimes \mathcal{F} \otimes 2 \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \left( C \otimes \mathcal{F} \otimes 3 \right)^{\mathcal{F}} & \xrightarrow{b_g^{\text{coinv}}} \ldots
\end{align*}
\]
We set \( \mathcal{N} := \lim_{k \to \infty} \mathcal{N}_k \), with \( \mathcal{N}_k := \text{Jet}_0^{(k)}(\mathcal{N}) \). Recall that the algebra \( \mathcal{F} \) is precisely the polynomial algebra generated by the components of such jets. By \( \psi \in \mathcal{N} \) we mean \( j_0^\infty(\psi) \in \mathcal{N} \) with \( \psi \in \mathcal{N} \).

Let \( C^q_{\text{pol}}(\mathcal{N}, \wedge^p g^*) \) be the set of polynomial functions

\[
c : \underbrace{\mathcal{N} \times \ldots \times \mathcal{N}}_{q+1 \text{ times}} \rightarrow \wedge^p g^*
\]

satisfying the covariance condition

\[
c(\psi_0 \psi, \ldots, \psi_q \psi) = \psi^{-1} \triangleright c(\psi_0, \ldots, \psi_q), \quad \forall \psi \in \mathcal{N},
\]

\[
b_{\text{pol}} c(\psi_0, \ldots, \psi_{q+1}) = \sum_{i=0}^{q+1} (-1)^i c(\psi_0, \ldots, \hat{\psi}_i, \ldots, \psi_{q+1}).
\]

\[
\tau_{\text{pol}}(c)(\psi_0, \ldots, \psi_q) = c(\psi_1, \ldots, \psi_q, \psi_0)
\]
We thus arrive at a bicomplex which mixes group cohomology of $H$ with coefficients in $\wedge^\bullet g^*$ and Lie algebra cohomology of $g$ with coefficients in $\mathcal{F}$, described by the diagram
In turn, this can be related to the bicomplex (4) via the obvious map $J : C_{\text{coinv}}^{\bullet,\bullet} (g^* , \mathcal{F}) \to C_{\text{pol}}^{\bullet} (\mathcal{N} , \wedge^\bullet g^*)$ defined, with the self-explanatory notation, by the formula

$$J \left( \sum \alpha \otimes \tilde{f} \right) (\psi_0 , \ldots , \psi_q) = \sum f^0 (\psi_0) \ldots f^q (\psi_q) \alpha .$$

(5)

Proposition

The map $J : C_{\text{coinv}}^{\bullet,\bullet} (g^* , \mathcal{F}) \to C_{\text{pol}}^{\bullet} (\mathcal{N} , \wedge^\bullet g^*)$ is an isomorphism of bicomplexes.
Wedge-Coinvariant Bicomplex

Since $\mathcal{F}$ is commutative we may consider the subcomplex $C_{c-w}^{p,q}(g^*, \mathcal{F}) \subset C_{\text{coinv}}^\bullet(g^*, \mathcal{F})$

\[
\begin{align*}
&\vdots \quad \vdots \quad \vdots \\
\partial_{c-w} &\downarrow & \partial_{c-w} &\downarrow & \partial_{c-w} \\
\wedge^2 g^* &\xrightarrow{b_{c-w}} (\wedge^2 g^* \otimes \wedge^2 \mathcal{F})^\mathcal{F} & b_{c-w} &\xrightarrow{b_{c-w}} (\wedge^2 g^* \otimes \wedge^3 \mathcal{F})^\mathcal{F} & b_{c-w} &\xrightarrow{b_{c-w}} \ldots \\
\partial_{c-w} &\downarrow & \partial_{c-w} &\downarrow & \partial_{c-w} &\downarrow & \partial_{c-w} \\
g^* &\xrightarrow{b_{c-w}} (g^* \otimes \wedge^2 \mathcal{F})^\mathcal{F} & b_{c-w} &\xrightarrow{b_{c-w}} (g^* \otimes \wedge^3 \mathcal{F})^\mathcal{F} & b_{c-w} &\xrightarrow{b_{c-w}} \ldots \\
\partial_{c-w} &\downarrow & \partial_{c-w} &\downarrow & \partial_{c-w} &\downarrow & \partial_{c-w} \\
\mathbb{C} &\xrightarrow{b_{c-w}} (\mathbb{C} \otimes \wedge^2 \mathcal{F})^\mathcal{F} & b_{c-w} &\xrightarrow{b_{c-w}} (\mathbb{C} \otimes \wedge^3 \mathcal{F})^\mathcal{F} & b_{c-w} &\xrightarrow{b_{c-w}} \ldots \\
\end{align*}
\]

Proposition

The antisym. map $\tilde{\alpha}_\mathcal{F} : C_{c-w}^{p,q}(g^*, \mathcal{F}) \to C_{\text{coinv}}^{p,q}(g^*, \mathcal{F})$ is a quasi-isomorphism.
To construct the desired homomorphism, we associate to any \( \omega \in C^r_{GF}(a) \) and any pair of integers \((p, q), p + m = q + r, 0 \leq q \leq m = \dim G\), a \(p\)-cochain \(C_{p,q}(\omega)(\psi_0, \ldots, \psi_p)\) on the group \(\mathcal{N}\) with values in \(q\)-currents on \(G\) as follows. For each \(q\)-form with compact support \(\zeta \in \Omega^q_c(G)\), one sets

\[
\langle C_{p,q}(\omega)(\psi_0, \ldots, \psi_p), \zeta \rangle = \int_{\Sigma(\psi_0, \ldots, \psi_p)} J_{\pi_1^{\infty}}^*(\zeta) \wedge \tilde{\omega};
\]

the integration is taken over the finite dimensional cycle in \(J_0^\infty\).

**Theorem**

The map \(C : C_{\text{top}}^\bullet(a) \to C_{\text{top}}^\bullet(\mathcal{N}, \Omega^\bullet(G))\) is a map of complexes. It lands in \(C_c^{\bullet, \bullet}(g^*, \mathcal{F})\) and induces an isomorphism in the level of cohomology.