In algebraic geometry, given an algebraic variety one forms the "canonical ring" $\oplus_{n=0}^{\infty} H^0(\otimes^n \Omega)$ where Ω is the canonical line bundle of the variety. For a von Neumann algebra N one can do the same thing with correspondences ${}_N\mathfrak{H}_N$. One forms

$$\oplus_{n=0}^{\infty} H^0(\otimes_M^n \mathfrak{H}).$$

The cohomology spaces are just invariant vectors and this vector space becomes an algebra just using the tensor product. This algebra could be called the canonical ring of the correspondence. In the trivial case $N = \mathbb{C}$, $\mathfrak{H} = \mathbb{C}^n$ one obtains the ring of noncommutative polynomials in a single variable.

Of considerable interest is the case where $\mathfrak{H} = L^2(M)$, and N is a subfactor of finite index of the II_1 factor M. Here one may introduce an involution so that the canonical ring

$$\mathfrak{R} = \oplus_{n=0}^{\infty} H^0(\otimes_M^n \mathfrak{H})$$

becomes a $*$-algebra with graded product given by the tensor product. The graded pieces of the canonical ring are,as vector spaces, the "tower of relative commutants" and the presence of the Temperley-Lieb algebra allows one to define a "Voiculescu trace" on the canonical ring which is positive definite. In the case of non-commutative polynomials this trace is the one defined by Voiculescu using the large N limit of a family of independent random $N \times N$ hermitian matrices. It is further possible to mix the graded product and the more usual algebra structure to obtain an infinite family $\mathfrak{R}_k$ of canonical rings with $\mathfrak{R}_0 = \mathfrak{R}$ and natural inclusions $\mathfrak{R}_k \subset \mathfrak{R}_{k+1}$. All the $\mathfrak{R}_k$ admit positive definite traces so they may be GNS completed and give C^*-algebras and II_1 factors. The projection structure in the C^*-algebras is interesting but not yet completely understood. The inclusion of II_1 factors coming from $\mathfrak{R}_0 \subset \mathfrak{R}_1$ gives a "canonical" subfactor whose standard invariant is the same as that of $N \subset M$. (Giving another proof of a result of Popa.) The completions of the $\mathfrak{R}_k$ form the tower of II_1 factors coming from the basic construction of subactor theory. For finite depth subfactors the isomorphism class of $\mathfrak{R}_0 \subset \mathfrak{R}_1$ is determined - they are interpolated free group factors (independently shown by Kodiyalam and Sunder). These algebra structures are natural in the planar algebra approach to the standard invariant and provide a link with random matrix theory via the Voiculescu trace.

The above work is joint with Guionnet and Shlyakhtenko and the lectures will be coordinated with those of Shlyakhtenko.

For many calculations it is much more convenient to change basis so that the graded pieces of the canonical rings are orthogonal. The cost is a more complicated multiplication that is filtered rather than graded. This was done independently by Kodiyalam-Sunder, and Jones-Shlyakhtenko-Walker. The new multiplication was actually indicated to the speaker by Roland Bacher in the early 1990's.