The Representation Ring

\[ K \text{ — compact group} \]

\[ \text{Rep}(K) = \left\{ \text{finite dimensional representations of } K \right\} \bigg/ \mathbb{R} \]

Rep\((K)\) is a semiring with unit:

\[ \oplus \text{ — direct sum} \]

\[ \otimes \text{ — tensor product} \]

\[ 1 \text{ — trivial representation on } \mathbb{C}. \]

Can extend it to a ring if we can institute a difference \(\ominus\).
K-Index of a Fredholm intertwiner

Consider data:

\[
\begin{array}{ccc}
\mathcal{H}_+ & F & \mathcal{H}_- \\
\pi_+(K) & & \pi_-(K)
\end{array}
\]

where

1. \(\pi_+, \pi_-\) are unitary representations of \(K\)

2. \(F : \mathcal{H}_+ \to \mathcal{H}_-\) is a Fredholm intertwiner
\hspace{1cm} (we will relax this condition soon)

\[\Rightarrow \ker F \text{ and } \coker F \text{ are fin. dim. rep'ns.}\]

Think of \(F\) as instituting their difference:

\[K\text{-Index}(F) = [\ker F] \oplus [\coker F]\]
A **Fredholm representation** of $G$ is

$$
\begin{array}{c}
\mathcal{H}_+ \\
\pi_+(G)
\end{array} \quad \begin{array}{c}
\mathcal{H}_- \\
\pi_-(G')
\end{array}
$$

- $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ — graded Hilbert space
- $\pi = \pi_+ \oplus \pi_-$ — rep’n of $G$ on $\mathcal{H}$ (by even operators)
- $F$ — odd, self-adjoint Fredholm op.

s.t.

(i) $g \mapsto \pi(g)F\pi(g^{-1})$ is norm-cts

(ii) $F^2 - 1 \in \mathcal{K}(\mathcal{H})$

(iii) $[F, \pi(g)] \in \mathcal{K}(\mathcal{H}) \quad (\forall g \in G)$
Kasparov representation ring $R(G)$

**Defn:** $R(G) = \{\text{Fredholm rep’ns}\}/\text{homotopy}.$

ie, 

$R(G) = KK^G(\mathbb{C}, \mathbb{C}).$

**Theorem** (Kasparov). $R(G)$ admits a product which makes it into a ring.

**Remark:** For K compact, $R(K)$ is isomorphic to the classical representation ring.
Two Theorems

Theorem 1. (Kasparov)

Let $\theta \in KK^G(C(\mathcal{X}), \mathbb{C})$. If

$$\theta \mapsto 1 \in KK^K(\mathbb{C}, \mathbb{C}),$$

then

$$\theta \mapsto \gamma \in KK^G(\mathbb{C}, \mathbb{C}).$$

Theorem 2. (Bernstein-Gelfand-Gelfand)

There is a differential complex comprised of

- (section spaces of) direct sums of $G$-homogeneous line bundles over $\mathcal{X}$,
- $G$-invariant differential operators between them,

which resolves the trivial representation.
Some geometry

$G$ — complex semisimple Lie group

**Defn.** A subgroup $P \leq G$ is called **parabolic** if $G/P$ is compact.

**Theorem.** There is a 1-1 correspondence between

1. subsets of the set $\Sigma$ of simple roots of $G$,
2. parabolic subgroups (up to conjugacy),
3. compact $G$-homogeneous spaces.
4. $G$-equivariant fibrations of $\mathcal{X} = G/B$. 
Harmonic Decompositions

Every \( G \)-homogeneous line bundle \( E \) over \( \mathcal{X} \) admits a decomposition:

\[
L^2(\mathcal{X}; E) \cong \bigoplus_{\pi \in \hat{K}} V^\pi \otimes (V^\pi)_\mu
\]

where \( (V^\pi)_\mu = \mu \)-weight space of \( \pi \), for some weight \( \mu \).

Notation.

\( S \subseteq \Sigma \) — subset of simple roots

\( P_S \) — associated parabolic subgroup

\( K_S \) := \( P_S \cap K \)

For each irrep \( \sigma \) of \( K_S \),

\( p_\sigma := \text{proj onto subspace of } L^2(\mathcal{X}; E) \)

of right \( K_S \)-type \( \sigma \).
Spectrally Proper Operators

$T : L^2(\mathcal{X}; E_1) \to L^2(\mathcal{X}; E_2)$ — bounded op.

Fix $S$ and write $T$ as a matrix w.r.t. harmonic decomposition $\{p_\sigma\}_{\sigma \in \hat{K}_S}$.

Definition.

- $T$ is $S$-spectrally finite if this matrix has finitely many nonzero entries.
- $T$ is $S$-spectrally proper if this matrix has finitely many nonzero entries in each row and column.

$\mathcal{A}_S := \{S\text{-spectrally proper op's }\}^{\|\cdot\|}$

$\mathcal{K}_S := \{S\text{-spectrally finite op's }\}^{\|\cdot\|}$
$C^*$-algebras associated to the fibrations

**Definition.** $\mathcal{A} := \bigcap_{S \subseteq \Sigma} \mathcal{A}_S$

**Proposition.** ($G = \text{SL}(n, \mathbb{C})$.) The following operators all belong to $\mathcal{A}$:

- $f \in C(\mathcal{X})$

- $\alpha(g)$ for $g \in G$

- $F_\alpha := \frac{X_\alpha}{|X_\alpha|}$, where $X_\alpha$ is a BGG operator along the fibration corresponding to $\alpha \in \Sigma$.

Moreover, $[F_\alpha, f]$ and $[F_\alpha, \alpha(g)]$ belong to $\mathcal{K}_{\{\alpha\}}$. 
Lattice of ideals

Definition.

\[ \mathcal{J}_S := \mathcal{K}_S \cap \mathcal{A} \]

Theorem. \( (G = \text{SL}(n, \mathbb{C})) \)

1. \( S \subset T \subset \Sigma \Rightarrow \mathcal{J}_T \triangleleft \mathcal{J}_S \).

2. For any \( S, T \subset \Sigma \), \( \mathcal{J}_S \cap \mathcal{J}_T = \mathcal{J}_{S \cup T} \).

3. \( \mathcal{J}_\Sigma = \mathcal{K} \) (compact operators).
'Normalized' BGG complex

Proposition. The phases $F_\alpha = \frac{X_\alpha}{|X_\alpha|}$ of the simple BGG operators can be completed to produce a complex modulo $\sum_{\alpha \in \Sigma} J_{\{\alpha\}}$. 