Kasparov’s operator K-theory and applications
4. Lafforgue’s approach

Georges Skandalis

Université Paris-Diderot Paris 7
Institut de Mathématiques de Jussieu

NCGOA Vanderbilt University
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Lafforgue’s approach

3 steps

1. Construction of a $KK$-theory for Banach algebras, and main properties.

2. Equality $\gamma = 1$ in $KK^\text{ban}_G(\mathbb{C}, \mathbb{C})$ in many cases. Almost all.

   The first two steps establish the Bost conjecture and some variants.

3. Construction of a suitable spectral dense subalgebra.
**B-pairs**

*B Bancah algebra.*

Right (resp. left) **Banach B-module**: Banach space $E$ endowed with a right (resp. left) action of $B$ such that, for all $x \in E$ and $a \in B$, we have $\|xa\| \leq \|x\|\|a\|$ (resp. $\|ax\| \leq \|a\|\|x\|$).

**B-pair:**

- left Banach $B$-module $E^<$,
- right Banach $B$-module $E^>$,
- bilinear map $\langle , \rangle : E^< \times E^> \to B$

satisfying: $\forall x \in E^>, \xi \in E^<$, the map $\eta \mapsto \langle \eta, x \rangle$ (resp. $y \mapsto \langle \xi, y \rangle$) left (resp. right) $B$-linear and $\|\langle \xi, x \rangle\| \leq \|\xi\|\|x\|$. Often $E^<$ not specified: take $E^<= (E^>)^* = \mathcal{L}(E,B)$. 
Morphisms of $B$-pairs $E$

- **Morphism** from $E = (E^<, E^>)$ to $F = (F^<, F^>)$ couple $f = (f^<, f^>)$ where $f^< : F^< \to E^<$ and $f^> : E^> \to F^> \mathbb{C}$-linear, left (resp. right) $B$-linear, continuous; $f^>$ and $f^<$ ajoint:
  $\langle \eta, f^>(x) \rangle = \langle f^<(\eta), x \rangle$.

- $\mathcal{L}(E, F)$ the Banach space of morphisms from $E$ to $F$ (norm $(f^<, f^>) \mapsto \sup(\|f^<\|, \|f^>\|))$).

- $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$ define morphism $gf = (f^< \circ g^<, g^> \circ f^>) \in \mathcal{L}(E, F)$.

- $\mathcal{L}(E) = \mathcal{L}(E, E)$ Banach algebra.

- Let $y \in F^>$ and $\xi \in E^<$. We note $\theta_{y, \xi} \in \mathcal{L}(E, F)$ (or $|y\rangle\langle\xi|$) the morphism given by
  - $F^< \ni \eta \mapsto \langle \eta, y \rangle \xi \in E^<$
  - $E^> \ni x \mapsto y\langle \xi, x \rangle \in F^>$.

- $\mathcal{K}(E, F)$ closed vector span in $\mathcal{L}(E, F)$ of morphisms $\theta_{y, \xi}$. 
Definition of the Banach $KK$-theory

$A, B$ be Banach algebras. Cycle for $KK^{\text{ban}}(A, B)$: triple $(E, \pi, f)$ where

- $E$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $B$-pair,
- $\pi : A \to \mathcal{L}(E)^{(0)}$ is a homomorphism
- $f \in \mathcal{L}(E)^{(1)}$

such that $f\pi(a) - \pi(a)f \in \mathcal{K}(E), \ (f^2 - 1)\pi(a) \in \mathcal{K}(E) \ (a \in A)$.

Sum of cycles and homotopy defined exactly as for Hilbert modules.

$KK^{\text{ban}}(A, B)$ set of homotopy classes of cycles. Abelian group.

Bifunctor: contravariant in $A$, covariant in $B$.

Action on $K$-theory

- $KK^{\text{ban}}(M_n(A), B)$ isomorphic $KK^{\text{ban}}(A, B)$.
- Idempotent $p \in A$, homomorphism $i_p : \mathbb{C} \to A$ setting $i_p(\lambda) = \lambda p$.
- Bilinear map $\varphi : K_0(A) \times KK^{\text{ban}}(A, B) \to KK^{\text{ban}}(\mathbb{C}, B)$
  
  $(p, x) \mapsto (i_p)^*(x)$. Can be constructed when $A$ not unital.
- $K_0(A) \simeq KK^{\text{ban}}(\mathbb{C}, A) \ (x \mapsto \varphi(x, 1_A))$.
- Banach-KK-theory acts on the $K$-theory.
The equivariant case

$G$ locally compact group acting on $A$ and $B$. Define $KK_G^{ban}(A, B)$ same way as the corresponding Kasparov group.

$A, B$ C*-algebras, natural homomorphism $KK_G(A, B) \to KK_G^{ban}(A, B)$.

**Morphism $j_G$**

Morphism $j_G^r : KK_G^{ban}(A, B) \to KK^{ban}(A \ltimes_r G, B \ltimes_r G)$ impossible. Need Banach algebras crossed product, extending reduced crossed product.

Natural crossed product for Banach algebras: $L^1(G; A)$. Morphism $j_G^{L^1} : KK_G^{ban}(A, B) \to KK^{ban}(L^1(G; A), L^1(G; B))$.

Using $j_G^{L^1} +$ equality $\gamma = 1$ in $KK_G^{ban}(\mathbb{C}, \mathbb{C}) \Rightarrow$ BC conjecture in $L^1(G)$. $L^1(G) \to C^*_r(G)$ isomorphism in $K$-theory? No general result (Bost).

Lafforgue: may perform other Banach crossed product.

**Definition**

Algebra norm $N$ on $C_c(G)$ is **unconditional** if $N(f) = N(|f|)$ ($f \in C_c(G)$).
Unconditional norms and crossed products

\[ N \text{ unconditional norm} \rightarrow \text{natural norm on } C_c(G; A): f \mapsto N(\|A \circ f\|) \]

Completion: Banach algebra \( A \rtimes_N G \).

Same construction for \( B \)-pairs: \( E \rtimes_N G \).

Morphism \( j^N_G : KK^\text{ban}_G(A, B) \rightarrow KK^\text{ban}(A \rtimes_N G, B \rtimes_N G) \).

**Conclusion of first step**

Baum-Connes’ conjecture for \( G \) with element \( \gamma \in KK_G(\mathbb{C}, \mathbb{C}) \):

1. Prove that \( \gamma = 1 \) in \( KK^\text{ban}_G(\mathbb{C}, \mathbb{C}) \).
2. Construct unconditional completion of \( C_c(G) \) with same \( K \)-theory as \( C^*_r(G) \).
Non isometric representations of $G$

Equality $\gamma = 1$ proved in slight generalization $KK^\text{ban}_G(\mathbb{C}, \mathbb{C})$.

$\ell$ length function on $G$: $\ell : G \rightarrow \mathbb{R}_+$ (continuous) such that $\ell(1) = 0$ and $\ell(xy) \leq \ell(x) + \ell(y)$ for all $x, y \in G$.

$KK^\text{ban}_G(A, B)$ same way as $KK^\text{ban}_G(A, B)$ action of $G$ in the $B$-pairs not isometric but is controlled by $\ell$ i.e. continuous and $\|x \cdot \xi\| \leq \exp(\ell(x))\|\xi\|$.

$\ell$ length function, $N$ unconditional norm.

- Put $N : f \mapsto N(e^\ell f)$: unconditional norm.
- $j^N_G,\ell : KK^\text{ban}_G(A, B) \rightarrow KK^\text{ban}(A \times_N G, B \times_N G)$.

$\ell$ length function. For $s \in \mathbb{R}^*_+$, define $N_s$ unconditional norm $f \mapsto N(e^{s\ell} f)$.

Subalgebra $\bigcup_{s \in \mathbb{R}^*_+} A \times_{N_s} G$ of $A \times_N G$ same $K$-theory as $A \times_N G$.

$\gamma = 1$ in $KK^\text{ban}_G(s\ell, \mathbb{C}, \mathbb{C})$ for all $s$, implies $\gamma$ identity in $K_0(A \times_N G)$. 
Homotopy between $\gamma$ and 1

Two different cases:

1. **“geometric”**: complete riemannian manifold with nonpositive sectional curvature; real Lie groups and closed subgroups.

2. **“Combinatoric”**: “strongly bolic” metric space; $p$-adic Lie groups and closed subgroups.

Concentrate here to the combinatoric case: case of buildings of type $\widetilde{A}_2$. 
Recall: Julg Vallette $\gamma$ element for $\tilde{A}_2$ buildings

$G$ acts properly, isometrically on $\tilde{A}_2$ building $X$.

$X^{(i)}$ $(0 \leq i \leq 2)$ set of faces of dimension $i$ in $X$.

$(e_x)_{x \in X^{(0)}}$ canonical Hilbert basis of $H_0 = \ell^2(X^{(0)})$.

$H_1 \subset \Lambda^2(H_0)$ vector span of $e_\sigma = e_x \wedge e_y$, $\sigma = (x, y) \in X^{(1)}$

$H_2 \subset \Lambda^3(H_0)$ vector span of $e_\sigma = e_x \wedge e_y \wedge e_z$, $\sigma = (x, y, z) \in X^{(2)}$.

- $H = (H_0 \oplus H_2) \oplus H_1$.
- $F = F_a = T_a + T_a^*$ depends on an origin $a \in X^{(0)}$.

Where $T_a(e_\sigma) = \nu_{a,\sigma} \wedge e_\sigma$

$\nu_a$: unit vector which points from $\sigma$ to $a$. 

Kasparov’s KK-theory - 4. Lafforgue
Abstract results: elliptic complexes

Let \( G \) be a locally compact group, length function \( \ell \), \( A, B \) Banach algebras, \( E \) a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( B \)-pair with actions of \( A \) and \( G \)-action “controled by \( \ell \)”. 

\[
D = \{ S \in \mathcal{L}(E); \ [S, a] \in \mathcal{K}(E), \ g.S - S \in \mathcal{K}(E); \ g \mapsto g.S \text{ continuous}\}.
\]

\( F \in D^{(1)} \) with \( \text{id}_E - F^2 \in \mathcal{K}(E); \ (E, F) \in KK^\text{ban}_{G,\ell}(A, B) \).

**Lemma**

Let \( S \in D^{(1)} \) such that \( S^2 \in \mathcal{K}(E) \) and \( \exists T \in D^{(1)} \) with 

\[
\text{id}_E - (TS + ST) \in \mathcal{K}(E) \ ((E, S) \text{ - or } S \text{ elliptic complex}).
\]

1. There exists such a \( T \) with \( T^2 \in \mathcal{K}(E) \). Then 
\[
(E, S + T) \in KK^\text{ban}_{G,\ell}(A, B).
\]

2. The class of \( (E, S + T) \) in \( KK^\text{ban}_{G,\ell}(A, B) \) does not depend on \( T \). 

Elliptic complexes define \( KK \)-elements.

1. \( TST \) is OK.

2. \( \{ T \in D^{(1)}; \ ST + TS - \text{id}_E \in \mathcal{K}(E) \} \) is affine.
Abstract results: Elliptic complexes (2)

Lemma

\( S, T \in D^{(1)} \) such that \( S^2 \in \mathcal{K}(E) \) and \( T^2 \in \mathcal{K}(E) \). Assume that the spectrum of \( S + T \) in \( D/\mathcal{K}(E) \) is disjoint from \( \mathbb{R}_- \).

1. \( S \) and \( T \) are elliptic complexes.
2. \( S \) and \( T \) define the same element of \( KK_{G,\ell}^{\text{ban}}(A, B) \).

1. In \( D/\mathcal{K}(E) \), \((ST + TS)\) commutes with \( S \) and \( T \).
   \( T(ST + TS)^{-1} \) and \( S(ST + TS)^{-1} \) ’quasi-inverses’.
2. We may define a logarithm of \( ST + TS \). The desired homotopy is
   \( S(ST + TS)^{-t} + T(ST + TS)^{t-1} \).
Abstract results: Elliptic complexes (3)

Lemma

\( S, T \in D^{(1)} \). Assume that \( S \) commutes exactly to \( A \) and to \( G \), that \( S^2 = 0 \) and \( T^2 \in \mathcal{K}(E) \), \( ST + TS = \text{id}_E \). Then the class of \((E, S + T)\) in \( KK_{\text{ban}}(G, \ell)(A, B) \) is zero.

May assume \( T^2 = 0 \) (replace \( T \) by \( TST \)). \( S(E) \subset E \) invariant by \( A \) and \( G \).

Decomposition \( E = S(E) \oplus T(E) \), matrix of these elements is of the form

\[
\begin{pmatrix}
c_{1,1} & c_{1,2} \\
0 & c_{2,2}
\end{pmatrix}
\]

Note that \( c_{1,2} \in \mathcal{K}(E) \) since \( T \in D \).

Change these actions through a homotopy \( \begin{pmatrix} c_{1,1} & tc_{1,2} \\ 0 & c_{2,2} \end{pmatrix} \) \((t \in [0, 1])\). At \( t = 0 \), \( S + T \) is degenerate.
Homotopy

\[ \varphi = \varphi_a : X \to \mathbb{R}_+ : \varphi(f) \text{ distance to } a \text{ of most remote point of } f. \]

Put \( \ell(g) = \varphi(g(a)) \).

It is a length function: indeed
\[ \ell(gh) = d(gh(a), a) \leq d(gh(a), g(a)) + d(g(a), a) = d(h(a), a) + d(g(a), a) \]
(g isometry).

**Theorem**

*For all \( s > 0 \), the images of \( \gamma \) and 1 in \( KK_{G,s\ell}^{\text{ban}}(\mathbb{C}, \mathbb{C}) \) coincide.*
Homotopy (2)

$E_p$ Banach space, graded by 0, 1, 2: replace $\ell^2$ norm by $\ell^p$ norm in construction of $H$: $\ell^p$ basis $e_\sigma$.

Consider $\partial : E_p \to E_p$ given by $\partial(e_x) = 0$, $\partial(e_x \wedge e_y) = e_y - e_x$ and $\partial(e_x \wedge e_y \wedge e_z) = (e_y \wedge e_z) - (e_x \wedge e_z) + (e_x \wedge e_y)$ (for all $(x, y, z) \in X^{(2)}$).

For $t > 0$, let $A_t$ multiplication by $e^{t\varphi}$ (unbounded) $\partial_t = A_t \circ \partial \circ A_t^{-1}$ (bounded).

$\partial_t - g.\partial_t$ is compact in every $E_p$ ($\varphi_a - \varphi_{ga}$ almost constant at infinity).

Proposition

1. For all $s > 0$, $\partial_s : E_1 \to E_1$ elliptic complex.
2. There exists $s > 0$ such that for all $p \in [1, 2]$, $\partial_s : E_p \to E_p$ elliptic complex.
Construction of a quasi-inverse

Let \( x \in X^{(0)} \). Points \( x \) and \( a = x_0 \) determine a parallelogram \( x_0, y, x, z \) in the building \( X \).

We set

\[
T_0(e_x) = \left(1 - \frac{j}{n}\right) \sum_{k=1}^{n} e_{x_{k-1}} \wedge e_{x_k} + \frac{j}{n} \sum_{k=1}^{n} e_{y_{k-1}} \wedge e_{y_k}
\]

where \( d(x, y) = j(= 6) \) and \( d(x, z) = n - j(= 3) \).

Clearly \( \partial \circ T_0(e_x) = e_x - e_0 \).
Construction of a quasi-inverse (2)

In order to define $T_0(e_x \wedge e_y)$, one uses the following lemma:

**Lemma**

\[ e_x \wedge e_y - T_0 \partial(e_x \wedge e_y) \text{ is in the image of } \partial. \]

Restrict to the parallelogram containing \( \{x_0, x, y\} \). \( \partial \) restricted to vertices, edges and faces of this parallelogram is exact in dimensions 1 and 2.

\( T_0(e_x \wedge e_y) \) is the element \( \xi \) such that \( \partial(\xi) = e_x \wedge e_y - T_0 \partial(e_x \wedge e_y) \) described above.

The desired quasi-inverse of \( \partial_s \) is \( T_s = A_s \circ T_0 \circ A_s^{-1} \).

1. For all \( s > 0 \), \( T_s \) is continuous from \( E_1 \) into \( E_1 \).
2. There exists \( t > 0 \) such that, for all \( p \in [1, 2] \), \( T_t \) is continuous from \( E_p \) in \( E_p \) and, for all \( g \in G \), \( T_t - g \cdot T_t \in K(E_p) \).
3. For all \( s > 0 \) and \( g \in G \), \( T_s - g \cdot T_s \in K(E_1) \).
Check $\| T_s(e_x) \|$ and $\| T_s(e_x \wedge e_y) \|$ bounded on $X$. Faces appearing located in parallelogram above; their number grows polynomially with the distance from $x$ to $x_0$; coefficients appearing bounded. Conjugation by $A_s$ multiplies by function with exponential decay.

In $T_s^*(e_x \wedge e_y)$ appear all points $z$ such that $x, y$ is on the path from $z$ to $x_0$. The coefficients appearing have exponential decay $\exp(-s\varphi(z))$; their number $z$ increases exponentially. Taking $s$ large enough, one may control the $\ell^1$ norm of $T_s^*(e_x \wedge e_y)$ and that of $T_s^*(e_x \wedge e_y \wedge e_z)$.

It follows that $T_s^*$ is continuous from $E_1$ into $E_1$, whence $T_s$ is continuous from $E_\infty$ into $E_\infty$. As it is continuous from $E_1$ into $E_1$, it is continuous from $E_p$ into $E_p$ for all $p$ (by interpolation).

Similar arguments show that $T_s - g.T_s \in K(E_p)$ for all $g \in G$. 

Check that when \( \{x, y\} \) goes to infinity, \( \|(T_s - g.T_s)(e_x)\| \) and \( \|(T_s - g.T_s)(e_x \wedge e_y)\| \) go to 0. \( g.T_0 \) is \( T_0 \) with \( x_0 \) replaced by \( g.x_0 \). For \( x \) far from \( x_0 \), from \( x \) to \( x_0 \) and from \( x \) to \( g.x_0 \) used in construction of \( T_0 \) coincide near \( x \). Since we conjugate by \( A_s \) only points near \( x \) count.
Homotopy: end

For $s$ small, $\partial_s$ almost invariant and we can use abstract results above to show that $(E_1, \partial_s)$ defines element 1. (Use $E^s_1$: where norm is changed - with $\partial$).

For $s$ large, $\partial_s$ almost quasi-inverse of $T_a$ appearing in $\gamma$ and we can use abstract results above to show that $(E_2, \partial_s)$ defines element $\gamma$.

These elements are homotopic!!
The last step... Case of Lie groups

$G$ (real or $p$-adic) reductive Lie group: slight modification of Schwarz space (Harish-Chandra algebra) *spectral unconditional completion* of $C_c(G)$.

In this way, we get more direct proof of results of Wassermann (real case) and a generalization of Baum, Higson and Plymen ($p$-adic case).
Property (RD) Haagerup-Jolissaint

$G$ discrete group $\ell$ length function on $G$. If $G$ is finitely generated, may take word length.

$H^\infty(G, \ell)$ vector space functions $f : G \to \mathbb{C}$ such that, for all $p \in \mathbb{R}_+$,

$$\sum_{x \in G} \ell(x)^p |f(x)|^2 < \infty.$$  

Unconditional.

Theorem (Haagerup)

$G$ finitely generated free group. $H^\infty(G)$ subalgebra of $C^*_r(G)$ stable under holomorphic functional calculus (spectral). In particular, inclusion $H^\infty(G) \to C^*_r(G)$ induces K-theory isomorphism.

Jolissaint:

Definition

Finitely generated group $G$ property (RD) if $H^\infty(G) \subset C^*_r(G)$.

Then $H^\infty(G)$ spectral subalgebra of $C^*_r(G)$. 
Groups with property (RD)

Jolissaint: many groups behaving like free groups (e.g. cocompact subgroups of simple real Lie groups of rank 1) have property (RD).

De la Harpe: Gromov’s hyperbolic groups.

Ramagge, Robertson and Steger: discrete groups acting properly with compact quotient on $\tilde{A}_2$ buildings (e.g. discrete subgroups of $SL_3(\mathbb{Q}_p)$).

Lafforgue adapts proof of Ramagge, Robertson and Steger to the case of cocompact subgroups of $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$.

Finally, these results extended by M. Talbi and I. Chaterji so to contain also the quaternionic case and products of above groups.