# Kasparov's operator K-theory and applications
# 2. KK-theory

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# Topics:

1. $K$-theory.
2. *$KK$-theory.*
3. Baum-Connes conjecture.
4. Lafforgue's Banach $KK$-theory.

# Recall: K-homology?

## Definition of $\mathrm{Ell}$ (via $\mathbb{Z}/2\mathbb{Z}$-graded formalism)

$\mathrm{Ell}(A)$ *(A unital $C^*$-algebra) set of pairs* $(\pi, F)$ *where*

- $\pi : A \to \mathcal{L}(H)^{(0)}$ *$*$-representation of $A$ on a graded Hilbert space $H$.*
- $F \in \mathcal{L}(H)^{(1)}$
- $1 - F^2 \in \mathcal{K}(H)$ *and* $[F, \pi(a)] \in \mathcal{K}(H)$ *(for $a \in A$).*

Equivalence relation?

*Brown-Douglas-Fillmore* studied extensions of $C(X)$ by $\mathcal{K}$ (of type $0 \to \mathcal{K} \to \mathcal{B} \to C(X) \to 0$): These extensions form a group $\mathrm{Ext}(X)$ recognized to be the $K$-homology group $K^1(C(X)) = K_1(X)$.

BDF $\mathrm{Ext}$: answer to Atiyah's problem of finding analytic definition of $K$-homology.

Many more contributions: Voiculescu, Arveson, Pimsner-Popa-Voiculescu...

# Kasparov's KK theory
## Hilbert C*-modules

'Ell' $\to$ $KK$: Hilbert spaces $\to$ Hilbert $C^*$-modules: Paschke, Rieffel, Kasparov.

### Definition

*Prehilbert B-module*: right $B$-module $E$ with $B$-valued scalar product $\langle\ ,\ \rangle : E \times E \to B$ such that:

1. $\langle x, \lambda y\rangle = \lambda\langle x, y\rangle$, $\langle x, yb\rangle = \langle x, y\rangle b$, $\quad (\forall x, y \in E,\ \lambda \in \mathbb{C},\ b \in B)$.
2. $\langle y, x\rangle = \langle x, y\rangle^*$, $\quad (\forall x, y \in E)$.
3. $\langle x, x\rangle \geq 0$, $\quad (\forall x \in E)$.

Set $\|x\| = \|\langle x, x\rangle\|^{1/2}$: semi-norm on $E$.
If $E$ is Hausdorff and complete *Hilbert B-module.*

# Hilbert C*-modules

## Remarks

1. $\langle \lambda x, y\rangle = \overline{\lambda}\langle x, y\rangle$ *and* $\langle xb, y\rangle = b^*\langle x, y\rangle$.
2. *$E$ prehilbert $B$-module. $N = \{x \in E,\ \langle x, x\rangle = 0\}$ submodule of $E$ Hausdorff completion of $E$ (completion of $E/N$) Hilbert $B$-module.*

## Examples

- $B$ is a Hilbert $B$-module: $\langle x, y\rangle = x^*y$ (same norm as $B$).
- $B^n$ Hilbert $B$-module ($n \in \mathbb{N}$): $\langle (x_i), (y_i)\rangle = \sum x_i^* y_i$.
- $(E_i)_{i\in I}$ Hilbert $B$-modules. Hilbert $B$-module $\bigoplus E_i$ scalar product $\langle (x_i), (y_i)\rangle = \sum \langle x_i, y_i\rangle$ (completion).
- $\bigoplus_{i\in\mathbb{N}} B$ fundamental Hilbert $B$-module noted $\mathcal{H}_B$.

# Operators; compact operators

Adjoint: not automatic

## Definition

$E_1, E_2$ Hilbert $B$-modules. $\mathcal{L}(E_1, E_2)$: maps $T : E_1 \to E_2$ with adjoint $T^* : E_2 \to E_1$ $\langle Tx_1, x_2\rangle = \langle x_1, T^*x_2\rangle$ $(\forall x_1 \in E_1,\ x_2 \in E_2)$.
Put $\mathcal{L}(E) = \mathcal{L}(E, E)$.

$T \in \mathcal{L}(E_1, E_2) \Rightarrow$ linear, $B$-linear and bounded; $T^*$ unique, $(T^*)^* = T$.

With norm of bounded operators, $\mathcal{L}(E)$ is a $C^*$-algebra.

## Definition

$E_1, E_2$ Hilbert $B$-modules and $x_1 \in E_1$ and $x_2 \in E_2$.

- Define $\theta_{x_1,x_2} \in \mathcal{L}(E_2, E_1)$ by $\theta_{x_1,x_2}(x) = x_1\langle x_2, x\rangle$. Note $\theta_{x_1,x_2} = \theta^*_{x_2,x_1}$.
- Norm closure of vector span of $\theta_{x_2,x_1}$: *compact operators* $\mathcal{K}(E_1, E_2)$.
- $\mathcal{K}(E) = \mathcal{K}(E, E)$ (closed two sided) ideal in $\mathcal{L}(E)$.

# Hilbert C*-modules

## Theorem (Kasparov)

*For every Hilbert B-module E,* $M(\mathcal{K}(E)) = \mathcal{L}(E)$*. In particular* $M(\mathcal{K} \otimes B) = \mathcal{L}(\mathcal{H}_B)$*.*

## Kasparov's Stabilization Theorem

*For every countably generated Hilbert B-module E,* $\mathcal{H}_B \oplus E$ *is isomorphic to* $\mathcal{H}_B$*.*

Adaptation of Gram-Schmidt orthonormalization method.

### Graded Hilbert modules

Decomposition $E = E^{(0)} \oplus E^{(1)}$. $\mathcal{L}(E)$ is then $\mathbb{Z}/2\mathbb{Z}$-graded.

# Kasparov's bifunctor

**Definition**

$A, B$ $C^*$-algebras.

- $\mathfrak{E}(A, B)$ triples $(E, \pi, F)$ where
  - $*$ $E$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert $B$-module.
  - $*$ $\pi : A \to \mathcal{L}(E)^{(0)}$ is a $*$-homomorphism.
  - $*$ $F \in \mathcal{L}(E)^{(1)}$ satisfies: $\forall a \in A,\ \pi(a)(F^2 - 1) \in \mathcal{K}(E)$ and $[\pi(a), F] \in \mathcal{K}(E)$.
- $\mathfrak{D}(A, B)$: *degenerate* $(E, \pi, F)$: $\pi(a)(F^2 - 1) = 0$ and $[\pi(a), F] = 0$.
- Addition: $(E, \pi, F) + (E', \pi', F') = (E \oplus E', \pi \oplus \pi', F \oplus F')$.
- *Homotopy* in $\mathfrak{E}(A, B)$: element of $\mathfrak{E}(A, B[0, 1])$.
- $KK(A, B)$ set of homotopy classes of elements of $\mathfrak{E}(A, B)$.

# The *KK*-groups

**Theorem**

*$KK(A, B)$ is an abelian group*

Degenerate elements are homotopic to 0.
The opposite of $(E, \pi, F)$ is $(-E, \pi, -F)$, where $-E$ is $E$ with opposite grading.

**Theorem**

*$KK(A, B)$ homotopy invariant bifunctor: covariant in $B$, contravariant in $A$.*

**Theorem**

*For every $C^*$-algebra $B$, $KK(\mathbb{C}, B) = K_0(B)$.*

# The KK-product (Kasparov)

## Definition

$1_A \in KK(A, A)$ class of $(A, i_A, 0)$ where $A^{(1)} = 0$ and
$i_A : A \to \mathcal{K}(A) \subset \mathcal{L}(A)$ $(i(a)b = ab,\ a, b \in A)$.

## Theorem

1. *There is a well defined bilinear coupling (Kasparov product)* $KK(A, D) \times KK(D, B) \to KK(A, B)$ *noted* $(x, y) \mapsto x \otimes_D y$.
2. *Covariant in* $B$*, contravariant in* $A$*;*
3. $f : D \to E$ *$*$-homomorphism,* $x \in KK(A, D)$ *and* $y \in KK(E, B)$ $f_*(x) \otimes_E y = x \otimes_D f^*(y)$.
4. $\forall x \in KK(A, B),\ x \otimes_B 1_B = 1_A \otimes x = x$.
5. *Associative:* $\forall x \in KK(A, D),\ y \in KK(D, E),\ z \in KK(E, B)$ *we have:* $(x \otimes_D y) \otimes_E z = x \otimes_D (y \otimes_E z)$.

# Extending KK-product

## Definition

- $\tau_D : KK(A, B) \to KK(A \otimes D, B \otimes D)$:
  $(E, \pi, F) \mapsto (E \otimes D, \pi \otimes i_D, F \otimes 1)$.
- $A_1, A_2, B_1, B_2, D$ $C^*$-algebras,
  $x \in KK(A_1, B_1 \otimes D)$, $y \in KK(D \otimes A_2, B_2)$.

  Put $x \otimes_D y = \tau_{A_2}(x) \otimes_{B_1 \otimes D \otimes A_2} \tau_{B_1}(y)$.

## Theorem

1. *Same properties as above.*
2. *Product over $\mathbb{C}$ commutative.*

# Abstract periodicity, abstract duality

## Theorem (abstract periodicity)

*$\alpha \in KK(D, E)$, $\beta \in KK(E, D)$ such that $\alpha \otimes_E \beta = 1_D$ and $\beta \otimes_D \alpha = 1_E$.*
*We say $D$ and $E$ are K-equivalent.*
*$\bullet \otimes_D \alpha : KK(A, B \otimes D) \to KK(A, B \otimes E)$ and*
*$\beta \otimes_D \bullet : KK(D \otimes A, B) \to KK(E \otimes A, B)$ isomorphisms*
*inverses $\bullet \otimes_E \beta$ and $\alpha \otimes_E \bullet$.*

$K$-equivalent $C^*$-algebras cannot be distinguished by $KK$-theory.

## Theorem (abstract duality)

*$\rho \in KK(D \otimes E, \mathbb{C})$, $\sigma \in KK(\mathbb{C}, E \otimes D)$ such that $\sigma \otimes_D \rho = 1_E$ and*
*$\sigma \otimes_E \rho = 1_D$. We say $D$ and $E$ are K-dual.*
*$\sigma \otimes_D \bullet : KK(D \otimes A, B) \to KK(A, E \otimes B)$ and*
*$\sigma \otimes_E \bullet : KK(A \otimes E, B) \to KK(A, B \otimes D)$ isomorphisms*
*inverses $\bullet \otimes_E \rho$ and $\bullet \otimes_D \rho$).*

# Bott periodicity

$\beta \in KK(\mathbb{C}, C_0(\mathbb{R}^2))$ class of $(E, \pi, F)$ where

- $E^{(0)} = E^{(1)} = C_0(\mathbb{R}^2)$;
- $\pi : \mathbb{C} \to \mathcal{L}(E)$
- $F(\xi, \eta) = (P^*\eta, P\xi)$ where $P : C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2)$ given $P\xi(s,t) = (1 + s^2 + t^2)^{-1/2}(s + it)\xi(s,t),\ (s,t) \in \mathbb{R}^2$.
  $1 - F^2 \in \mathcal{K}(E)$ (multiplication by $(1 + s^2 + t^2)^{-1} \in C_0(\mathbb{R}^2)$).

$\alpha \in KK((C_0(\mathbb{R}^2), \mathbb{C})$ class of $(H, \pi, D)$ where

- $H^{(0)} = H^{(1)} = L^2(\mathbb{R}^2)$;
- $\pi$ action by multiplication;
- $D(\xi, \eta) = (T\eta, T^*\xi)$ where $T$ Fourier transform of $P$ ($\widehat{T\xi} = P\widehat{\xi}$).

**Theorem (Bott periodicity)**

*$KK(A, B(\mathbb{R}^2)) \simeq KK(A, B)$ and $KK(A(\mathbb{R}^2), B) \simeq KK(A, B)$.*

# Other consequences of KK-product

## Theorem (Bott periodicity)

*$m, n \geq 0$ we have:*

- *if $m+n$ is even, $KK(A(\mathbb{R}^m), B(\mathbb{R}^n)) \simeq KK(A, B)$;*
- *if $m+n$ is odd,*
  $KK(A(\mathbb{R}^m), B(\mathbb{R}^n)) \simeq KK(A, B(\mathbb{R})) \simeq KK(A(\mathbb{R}), B) := KK^1(A, B)$.

## Theorem (Thom isomorphism)

*$X$ locally compact space and let $E$ (total space) complex vector bundle over $X$. $C_0(X)$ and $C_0(E)$ are $K$-equivalent.*

## Proposition (Stability)

*For every $C^*$-algebra $A$, $A$ and $A \otimes \mathcal{K}$ are $K$-equivalent.*

Generalization: Morita equivalence.

# Equivariant KK-theory

Actions of $G$ locally compact group on $A$ and $B$ by same letter $\alpha$.

## Definition

1. *Equivariant Hilbert B-module* $E$: (pointwise) cont. action $\alpha$ of $G$: $\alpha_g(\xi b) = \alpha_g(\xi)\alpha_g(b), \quad \alpha_g(\langle \xi, \eta\rangle) = \langle \alpha_g(\xi), \alpha_g(\eta)\rangle$, $\partial(\alpha_g(\xi)) = \partial(\xi)$ ($b \in B$, $\xi$, $\eta \in E$ and $g \in G$).
2. *equivariant bimodule*: $\pi : A \to \mathcal{L}(E)$: $\alpha_g(\pi(a)\xi) = \pi(\alpha_g(a))\alpha_g(\xi)$ ($a \in A$, $g \in G$, $\xi \in E$).
3. $\alpha_g(F)\xi = \alpha_g(F\alpha_{g^{-1}}(\xi))$.
4. $F$ said to be *G-continuous*: $g \mapsto \alpha_g(F)$ *norm* continuous.
5. $\mathfrak{E}_G(A, B)$
    - $(E, \pi, F) \in \mathfrak{E}(A, B)$;
    - $E$ equivariant bimodule;
    - $F$ $G$-continuous and $\alpha_g(F) - F \in \mathcal{K}(E)$ ($g \in G$).
6. $KK_G(A, B)$ group of homotopy classes of elements of $\mathfrak{E}_G(A, B)$.

# Constructions

1. *KK-product* extends to equivariant case.
2. *Restriction.* $r : G \to H$ continuous group homomorphism. Restriction $r^* : KK_H(A, B) \to KK_G(A, B)$.
3. *Induction* If $H$ is a closed subgroup of $G$, Kasparov constructs an induction homomorphism $\mathrm{ind}_H^G$ maps $KK_H(A, B)$ to $KK_G(C(G \times_H A), C(G \times_H B))$.

- Both restriction and induction have a good behaviour with respect to Kasparov product.

4. *Descent morphism $j_G$* $A, B$-bimodule $(E, \pi)$ we associate an $A \rtimes_\alpha G$, $B \rtimes_\alpha G$-bimodule $(E \rtimes_\alpha G, \pi_\alpha)$.

**Theorem**

1. *If $(E, \pi, F) \in \mathfrak{E}_G(A, B)$ then $(E \rtimes_\alpha G, \pi_\alpha, F \otimes_B 1) \in \mathfrak{E}(A \rtimes_\alpha G, B \rtimes_\alpha G)$.*
2. *Gives group homomorphism $j_G : KK_G(A, B) \to KK(A \rtimes_\alpha G, B \rtimes_\alpha G)$.*
3. *Kasparov product: $j_G(x \otimes_B y) = j_G(x) \otimes_{B \rtimes_\alpha G} j_G(y)$ and $j_G(1_A) = 1_{A \rtimes_\alpha G}$.*

# Generalizations

- Groupoids: Le Gall.
- Hopf algebras: Baaj-Sk

# Index Theory

Ideal setting for Atiyah-Singer index theorems and generalizations.

*Atiyah-Singer index theorem*
Elliptic pseudodifferential operator $P$ of order 0 on manifold $M$ defines an element $[P] \in KK(C(M, \mathbb{C})$ (starting point of $\mathrm{Ell}$)

*Index* of $P$ *i.e.* homomorphism from $K_0(C(M))$ to $\mathbb{Z} = K_0(\mathbb{C})$ it defines is computed in terms of $[\sigma_P]$.

Correspondence between $[P] \leftrightarrow [\sigma_P]$ isomorphism (*Poincaré duality*) $KK(C(M), \mathbb{C}) \leftrightarrow KK(\mathbb{C}, C_0(T^*M))$.

*Topological index* also seen as an element of $KK$.

**Theorem**

*Topological index=Analytical index. Equality of $KK$-elements.*

# Generalizations

Many more examples of natural $KK$-elements related with index problems:

- The Atiyah-Singer index theorem for families: Families of manifolds indexed by a locally compact space $Y$: fibration $M \to Y$ with fibers manifolds. Pseudodifferential family: $KK(C(M), C(Y))$.
- Longitudinal index theorem for foliations (Connes+Sk)
- Atiyah's index theorem for covering spaces
- Mishchenko and Fomenko index theorem
- Signature operators on Lipschitz manifolds (Teleman, Hilsum)
- Transversally elliptic operators for foliations (Hilsum-Sk)

# C*-algebraic extensions

Many $C^*$-algebras are formed out of simpler blocks through extensions.

Important to give invariants that classify different possible extensions of a pair of given $C^*$-algebras.

Already mentioned important work of *Brown-Douglas-Fillmore* extensions $0 \to \mathcal{K} \to \mathcal{B} \to C(X) \to 0$.

# Kasparov's Bivariant Ext

Extensions of $A$ by $B \otimes \mathcal{K}$.

- *Trivial:* if split.

  Consider extensions: $0 \to B \otimes \mathcal{K} \xrightarrow{i_j} D_j \xrightarrow{p_j} A \to 0 \quad (j = 0, 1)$

- *Sum* given by set of matrices $\begin{pmatrix} x_1 & y \\ z & x_2 \end{pmatrix}$ with $y, z \in B \otimes \mathcal{K}$ and $x_i \in D_i$ satisfying $p_1(x_1) = p_2(x_2)$.

- *Unitary equivalence:*

$$\begin{array}{ccccccccc} 0 & \to & B \otimes \mathcal{K} & \to & D_1 & \to & A & \to & 0 \\ & & Ad(u) \downarrow & & h \downarrow & & \| & & \\ 0 & \to & B \otimes \mathcal{K} & \to & D_2 & \to & A & \to & 0 \end{array}$$

- *Equivalence of extensions* $\sigma_1 \sim \sigma_2$ if there exist trivial extensions $\tau_1, \tau_2$ with $\sigma_1 + \tau_1$ and $\sigma_2 + \tau_2$ unitarily equivalent.
  Extensions form a semigroup $\mathrm{Ext}(A, B)$.

# Results

**Theorem (Kasparov)**

*Extension* $0 \to B \otimes \mathcal{K} \to D \to A$ *is invertible in* $\mathrm{Ext}(A, B)$ *iff it admits a completely positive splitting (generalized 'Stinespring theorem').*

**Theorem (Kasparov)**

*The group of invrtible elements* $\mathrm{Ext}(A, B)^{-1}$ *is* $KK^1(A, B) := KK(A, B(\mathbb{R}))$.

In particular $\mathrm{Ext}^{-1}$ is homotopy invariant and satisfies all nice periodicities...

Finally, Kasparov shows that in each class of $\mathrm{Ext}(A, B)$ there is a (unique) element which absorbs all trivial extensions (generalized 'Voiculescu theorem' - asumption of nuclearity).

# More properties of *KK*

- Exact sequences in both variables. Assumption of nuclearity (or cp lifting).
- Roseberg-Schochet: computation in terms of $K$-theory. *Universal Coefficient Formula.* (Good cases).
- Connes-Higson: $E(A, B)$ exact in all cases. In a way: group of formal differences of $\mathrm{Ext}(A, B)$.