**C*-ALGEBRAS OVER TOPOLOGICAL SPACES:**
**THE BOOTSTRAP CLASS**

RALF MEYER AND RYSZARD NEST

Abstract. We carefully define and study C*-algebras over topological spaces, possibly non-Hausdorff, and review some relevant results from point-set topology along the way. We explain the triangulated category structure on the bivariant Kasparov theory over a topological space and study the analogue of the bootstrap class for C*-algebras over a finite topological space.

1. Introduction

If $X$ is a locally compact Hausdorff space, then there are various equivalent characterisations of what it means for $X$ to act on a C*-algebra $A$. The most common definition uses an essential *-homomorphism from $C_0(X)$ to the centre of the multiplier algebra of $A$. An action of this kind is equivalent to a continuous map from the primitive ideal space $\text{Prim}(A)$ of $A$ to $X$. This makes sense in general: A C*-algebra over a topological space $X$, which may be non-Hausdorff, is a pair $(A, \psi)$, where $A$ is a C*-algebra and $\psi: \text{Prim}(A) \to X$ is a continuous map. One of the purposes of this article is to discuss this definition and relate it to other notions due to Eberhard Kirchberg and Alexander Bonkat [3,7].

An analogue of Kasparov theory for C*-algebras over locally compact Hausdorff spaces was defined already by Gennadi Kasparov in [6]. He used it in his proof of the Novikov conjecture for subgroups of Lie groups. Kasparov’s definition was extended by Eberhard Kirchberg to the non-Hausdorff case in [7], in order to generalise classification results for simple, purely infinite, nuclear C*-algebras to the non-simple case. In his thesis [3], Alexander Bonkat studies an even more general theory and extends the basic results of Kasparov theory to this setting.

This article is part of an ongoing project to compute the Kasparov groups $KK_*(X; A, B)$ for a topological space $X$ and C*-algebras $A$ and $B$ over $X$. The aim is a Universal Coefficient Theorem in this context that is useful for the classification programme. At the moment, we can achieve this goal for some finite topological spaces (see [13]), but the general situation, even in the finite case, is still unclear.

The main theme of this article is to describe an analogue of the bootstrap class for C*-algebras over a topological space. Although we propose a definition for infinite spaces in §4.4, most of our results are limited to finite spaces.

Our first task is to clarify the definition of C*-algebras over $X$; this is the main point of Section 2. Our definition is quite natural, but more restrictive than the definitions in [3,7]. The approach there is to use the map $\mathcal{O}(X) \to \mathcal{O}(\text{Prim}A)$ induced by $\psi: \text{Prim}(A) \to X$, where $\mathcal{O}(X)$ denotes the complete lattice of open subsets of $X$. If $X$ is a sober space—this is a very mild assumption that is also made under a different name in [3,7]—then we can recover it from the lattice $\mathcal{O}(X)$, and a continuous map $\text{Prim}(A) \to X$ is equivalent to a map $\mathcal{O}(X) \to \mathcal{O}(\text{Prim}A)$ that commutes with arbitrary unions and finite intersections.
The definition of the Kasparov groups KK_*(X; A, B) still makes sense for any map \( \mathcal{O}(X) \to \mathcal{O}(\text{Prim } A) \) (in the category of sets), that is, even the restrictions imposed in [3,7] can be removed. But such a map \( \mathcal{O}(X) \to \mathcal{O}(\text{Prim } A) \) corresponds to a continuous map \( \text{Prim}(A) \to Y \) for another, more complicated space \( Y \) that contains \( X \) as a subspace. Hence the definitions in [3,7] are, in fact, not more general. But they make computations much more complicated because the discontinuities add further input data which must be taken into account even for examples where they vanish because the action is continuous.

Since the relevant point-set topology is widely unknown among operator algebraists, we also recall some basic notions such as sober spaces and Alexandrov spaces. The latter are highly non-Hausdorff spaces—Alexandrov categories: the subcategory of nuclear \( \ast \)-algebras over a topological space. We omit most proofs because they are similar to the familiar arguments for ordinary Kasparov theory and because the technical details are already dealt with in [3]. We emphasise the triangulated category structure on the Kasparov category over \( X \) because it plays an important role in connection with the bootstrap class.

In Section 3, we briefly recall the definition and the basic properties of bivariant Kasparov theory for \( \ast \)-algebras over a topological space. We omit most proofs because they are similar to the familiar arguments for ordinary Kasparov theory and because the technical details are already dealt with in [3]. We emphasise the triangulated category structure on the Kasparov category over \( X \) because it plays an important role in connection with the bootstrap class.

In Section 4, we define the bootstrap class over a finite topological space \( X \) and give criteria for a \( \ast \)-algebra over \( X \) to belong to it. Our constructions depend heavily on the relation between Alexandrov spaces and preordered sets and therefore do not extend directly to infinite spaces.

We define the \( X \)-equivariant bootstrap class \( B(X) \) as the localising subcategory of the Kasparov category of \( \ast \)-algebras over \( X \) that is generated by the basic objects \( (\mathbb{C}, x) \) for \( x \in X \), where we identify \( x \in X \) with the corresponding constant map \( \text{Prim}(\mathbb{C}) \to X \). Notice that this is exactly the list of all \( \ast \)-algebras over \( X \) with underlying \( \ast \)-algebra \( \mathbb{C} \).

We show that a nuclear \( \ast \)-algebra \( (A, \psi) \) over \( X \) belongs to the \( X \)-equivariant bootstrap class if and only if its “fibres” \( A(x) \) belong to the usual bootstrap class for all \( x \in X \). These fibres are certain subquotients of \( A \); if \( \psi : \text{Prim}(A) \to X \) is a homeomorphism, then they are exactly the simple subquotients of the \( \ast \)-algebra \( A \).

The bootstrap class we define is the class of objects where we expect a Universal Coefficient Theorem to hold. If \( A \) and \( B \) belong to the bootstrap class, then an element of \( \text{KK}_*(X; A, B) \) is invertible if and only if it is fibrewise invertible on K-theory, that is, the induced maps \( K_*(A(x)) \to K_*(B(x)) \) are invertible for all \( x \in X \). This follows easily from our definition of the bootstrap class. The proof of our criterion for a \( \ast \)-algebra over \( X \) to belong to the bootstrap class already provides a spectral sequence that computes \( \text{KK}_*(X; A, B) \) in terms of non-equivariant Kasparov groups. Unfortunately, this spectral sequence is not useful for classification purposes because it rarely degenerates to an exact sequence.

We call a \( \ast \)-algebra over \( X \) tight if the map \( \text{Prim}(A) \to X \) is a homeomorphism. This implies that its fibres are simple. We show in Section 5 that any separable nuclear \( \ast \)-algebra over \( X \) is KK(\( X \))-equivalent to a tight, separable, nuclear, purely infinite, stable \( \ast \)-algebra over \( X \). The main issue is tightness. By Kirchberg’s classification result, this model is unique up to \( X \)-equivariant \( \ast \)-isomorphism. In this sense, tight, separable, nuclear, purely infinite, stable \( \ast \)-algebra over \( X \) are classified up to isomorphism by the isomorphism classes of objects in a certain triangulated category: the subcategory of nuclear \( \ast \)-algebras over \( X \) in the Kasparov category.
The difficulty is to replace this complete “invariant” by a more tractable one that classifies objects of the—probably smaller—bootstrap category $\mathcal{B}(X)$ by K-theoretic data.

If $\mathcal{C}$ is a category, then we write $A \in \mathcal{C}$ to denote that $A$ is an object of $\mathcal{C}$—as opposed to a morphism in $\mathcal{C}$.

2. $C^*$-algebras over a topological space

We define the category $\mathcal{C}^\ast\text{alg}(X)$ of $C^*$-algebras over a topological space $X$. In the Hausdorff case, this amounts to the familiar category of $C_0(X)$-$C^*$-algebras. For non-Hausdorff spaces, our notion is related to another one by Eberhard Kirchberg. For the Universal Coefficient Theorem, we must add some continuity conditions to Kirchberg’s definition of $\mathcal{C}^\ast\text{alg}(X)$. We explain in §2.9 why these conditions result in essentially no loss of generality. Furthermore, we explain briefly why it is allowed to restrict to the case where the underlying space $X$ is sober, and we consider some examples, focusing on special properties of finite spaces and Alexandrov spaces.

2.1. The Hausdorff case. Let $A$ be a $C^*$-algebra and let $X$ be a locally compact Hausdorff space. There are various equivalent additional structures on $A$ that turn it into a $C^*$-algebra over $X$ (see [14] for the proofs of most of the following assertions). The most common definition is the following one from [9]:

**Definition 2.1.** A $C_0(X)$-$C^*$-algebra is a $C^*$-algebra $A$ together with an essential $^*$-homomorphism $\varphi$ from $C_0(X)$ to the centre of the multiplier algebra of $A$. We abbreviate $h \cdot a := \varphi(h) \cdot a$ for $h \in C_0(X)$.

A $^*$-homomorphism $f: A \to B$ between two $C_0(X)$-$C^*$-algebras is $C_0(X)$-linear if $f(h \cdot a) = h \cdot f(a)$ for all $h \in C_0(X)$, $a \in A$.

Let $\mathcal{C}^\ast\text{alg}(C_0(X))$ be the category of $C_0(X)$-$C^*$-algebras, whose morphisms are the $C_0(X)$-linear $^*$-homomorphisms.

A map $\varphi$ as above is equivalent to an essential $^*$-homomorphism $\bar{\varphi}: C_0(X, A) \cong C_0(X) \otimes_{\text{max}} A \to A$, $f \otimes a \mapsto \varphi(f) \cdot a$, which exists by the universal property of the maximal tensor product; the centrality of $\varphi$ ensures that $\bar{\varphi}$ is a $^*$-homomorphism and well-defined. Of course, we get $\varphi$ back from $\bar{\varphi}$ by restricting to elementary tensors. The description via $\bar{\varphi}$ has two advantages: it requires no multipliers, and the resulting class in $\text{KK}(C_0(X, A), A)$ plays a role in connection with duality in bivariant Kasparov theory (see [5]).

Any $C_0(X)$-$C^*$-algebra is isomorphic to the $C^*$-algebra of $C_0(X)$-sections of an upper semi-continuous $C^*$-algebra bundle over $X$ (see [14]). Even more, this yields an equivalence of categories between $\mathcal{C}^\ast\text{alg}(C_0(X))$ and the category of upper semi-continuous $C^*$-algebra bundles over $X$.

**Definition 2.2.** Let $\text{Prim}(A)$ denote the primitive ideal space of $A$, equipped with the usual hull-kernel or Jacobson topology.

The Dauns–Hofmann Theorem identifies the centre of the multiplier algebra of $A$ with the $C^*$-algebra $C_b(\text{Prim}(A))$ of bounded continuous functions on the primitive ideal space of $A$. Therefore, the map $\varphi$ in Definition 2.1 is of the form $\psi^*: C_0(X) \to C_0(\text{Prim}(A))$, $f \mapsto f \circ \psi$, for some continuous map $\psi: \text{Prim}(A) \to X$ (see [14]). Thus $\varphi$ and $\psi$ are equivalent additional structures. We use such maps $\psi$ to generalise Definition 2.1 to the non-Hausdorff case.
2.2. The general definition. Let \( X \) be an arbitrary topological space.

**Definition 2.3.** A \( C^* \)-algebra over \( X \) is a pair \( (A, \psi) \) consisting of a \( C^* \)-algebra \( A \) and a continuous map \( \psi : \text{Prim}(A) \to X \).

Our next task is to define morphisms between \( C^* \)-algebras \( A \) and \( B \) over the same space \( X \). This requires some care because the primitive ideal space is not functorial for arbitrary \( * \)-homomorphisms.

**Definition 2.4.** For a topological space \( X \), let \( \mathcal{O}(X) \) be the set of open subsets of \( X \), partially ordered by \( \subseteq \).

**Definition 2.5.** For a \( C^* \)-algebra \( A \), let \( \mathbb{I}(A) \) be the set of all closed \( * \)-ideals in \( A \), partially ordered by \( \subseteq \).

The partially ordered sets \( (\mathcal{O}(X), \subseteq) \) and \( (\mathbb{I}(A), \subseteq) \) are complete lattices, that is, any subset in them has both an infimum \( \bigwedge I \) and a supremum \( \bigvee I \). Namely, in \( \mathcal{O}(X) \), the supremum is \( \bigcup S \), and the infimum is the interior of \( \bigcap S \); in \( \mathbb{I}(A) \), the infimum and supremum are

\[
\bigwedge_{I \in S} I = \bigcap_{I \in S} I, \quad \bigvee_{I \in S} I = \bigcup_{I \in S} I.
\]

We always identify \( \mathcal{O}(\text{Prim}(A)) \) and \( \mathbb{I}(A) \) using the familiar isomorphism

\[
(2.6) \quad \mathcal{O}(\text{Prim}(A)) \cong \mathbb{I}(A), \quad U \mapsto \bigcap_{p \in \text{Prim}(A) \setminus U} p.
\]

This is a lattice isomorphism and hence preserves infima and suprema.

Let \( (A, \psi) \) be a \( C^* \)-algebra over \( X \). We get a map

\[
\psi^* : \mathcal{O}(X) \to \mathcal{O}(\text{Prim}(A)) \cong \mathbb{I}(A), \quad U \mapsto \{ p \in \text{Prim}(A) \mid \psi(p) \in U \} \cong A(U).
\]

We usually write \( A(U) \in \mathbb{I}(A) \) for the ideal and \( \psi^*(U) \) or \( \psi^{-1}(U) \) for the corresponding open subset of \( \text{Prim}(A) \). If \( X \) is a locally compact Hausdorff space, then \( A(U) := C_0(U) \cdot A \) for all \( U \in \mathcal{O}(X) \).

**Example 2.7.** For any \( C^* \)-algebra \( A \), the pair \( (A, \text{id}_{\text{Prim}(A)}) \) is a \( C^* \)-algebra over \( \text{Prim}(A) \); the ideals \( A(U) \) for \( U \in \mathcal{O}(\text{Prim}(A)) \) are given by \( (2.6) \). \( C^* \)-algebras over topological spaces of this form play an important role in §5.

**Lemma 2.8.** The map \( \psi^* \) is compatible with arbitrary suprema (unions) and finite infima (intersections), so that

\[
A \left( \bigcup_{U \in S} U \right) = \bigcup_{U \in S} A(U), \quad A \left( \bigcap_{U \in F} U \right) = \bigcap_{U \in F} A(U)
\]

for any subset \( S \subseteq \mathcal{O}(X) \) and for any finite subset \( F \subseteq \mathcal{O}(X) \).

**Proof.** This is immediate from the definition. \( \square \)

Taking for \( S \) and \( F \) the empty set, this specialises to \( A(\emptyset) = \{0\} \) and \( A(X) = A \). Taking \( S = \{U, V\} \) with \( U \subseteq V \), this specialises to the monotonicity property

\[
U \subseteq V \implies A(U) \subseteq A(V);
\]

We will implicitly use later that these properties follow from compatibility with finite infima and suprema.

The following lemma clarifies when the map \( \psi^* \) is compatible with infinite infima.

**Lemma 2.9.** If the map \( \psi : \text{Prim}(A) \to X \) is open or if \( X \) is finite, then the map \( \psi^* : \mathcal{O}(X) \to \mathbb{I}(A) \) preserves infima—that is, it maps the interior of \( \bigcap_{U \in S} U \) to the ideal \( \bigcap_{U \in S} A(U) \) for any subset \( S \subseteq \mathcal{O}(X) \). Conversely, if \( X \) is a \( T_1 \)-space, that is, points in \( X \) are closed, and \( \psi^* \) preserves infima, then \( \psi \) is open.
Since preservation of infinite infima is automatic for finite $X$, the converse assertion cannot hold for general $X$.

**Proof.** If $X$ is finite, then any subset of $\mathcal{O}(X)$ is finite, and there is nothing more to prove. Suppose that $\psi$ is open. Let $V$ be the interior of $\bigcap_{U \in S} U$. Let $W \subseteq \text{Prim}(A)$ be the open subset that corresponds to the ideal $\bigcap_{U \in S} \psi^*(U)$. We must show $\psi^*(V) = W$. Monotonicity yields $\psi^*(V) \subseteq W$. Since $\psi$ is open, $\psi(W)$ is an open subset of $X$. By construction, $\psi(W) \subseteq U$ for all $U \in S$ and hence $\psi(W) \subseteq V$. Thus $\psi^*(V) \supseteq \psi^*(\psi(W)) \supseteq W \supseteq \psi^*(V)$.

Now suppose that $\psi^*$ preserves infima and that points in $X$ are closed. Assume that $\psi$ is not open. Then there is an open subset $W$ in $\text{Prim}(A)$ for which $\psi(W)$ is not open in $X$. Let $S := \{X \setminus \{x\} \mid x \in X \setminus \psi(W)\} \subseteq \mathcal{O}(X)$; this is where we need points to be closed. We have $\bigcap_{U \in S} U = \psi(W)$ and $\bigcap_{U \in S} \psi^*(U) = \psi^{-1}(\psi(W))$. Since $\psi(W)$ is not open, the infimum $V$ of $S$ in $\mathcal{O}(X)$ is strictly smaller than $\psi(W)$. Hence $\psi^*(V)$ cannot contain $W$. But $W$ is an open subset of $\psi^{-1}(\psi(W))$ and hence contained in the infimum of $\psi^*(S)$ in $\mathcal{O}(\text{Prim } A)$. Therefore, $\psi^*$ does not preserve infima, contrary to our assumption. Hence $\psi$ must be open. \qed

For a locally compact Hausdorff space $X$, the map $\text{Prim}(A) \to X$ is open if and only if $A$ corresponds to a continuous $C^*$-algebra bundle over $X$.

**Definition 2.10.** Let $A$ and $B$ be $C^*$-algebras over a topological space $X$. A $^*$-homomorphism $f: A \to B$ is $X$-equivariant if $f(A(U)) \subseteq B(U)$ for all $U \in \mathcal{O}(X)$.

For locally compact Hausdorff spaces, this is equivalent to $C_0(X)$-linearity by the following variant of [3, Proposition 5.4.7]:

**Proposition 2.11.** Let $A$ and $B$ be $C^*$-algebras over a locally compact Hausdorff space $X$, and let $f: A \to B$ be a $^*$-homomorphism. The following assertions are equivalent:

1. $f$ is $C_0(X)$-linear;
2. $f$ is $X$-equivariant, that is, $f(A(U)) \subseteq B(U)$ for all $U \in \mathcal{O}(X)$;
3. $f$ descends to the fibres, that is, $f(A(X \setminus \{x\})) \subseteq B(X \setminus \{x\})$ for all $x \in X$.

To understand the last condition, recall that the fibres of the $C^*$-algebra bundle associated to $A$ are $A_x := A/A(X \setminus \{x\})$. Condition (3) means that $f$ descends to maps $f_x: A_x \to B_x$ for all $x \in X$.

**Proof.** It is clear that (1) $\implies$ (2) $\implies$ (3). The equivalence (3) $\iff$ (1) is the assertion of [3, Proposition 5.4.7]. To check that (3) implies (1), take $h \in C_0(X)$ and $a \in A$. We get $f(h \cdot a) = h \cdot f(a)$ provided both sides have the same values at all $x \in X$ because the map $A \to \prod_{x \in X} A_x$ is injective. Now (3) implies $f(h \cdot a)_x = h(x) \cdot f(a)_x$ because $(h - h(x)) \cdot a \in A(X \setminus \{x\})$. \qed

**Definition 2.12.** Let $\mathcal{C}^*\text{alg}(X)$ be the category whose objects are the $C^*$-algebras over $X$ and whose morphisms are the $X$-equivariant $^*$-homomorphisms. We write $\text{Hom}_X(A, B)$ for this set of morphisms.

Proposition 2.11 yields an isomorphism of categories $\mathcal{C}^*\text{alg}(C_0(X)) \cong \mathcal{C}^*\text{alg}(X)$. In this sense, our theory for general spaces extends the more familiar theory of $C_0(X)$-$C^*$-algebras.

### 2.3. Locally closed subsets and subquotients.

**Definition 2.13.** A subset $C$ of a topological space $X$ is called locally closed if it is the intersection of an open and a closed subset or, equivalently, of the form $C = U \setminus V$ with $U, V \in \mathcal{O}(X)$; we can also assume $V \subseteq U$ here. We let $\text{LC}(X)$ be the set of locally closed subsets of $X$. 
It is easy to see that a subset is locally closed if and only if it is relatively open in its closure. Being locally closed is inherited by finite intersections, but not by unions or complements.

**Definition 2.14.** Let $X$ be a topological space and let $(A, \psi)$ be a $C^*$-algebra over $X$. Write $C \in \mathcal{LC}(X)$ as $C = U \setminus V$ for open subsets $U, V \subseteq X$ with $V \subseteq U$. We define

$$A(C) := A(U)/A(V).$$

**Lemma 2.15.** The subquotient $A(C)$ does not depend on $U$ and $V$ above.

**Proof.** Let $U_1, V_1, U_2, V_2 \in \mathcal{O}(X)$ satisfy $V_1 \subseteq U_1, V_2 \subseteq U_2$, and $U_1 \setminus V_1 = U_2 \setminus V_2$. Then $V_1 \cup U_2 = U_1 \cup U_2 = U_1 \cup V_2$ and $V_1 \cap U_2 = V_1 \cap V_2 = U_1 \cap V_2$. Since $U \mapsto A(U)$ preserves unions, this implies

$$A(U_2) + A(V_1) = A(U_1) + A(V_2).$$

We divide this equation by $A(V_1 \cup V_2) = A(V_1) + A(V_2)$. This yields

$$\frac{A(U_2) + A(V_1)}{A(V_1 \cup V_2)} \cong \frac{A(U_2)}{A(U_2 \cap A(V_1 \cup V_2))} + \frac{A(U_2)}{A(V_1 \cup V_2)} = \frac{A(U_2)}{A(V_1 \cup V_2)}$$

on the left hand side and, similarly, $A(U_1)/A(V_1)$ on the right hand side. Hence $A(U_1)/A(V_1) \cong A(U_2)/A(V_2)$ as desired. \qed

Now assume that $X = \text{Prim}(A)$ and $\psi = \text{id}_{\text{Prim}(A)}$. Lemma 2.15 associates a subquotient $A(C)$ of $A$ to each locally closed subset of $\text{Prim}(A)$. Equation 2.6 shows that any subquotient of $A$ arises in this fashion; here subquotient means a quotient of one ideal in $A$ by another ideal in $A$. Open subsets of $X$ correspond to ideals, closed subsets to quotients of $A$. For any $C \in \mathcal{LC} \text{(Prim } A\text{)},$ there is a canonical homeomorphism $\text{Prim}(A(C)) \cong C$. This is well-known if $C$ is open or closed, and the general case reduces to these special cases.

**Example 2.16.** If $\text{Prim}(A)$ is a finite topological $T_0$-space, then any singleton $\{p\}$ in $\text{Prim}(A)$ is locally closed; this holds more generally for the Alexandrov $T_0$-spaces introduced in §2.7 and is easy to prove using the description of closed subsets in terms of the specialisation preorder.

Since $\text{Prim}(A(C)) \cong C$, the subquotients $A_p := A(\{p\})$ for $p \in \text{Prim}(A)$ are precisely the simple subquotients of $A$.

**Example 2.17.** Consider the interval $[0, 1]$ with the topology where the non-empty closed subsets are the closed intervals $[a, 1]$ for all $a \in [0, 1]$. A non-empty subset is locally closed if and only if it is either of the form $[a, 1]$ or $[a, b)$ for $a, b \in [0, 1]$ with $a < b$. In this space, singletons are not locally closed. Hence a $C^*$-algebra with this primitive ideal space has no simple subquotients.

### 2.4. Functoriality and tensor products.

**Definition 2.18.** Let $X$ and $Y$ be topological spaces. A continuous map $f : X \to Y$ induces a functor

$$f_* : \mathcal{C}^\text{alg}(X) \to \mathcal{C}^\text{alg}(Y), \quad (A, \psi) \mapsto (A, f \circ \psi).$$

Thus $X \mapsto \mathcal{C}^\text{alg}(X)$ is a functor from the category of topological spaces to the category of categories (up to the usual issues with sets and classes).

Since $(f \circ \psi)^{-1} = \psi^{-1} \circ f^{-1}$, we have

$$(f_* A)(C) = A(f^{-1}(C))$$

for all $C \in \mathcal{LC}(Y)$.

If $f : X \to Y$ is the embedding of a subset with the subspace topology, we also write

$$i^Y_X := f_* : \mathcal{C}^\text{alg}(X) \to \mathcal{C}^\text{alg}(Y)$$
and call this the extension functor from \( X \) to \( Y \). We have \((i_X^Y A)(C) = A(C \cap X)\) for all \( C \in \mathcal{L}C(Y)\).

**Definition 2.19.** Let \( X \) be a topological space and let \( Y \) be a locally closed subset of \( X \), equipped with the subspace topology. Let \((A, \psi)\) be a \( C^*\)-algebra over \( X \). Its restriction to \( Y \) is a \( C^*\)-algebra \( A|_Y \) over \( Y \), consisting of the \( C^*\)-algebra \( A(Y) \) defined as in Definition 2.14, equipped with the canonical map

\[
\text{Prim } A(Y) \xrightarrow{\cong} \psi^{-1}(Y) \xrightarrow{\cong} Y.
\]

Thus \( A|_Y(C) = A(C) \) for \( C \in \mathcal{L}C(Y) \subseteq \mathcal{L}C(X) \).

It is clear that the restriction to \( Y \) provides a functor

\[
r_X^Y : \mathcal{C}^*\text{alg}(X) \to \mathcal{C}^*\text{alg}(Y)
\]

that satisfies \( r^X \circ r^Y_X = r^X_Y \) if \( Z \subseteq Y \subseteq X \) and \( r^X_X = \text{id} \).

If \( Y \) and \( X \) are Hausdorff and locally compact, then a continuous map \( f : Y \to X \)
also induces a pull-back functor

\[
f^* : \mathcal{C}^*\text{alg}(X) \cong \mathcal{C}^*\text{alg}(C_0(X)) \to \mathcal{C}^*\text{alg}(C_0(Y)) \cong \mathcal{C}^*\text{alg}(Y),
\]

\[
A \mapsto C_0(Y) \otimes_{C_0(X)} A.
\]

For the constant map \( Y \to \ast \), this functor \( \mathcal{C}^*\text{alg} \to \mathcal{C}^*\text{alg}(Y) \) maps a \( C^*\)-algebra \( A \)
to \( f^*(A) := C_0(Y, A) \) with the obvious \( C_0(Y)\)-\( C^*\)-algebra structure. This functor has
no analogue for a non-Hausdorff space \( Y \). Therefore, a continuous map \( f : Y \to X \)
need not induce a functor \( f^* : \mathcal{C}^*\text{alg}(X) \to \mathcal{C}^*\text{alg}(Y) \).

For embeddings of locally closed subsets, the functor \( r^Y_X \) plays the role of \( f^* \).

**Lemma 2.20.** Let \( X \) be a topological space and let \( Y \subseteq X \).

(a) If \( Y \) is open, then there are natural isomorphisms

\[
\text{Hom}_X(i_X^Y(A), B) \cong \text{Hom}_Y(A, r^Y_X(B))
\]

if \( A \) and \( B \) are \( C^*\)-algebras over \( Y \) and \( X \), respectively.

In other words, \( i_X^Y \) is left adjoint to \( r^Y_X \).

(b) If \( Y \) is closed, then there are natural isomorphisms

\[
\text{Hom}_Y(r^Y_X(A), B) \cong \text{Hom}_X(A, i_X^Y(B))
\]

if \( A \) and \( B \) are \( C^*\)-algebras over \( X \) and \( Y \), respectively.

In other words, \( i_X^Y \) is right adjoint to \( r^Y_X \).

(c) For any locally closed subset \( Y \subseteq X \), we have \( r^Y_X \circ i_X^Y(A) = A \) for all \( C^*\)-algebras \( A \) over \( Y \).

**Proof.** We first prove (a). We have \( i_X^Y(A)(U) = A(U \cap Y) \) for all \( U \in \mathcal{O}(X) \), and this is an ideal in \( A(U) \). A morphism \( \varphi : i_X^Y(A) \to B \) is equivalent to a \( \ast \)-homomorphism \( \varphi : A(Y) \to B(X) \) that maps \( A(U \cap Y) \to B(U) \) for all \( U \in \mathcal{O}(X) \). This holds for all \( U \in \mathcal{O}(X) \) once it holds for \( U \in \mathcal{O}(Y) \subseteq \mathcal{O}(X) \). Hence \( \varphi \) is equivalent to a \( \ast \)-homomorphism \( \varphi' : A(Y) \to B(Y) \) that maps \( A(U) \to B(U) \) for all \( U \in \mathcal{O}(Y) \). The latter is nothing but a morphism \( A \to r^Y_X(B) \). This proves (a).

Now we turn to (b). Again, we have \( i_X^Y(B)(U) = B(U \cap Y) \) for all \( U \in \mathcal{O}(X) \),
but now this is a quotient of \( B(U) \). A morphism \( \varphi : A \to i_X^Y(B) \) is equivalent to a \( \ast \)-homomorphism \( \varphi : A(X) \to B(Y) \) that maps \( A(U \cap Y) \to B(U) \) for all \( U \in \mathcal{O}(X) \). Hence \( A(X \setminus Y) \) is mapped to \( B(\emptyset) = 0 \), so that \( \varphi \) descends to a map \( \varphi' \) from \( A/A(X \setminus Y) \cong A(Y) \) to \( B(Y) \) that maps \( A(U \cap Y) \to B(U) \) for all \( U \in \mathcal{O}(X) \). The latter is equivalent to a morphism \( r^Y_X(A) \to B \) as desired. This finishes the proof of (b).

Assertion (c) is trivial. \( \square \)
Example 2.21. For each \( x \in X \), we get a map \( i_x = i_x^X : * \cong \{ x \} \subseteq X \) from the one-point space to \( X \). The resulting functor \( \mathcal{C}^{\ast \mathrm{alg}} \to \mathcal{C}^{\ast \mathrm{alg}}(X) \) maps a \( C^\ast \)-algebra \( A \) to the \( C^\ast \)-algebra \( i_x(A) = (A, x) \) over \( X \), where \( x \) also denotes the constant map \( x : \text{Prim}(A) \to X \), \( p \mapsto x \) for all \( p \in \text{Prim}(A) \).

If \( C \in \mathbb{L} \mathcal{C}(X) \), then

\[
i_x(A)(C) = \begin{cases} A & \text{if } x \in C; \\ 0 & \text{otherwise.} \end{cases}
\]

The functor \( i_x \) plays an important role if \( X \) is finite. The generators of the bootstrap class are of the form \( i_x(C) \). Each \( C^\ast \)-algebra over \( X \) carries a canonical filtration whose subquotients are of the form \( i_x(A) \).

**Lemma 2.22.** Let \( X \) be a topological space and let \( x \in X \). Then

\[
\text{Hom}_X(A, i_x^X(B)) \cong \text{Hom}(A(\{x\}), B)
\]

for all \( A \in \mathcal{C}^{\ast \mathrm{alg}}, B \in \mathcal{C}^{\ast \mathrm{alg}} \), and

\[
\text{Hom}_X(i_x^X(A), B) \cong \text{Hom}\left( A, \bigcap_{U \in \mathcal{U}_x} B(U) \right).
\]

for all \( A \in \mathcal{C}^{\ast \mathrm{alg}}, B \in \mathcal{C}^{\ast \mathrm{alg}}(X) \), where \( \mathcal{U}_x \) denotes the open neighbourhood filter of \( x \) in \( X \). If \( x \) has a minimal open neighbourhood \( U_x \), then this becomes

\[
\text{Hom}_X(i_x^X(A), B) \cong \text{Hom}(A, B(U_x)).
\]

Recall that \( A \in \mathcal{C} \) means that \( A \) is an object of \( \mathcal{C} \).

**Proof.** Let \( C := \{ x \} \). Then any non-empty open subset \( V \subseteq C \) contains \( x \), so that \( i_x^C(B)(V) = B \). This implies \( \text{Hom}_C(A, i_x^C(B)) \cong \text{Hom}(A(C), B) \). Combining this with \( i_x^X = i_x^C \circ i_x^C \) and the adjointness relation in Lemma 2.20.(b) yields

\[
\text{Hom}_X(A, i_x^X(B)) \cong \text{Hom}_C(i_x^C(A), i_x^C(B)) \cong \text{Hom}(A(C), B).
\]

An \( X \)-equivariant \( * \)-homomorphism \( i_x^X A \to B \) restricts to a family of compatible maps \( A = (i_x^X A)(U) \to B(U) \) for all \( U \in \mathcal{U}_x \), so that we get a \( * \)-homomorphism from \( A \) to \( \bigcap_{U \in \mathcal{U}_x} B(U) \). Conversely, any such \( * \)-homomorphism \( A \to \bigcap_{U \in \mathcal{U}_x} B(U) \) provides an \( X \)-equivariant \( * \)-homomorphism \( i_x^X A \to B \). This yields the second assertion.

Let \( A \) and \( B \) be \( C^\ast \)-algebras and let \( A \otimes B \) be their minimal (or spatial) \( C^\ast \)-tensor product. Then there is a canonical continuous map

\[
\text{Prim}(A) \times \text{Prim}(B) \to \text{Prim}(A \otimes B),
\]

Therefore, if \( A \) and \( B \) are \( C^\ast \)-algebras over \( X \) and \( Y \), respectively, then \( A \otimes B \) is a \( C^\ast \)-algebra over \( X \times Y \). This defines a bifunctor

\[
\otimes : \mathcal{C}^{\ast \mathrm{alg}}(X) \times \mathcal{C}^{\ast \mathrm{alg}}(Y) \to \mathcal{C}^{\ast \mathrm{alg}}(X \times Y).
\]

We get \( (A \otimes B)(U \times V) = A(U) \otimes B(V) \) for all \( U \in \mathcal{O}(X), V \in \mathcal{O}(Y) \).

In particular, if \( Y = * \) is the one-point space, then we get endofunctors \( \cup \otimes B \) on \( \mathcal{C}^{\ast \mathrm{alg}}(X) \) for \( B \in \mathcal{C}^{\ast \mathrm{alg}} \) because \( X \times * \cong X \).

If \( X \) is a Hausdorff space, then the diagonal in \( X \times X \) is closed and we get an internal tensor product functor \( \otimes_X \) in \( \mathcal{C}^{\ast \mathrm{alg}}(X) \) by restricting the external tensor product in \( \mathcal{C}^{\ast \mathrm{alg}}(X \times X) \) to the diagonal. This operation has no analogue for general \( X \).
2.5. **Restriction to sober spaces.** A space is sober if and only if it can be recovered from its lattice of open subsets. Any topological space can be completed to a sober space with the same lattice of open subsets. Therefore, it usually suffices to study $C^*$-algebras over sober topological spaces.

**Definition 2.23.** A topological space is **sober** if each irreducible closed subset of $X$ is the closure $\{x\}$ of exactly one singleton of $X$. Here an **irreducible** closed subset of $X$ is a non-empty closed subset of $X$ which is not the union of two proper closed subsets of itself.

If $X$ is not sober, let $\hat{X}$ be the set of all irreducible closed subsets of $X$. There is a canonical map $\iota: X \to \hat{X}$ which sends a point $x \in X$ to its closure. If $S \subseteq X$ is closed, let $\hat{S} \subseteq \hat{X}$ be the set of all $A \in \hat{X}$ with $A \subseteq S$. The map $S \mapsto \hat{S}$ commutes with finite unions and arbitrary intersections; in particular, it maps $X$ itself to all of $\hat{X}$ and $\emptyset$ to $\emptyset = \emptyset$. Hence the subsets of $\hat{X}$ of the form $\hat{S}$ for closed subsets $S \subseteq X$ form the closed subsets of a topology on $\hat{X}$.

The map $\iota$ induces a bijection between the families of closed subsets of $X$ and $\hat{X}$. Hence $\iota$ is continuous, closed, and open, and it induces a bijection $\mathcal{O}(X) \to \mathcal{O}(\hat{X})$. It also follows that $\hat{X}$ is a sober space because $X$ and $\hat{X}$ have the same irreducible closed subsets.

Since the morphisms in $\mathcal{C}^*\text{-algs}(X)$ only use $\mathcal{O}(X)$, the functor

$$\iota_*: \mathcal{C}^*\text{-algs}(X) \to \mathcal{C}^*\text{-algs}(\hat{X})$$

is fully faithful. Therefore, we do not lose much if we assume our topological spaces to be sober.

The following example shows a pathology that can occur if the separation axiom $T_0$ fails:

**Example 2.24.** Let $X$ carry the chaotic topology $\mathcal{O}(X) = \{\emptyset, X\}$. Then $\hat{X} = \ast$ is the space with one point. By definition, an action of $X$ on a $C^*$-algebra $A$ is a map $\text{Prim}(A) \to X$. But for a $\ast$-homomorphism $A \to B$ between two $C^*$-algebras over $X$, the $X$-equivariance condition imposes no restriction. Hence all maps $\text{Prim}(A) \to X$ yield isomorphic objects of $\mathcal{C}^*\text{-algs}(X)$.

**Lemma 2.25.** If $X$ is a sober topological space, then there is a bijective correspondence between continuous maps $\text{Prim}(A) \to X$ and maps $\mathcal{O}(X) \to \mathcal{I}(A)$ that commute with arbitrary suprema and finite infima; it sends a continuous map $\psi: \text{Prim}(A) \to X$ to the map

$$\psi^*: \mathcal{O}(X) \to \mathcal{O}(\text{Prim}(A)) = \mathcal{I}(A).$$

**Proof.** We have already seen that a continuous map $\psi: \text{Prim}(A) \to X$ generates a map $\psi^*$ with the required properties for any space $X$.

Conversely, let $\psi^*: \mathcal{O}(X) \to \mathcal{I}(A)$ be a map that preserves arbitrary unions and finite intersections. Given $p \in \text{Prim}(A)$, let $U_p$ be the union of all $U \in \mathcal{O}(X)$ with $p \notin \psi^*(U)$. Then $p \notin \psi^*(U_p)$ because $\psi^*$ preserves unions, and $U_p$ is the maximal open subset with this property. The complement $A_p := X \setminus U_p$ is the minimal closed subset with $p \notin \psi^*(X \setminus A_p)$. This subset is non-empty because $\psi^*(X) = \text{Prim}(A)$ contains $p$, and irreducible because $\psi^*$ preserves finite intersections.

Since $X$ is sober, there is a unique $\psi(p) \in X$ with $A_p = \{\psi(p)\}$. This defines a map $\psi: \text{Prim}(A) \to X$. If $U \subseteq X$ is open, then $\psi(p) \notin U$ if and only if $A_p \cap U = \emptyset$, if and only if $p \notin \psi^*(U)$. Hence $\psi^*(U) = \psi^{-1}(U)$. This shows that $\psi$ is continuous and generates $\psi^*$. Thus the map $\psi \to \psi^*$ is surjective.

Since sober spaces are $T_0$, two different continuous maps $\psi_1, \psi_2: \text{Prim}(A) \to X$ generate different maps $\psi_1^*, \psi_2^*: \mathcal{O}(X) \to \mathcal{I}(A)$. Hence the map $\psi \to \psi^*$ is also injective. \qed
2.6. Some very easy examples. Here we describe the categories of $C^*$-algebras over the three sober topological spaces with at most two points.

Example 2.26. If $X$ is a single point, then $\mathcal{C}^\ast\text{alg}(X)$ is isomorphic to the category of $C^*$-algebras (without any extra structure).

Up to homeomorphism, there are two sober topological spaces with two points. The first one is the discrete space.

Example 2.27. The category of $C^*$-algebras over the discrete two-point space is equivalent to the product category $\mathcal{C}^\ast\text{alg} \times \mathcal{C}^\ast\text{alg}$ of pairs of $C^*$-algebras.

More generally, if $X = X_1 \sqcup X_2$ is a disjoint union of two subspaces, then

\[ \mathcal{C}^\ast\text{alg}(X) \simeq \mathcal{C}^\ast\text{alg}(X_1) \times \mathcal{C}^\ast\text{alg}(X_2). \]

Thus it usually suffices to study connected spaces.

Example 2.29. Another sober topological space with two points is $X = \{1, 2\}$ with $\mathcal{O}(X) = \{\emptyset, \{1\}, \{1, 2\}\}$. A $C^*$-algebra over this space comes with a single distinguished ideal $A(1)/A$, which is arbitrary. Thus we get the category of pairs $(I, A)$ where $I$ is an ideal in $A$. We may associate to this data the $C^*$-algebra extension $I \hookrightarrow A \twoheadrightarrow A/I$. In fact, the morphisms in $\text{Hom}_X(A, B)$ are the morphisms of extensions

\[
\begin{array}{ccccccc}
A(1) & \xrightarrow{\sim} & A & \longrightarrow & A/A(1) \\
\downarrow & & \downarrow & & \downarrow \\
B(1) & \xrightarrow{\sim} & B & \longrightarrow & B/B(1).
\end{array}
\]

Thus $\mathcal{C}^\ast\text{alg}(X)$ is equivalent to the category of $C^*$-algebra extensions. This example is also studied in [9].

2.7. Topologies and partial orders. Certain non-Hausdorff spaces are closely related to partially ordered sets. In particular, there is a bijection between sober topologies and partial orders on a finite set. Here we recall the relevant constructions.

Definition 2.30. Let $X$ be a topological space. The specialisation preorder $\preceq$ on $X$ is defined by $x \preceq y$ if the closure of $\{x\}$ is contained in the closure of $\{y\}$ or, equivalently, if $y$ is contained in all open subsets of $X$ that contain $x$. Two points $x$ and $y$ are called topologically indistinguishable if $x \preceq y$ and $y \preceq x$, that is, the closures of $\{x\}$ and $\{y\}$ are equal.

The separation axiom $T_0$ means that topologically indistinguishable points are equal. Since this is automatic for sober spaces, $\preceq$ is a partial order on $X$ in all cases we need. As usual, we write $x \prec y$ if $x \preceq y$ and $x \neq y$, and $x \succeq y$ and $x \succ y$ are equivalent to $y \preceq x$ and $y \prec x$, respectively.

The separation axiom $T_1$ requires points to be closed. This is equivalent to the partial order $\preceq$ being trivial, that is, $x \preceq y$ if and only if $x = y$. Thus our partial order is only meaningful for highly non-separated spaces.

The following notion goes back to an article by Paul Alexandrov from 1937 ([1]); see also [2] for a more recent reference, or the English Wikipedia entry on the Alexandrov topology.

Definition 2.31. Let $(X, \preceq)$ be a preordered set. A subset $S \subseteq X$ is called Alexandrov-open if $S \ni x \preceq y$ implies $y \in S$. The Alexandrov-open subsets form a topology on $X$ called the Alexandrov topology.
A subset of $X$ is closed in the Alexandrov topology if and only if $S \ni x$ and $x \geq y$ imply $S \ni y$. It is locally closed if and only if it is convex, that is, $x \leq y \leq z$ and $x,z \in S$ imply $y \in S$. In particular, singletons are locally closed (compare Example 2.16).

The specialisation preorder for the Alexandrov topology is the given preorder. Moreover, a map $(X, \leq) \rightarrow (Y, \leq)$ is continuous for the Alexandrov topology if and only if it is monotone. Thus we have identified the category of preordered sets with monotone maps with a full subcategory of the category of topological spaces.

It also follows that if a topological space carries an Alexandrov topology for some preorder, then this preorder must be the specialisation preorder. In this case, we call the space an Alexandrov space or a finitely generated space. The following lemma provides some equivalent descriptions of Alexandrov spaces; the last two explain in what sense these spaces are finitely generated.

**Lemma 2.32.** Let $X$ be a topological space. The following are equivalent:

- $X$ is an Alexandrov space;
- an arbitrary intersection of open subsets of $X$ is open;
- an arbitrary union of closed subsets of $X$ is closed;
- every point of $X$ has a smallest neighbourhood;
- a point $x$ lies within the closure of a subset $S$ of $X$ if and only if $x \in \overline{\{y\}}$ for some $y \in S$;
- $X$ is the inductive limit of the inductive system of its finite subspaces.

**Corollary 2.33.** Any finite topological space is an Alexandrov space. Thus the construction of Alexandrov topologies and specialisation preorders provides a bijection between preorders and topologies on a finite set.

**Definition 2.34.** Let $X$ be an Alexandrov space. We denote the minimal open neighbourhood of $x \in X$ by $U_x \in \mathcal{O}(X)$.

We have

$$x \in U_y \iff U_x \subseteq U_y \iff y \in \overline{x} \iff \overline{\{y\}} \subseteq \overline{x} \iff y \preceq x.$$  

If $X$ is an Alexandrov space, then we can simplify the data for a $C^*$-algebra over $X$ as follows:

**Lemma 2.35.** A $C^*$-algebra over a sober Alexandrov space $X$ is determined uniquely by a $C^*$-algebra $A$ together with ideals $A(U_x) \subseteq A$ for all $x \in X$, subject to the two conditions $\sum_{x \in X} A(U_x) = A$ and

$$(2.36) \quad A(U_x) \cap A(U_y) = \sum_{z \in U_x \cap U_y} A(U_z) \quad \text{for all } x,y \in X.$$  

**Proof.** A map $\mathcal{O}(X) \rightarrow \mathbb{I}(A)$ that preserves suprema and maps $U_x$ to $A(U_x)$ for all $x \in X$ must map $U = \bigvee_{x \in U} U_x$ to $\bigvee_{x \in U} A(U_x) = \sum_{x \in U} A(U_x)$. The map so defined preserves suprema by construction. The two hypotheses of the lemma ensure $A(X) = A$ and $A(U_x \cap U_y) = A(U_x) \cap A(U_y)$ for all $x,y \in X$. Hence they are necessary for preservation of finite infima.

Since the lattice $\mathbb{I}(A) \cong \mathcal{O}({\text{Prim}} \ A)$ is distributive, (2.36) implies

$$A(U) \wedge A(V) = \bigvee_{x \in U} A(U_x) \wedge \bigvee_{y \in V} A(V_y) = \bigvee_{(x,y) \in U \times V} A(U_x) \wedge A(V_y) = \bigvee_{(x,y) \in U \times V} A(U_x \cap V_y) = A(U \cap V);$$

the last step uses that $U \mapsto A(U)$ commutes with suprema. We clearly have $A(\emptyset) = \{0\}$ as well, so that $U \mapsto A(U)$ preserves arbitrary finite intersections.
Therefore, our map $\mathcal{O}(X) \to \mathbb{I}(A)$ satisfies the conditions in Lemma 2.25 and hence comes from a continuous map $\operatorname{Prim} A \to X$.  

Of course, a $\ast$-homomorphism $A \to B$ between two $C^*$-algebras over $X$ is $X$-equivariant if and only if it maps $A(U_x) \to B(U_x)$ for all $x \in X$.

By the way, equation (2.36) implies $A(U_x) \subseteq A(U_y)$ if $U_x \subseteq U_y$, that is, if $x \succ y$. Thus the map $x \mapsto A(U_x) \subseteq A(U_y)$ is order-reversing. It sometimes happens that $U_x \cap U_y = U_z$ for some $x, y, z \in X$. In such a case, we can drop the ideal $A(U_z)$ from the description of a $C^*$-algebra over $X$ and replace the condition (2.36) for $x, y$ by $A(U_w) \subseteq A(U_x) \cap A(U_y)$ for all $w \in U_x \cap U_y$.

2.8. Some more examples. A useful way to represent finite partially ordered sets and hence finite sober topological spaces is via finite directed acyclic graphs.

To a partial order $\preceq$ on $X$, we associate the finite directed acyclic graph with vertex set $X$ and with an arrow $x \to y$ if and only if $x \prec y$ and there is no $z \in X$ with $x \prec z \prec y$. We can recover the partial order from this graph by letting $x \preceq y$ if and only if the graph contains a directed path $x \leftarrow x_1 \leftarrow \cdots \leftarrow x_n \leftarrow y$.

We have reversed arrows here because for an Alexandrov space, an arrow $x \to y$ implies $A(U_x) \subseteq A(U_y)$. Furthermore, $x \in U_y$ if and only if there is a directed path from $x$ to $y$. Thus we can read the meaning of the relations (2.36) from the graph.

Example 2.37. Let $(X, \leq)$ be a set with a total order, such as $\{1, \ldots, n\}$ with the usual order. The corresponding graph is

$$
1 \to 2 \to 3 \to \cdots \to n.
$$

In this case, (2.36) is equivalent to monotonicity of the map $x \mapsto A(U_x)$. As a consequence, a $C^*$-algebra $A$ over $X$ is nothing but a $C^*$-algebra $A$ together with a monotone map $X \to \mathbb{I}(A), x \mapsto A(U_x)$, such that $\bigvee_{x \in X} A(U_x) = A$. In the finite case above, the latter condition just means $A(U_n) = A$, so that we can drop this ideal. Thus we get $C^*$-algebras with an increasing chain of $n-1$ ideals $I_1 \lhd I_2 \lhd \cdots \lhd I_{n-1} \lhd A$. This situation is studied in detail in [13].

Using that any finite topological space is an Alexandrov space, we can easily list all homeomorphism classes of finite topological spaces with, say, three or four elements. We only consider sober spaces here, and we assume connectedness to further reduce the number of cases. Under these assumptions, we get the cases listed in Figure 1. In each case, we can use Lemma 2.35 to describe $C^*$-algebras over $X$ as $C^*$-algebras equipped with three or four ideals $A(U_x)$ for $x \in X$, subject to some conditions; these conditions often make some of the ideals redundant. The first and fourth case are already contained in Example 2.37.

Example 2.38. The second graph in Figure 1 describes $C^*$-algebras with three ideals $A(U_j)$, $j = 1, 2, 3$, subject to the conditions $A(U_2) \cap A(U_3) = A(U_1)$ and $A(U_2) + A(U_3) = A$. This is equivalent to prescribing only two ideals $A(U_2)$ and $A(U_3) \subseteq A(U_1)$ subject to the single condition $A(U_2) + A(U_3) = A$.

Example 2.39. Similarly, the third graph in Figure 1 describes $C^*$-algebras with two distinguished ideals $A(U_1)$ and $A(U_2)$ subject to the condition $A(U_1) \cap A(U_2) = \{0\}$; here $U_3 = X$ implies $A(U_3) = A$.

Example 2.40. The ninth case above is more complicated. We label our points by $1, 2, 3, 4$ such that $1 \to 3 \leftarrow 2 \to 4$. Here we have a $C^*$-algebra $A$ with four ideals $I_j := A(U_j)$ for $j = 1, 2, 3, 4$, subject to the conditions

$$
I_1 \subseteq I_3, \quad I_1 \cap I_4 = \{0\}, \quad I_2 = I_3 \cap I_4, \quad I_3 + I_4 = A.
$$

Thus the ideal $I_2$ is redundant, and we are left with three ideals $I_1, I_3, I_4$ subject to the conditions $I_1 \subseteq I_3, I_1 \cap I_4 = \{0\}$, and $I_3 + I_4 = A$. 

2.9. **How to treat discontinuous bundles.** The construction of $X$-equivariant Kasparov theory in \cite{3,7} works for any map $\psi^* : \mathcal{O}(X) \to \mathcal{I}(A)$, we do not need the conditions in Lemma 2.25. Here we show how to reduce this more general situation to the case considered above: discontinuous actions of $\mathcal{O}(X)$ as in \cite{3,7} are equivalent to continuous actions of another space $Y$ that contains $X$ as a subspace.

The category $\mathcal{C}^*\text{alg}(Y)$ contains $\mathcal{C}^*\text{alg}(X)$ as a full subcategory, and a similar statement holds for the associated Kasparov categories. As a result, allowing general maps $\psi^*$ merely amounts to replacing the space $X$ by the larger space $Y$. For $C^*$-algebras that really live over the subspace $X$, the extension to $Y$ significantly complicates the computation of the Kasparov groups. This is why we always require $\psi^*$ to satisfy the conditions in Lemma 2.25 which ensure that it comes from a continuous map $\text{Prim}(A) \to X$.

**Example 2.41.** Let $X = \{1, 2\}$ with the discrete topology. A monotone map $\psi^* : \mathcal{O}(X) \to A$ with $\psi^*(\emptyset) = \{0\}$ and $\psi^*(X) = A$ as considered in \cite{3,7} is equivalent to specifying two arbitrary ideals $A(1)$ and $A(2)$. This automatically generates the ideals $A(1) \cap A(2)$ and $A(1) \cup A(2)$. We can encode these four ideals in an action of a topological space $Y$ with four points $\{1 \cap 2, 1, 2, 3\}$ and open subsets

$$\emptyset, \quad \{1 \cap 2\}, \quad \{1 \cap 2, 1\}, \quad \{1 \cap 2, 2\}, \quad \{1 \cap 2, 1, 2\}, \quad \{1 \cap 2, 1, 2, 3\}.$$ 

The corresponding graph is the seventh one in Figure 1. The map $\psi^*$ maps these open subsets to the ideals

$$\{0\}, \quad A(1) \cap A(2), \quad A(1), \quad A(2), \quad A(1) \cup A(2), \quad A,$$

respectively. This defines a complete lattice morphism $\mathcal{O}(Y) \to \mathcal{I}(A)$, and any complete lattice morphism is of this form for two ideals $A(1)$ and $A(2)$. Thus an action of $\mathcal{O}\{1, 2\}$ in the generalised sense considered in \cite{3,7} is equivalent to an action of $Y$ in our sense.

Any $X$-equivariant $^*$-homomorphism $A \to B$ between two such discontinuous $C^*$-algebras over $X$ will also preserve the ideals $A(1) \cap A(2)$ and $A(1) \cup A(2)$. Hence it is $Y$-equivariant as well. Therefore, the above construction provides an equivalence.
We describe the topology on \( \mathcal{O}(X) \) and the category of \( C^* \)-algebras with an action of \( \mathcal{O}(X) \) in the sense of \[3,7\].

Whereas the computation of

\[
\text{KK}_*(X; A, B) \cong \text{KK}_*(A(1), B(1)) \times \text{KK}_*(A(2), B(2))
\]

for two \( C^* \)-algebras \( A \) and \( B \) over \( X \) is trivial, the corresponding problem for \( C^* \)-algebras over \( Y \) is an interesting problem: this is one of the small examples where filtrated K-theory does not suffice for classification.

This simple example generalises as follows. Let \( f: \mathcal{O}(X) \to \mathcal{I}(A) \) be an arbitrary map. Let \( Y := \mathcal{P}(\mathcal{O}(X)) \) be the power set of \( \mathcal{O}(X) \), partially ordered by inclusion. We describe the topology on \( Y \) below. We embed the original space \( X \) into \( Y \) by mapping \( x \in X \) to its open neighbourhood filter:

\[
\mathcal{U}: X \to Y, \quad x \mapsto \{ U \in \mathcal{O}(X) \mid x \in U \}.
\]

We define a map

\[
\psi: \text{Prim}(A) \to Y, \quad p \mapsto \{ U \in \mathcal{O}(X) \mid p \in f(U) \}.
\]

For \( y \in Y \), let \( Y_{2y} := \{ x \in Y \mid x \supseteq y \} \). For a singleton \( \{ U \} \) with \( U \in \mathcal{O}(X) \), we easily compute

\[
\psi^{-1}(Y_{2y}) = f(U) \in \mathcal{I}(A) \cong \mathcal{O}(\text{Prim } A).
\]

Moreover, \( Y_{2y} \cap Y_{2z} = Y_{2y} \cap Y_{2z} \), so that we get

\[
\psi^{-1}(Y_{2(U_1, \ldots, U_n)}) = f(U_1) \cap \cdots \cap f(U_n).
\]

A similar argument shows that

\[
\mathcal{U}^{-1}(Y_{2(U_1, \ldots, U_n)}) = \mathcal{U}^{-1}(Y_{2U_1}) \cap \cdots \cap \mathcal{U}^{-1}(Y_{2U_n}) = U_1 \cap \cdots \cap U_n.
\]

We equip \( Y \) with the topology that has the sets \( Y_{2F} \) for finite subsets \( F \) of \( \mathcal{O}(X) \) as a basis. It is clear from the above computations that this makes the maps \( \psi \) and \( \mathcal{U} \) continuous; even more, the subspace topology on the range of \( \mathcal{U} \) is the given topology on \( X \).

As a consequence, any map \( f: \mathcal{O}(X) \to \mathcal{I}(A) \) turns \( A \) into a \( C^* \)-algebra over the space \( Y \supseteq X \). Conversely, given a \( C^* \)-algebra over \( Y \), we define \( f: \mathcal{O}(X) \to \mathcal{I}(A) \) by \( f(U) := \psi^{-1}(Y_{2U}) \). This construction is inverse to the one above. Furthermore, a \( * \)-homomorphism \( A \to B \) that maps \( f_A(U) \) to \( f_B(U) \) for all \( U \in \mathcal{O}(X) \) also maps \( \psi_A(U) \to \psi_B(U) \) for all \( U \in \mathcal{O}(Y) \). We can sum this up as follows:

**Theorem 2.42.** The category of \( C^* \)-algebras equipped with a map \( f: \mathcal{O}(X) \to \mathcal{I}(A) \) is isomorphic to the category of \( C^* \)-algebras over \( Y \).

If \( f \) has some additional properties like monotonicity, or is a lattice morphism, then this limits the range of the map \( \psi \) above and thus allows us to replace \( Y \) by a smaller subset.

In \[3,7\], the map \( f \) is assumed to be monotone and to satisfy \( f(\emptyset) = \{ 0 \} \) and \( f(X) = A \). These assumptions are equivalent to

\[
U \in \psi(p), \quad U \subseteq V \implies V \in \psi(p)
\]

and \( \emptyset \notin \psi(p) \) and \( X \in \psi(p) \) for all \( p \in \text{Prim}(A) \). Hence actions of \( \mathcal{O}(X) \) that satisfy the extra conditions required in \[3,7\] are equivalent to \( C^* \)-algebras over

\[
Y' := \{ y \subseteq \mathcal{O}(X) \mid y \owns U \subseteq V \implies V \in y, \emptyset \notin y, X \in Y \},
\]

equipped with the subspace topology from \( Y \).
3. Bivariant $K$-theory for $C^*$-algebras over topological spaces

Let $X$ be a topological space. Eberhard Kirchberg and Alexander Bonkat define Kasparov groups $KK^*_X(A;B)$ for separable $C^*$-algebras $A$ and $B$ over $X$. They use a somewhat different setup and allow more singular actions of $\mathbb{G}(X)$. We have explained in [2.9] why it is reasonable to use Definition 2.3 instead.

If $X$ is Hausdorff and locally compact, $KK^*_X(A;B)$ agrees with Gennadi Kasparov’s theory $\mathcal{R}KK^*_X(A;B)$ defined in [6]. In this section, we recall the definition and some basic properties of the functor $KK^*_X(A;B)$ and the resulting category $\mathcal{R}\mathcal{K}(X)$, and we equip the latter with a triangulated category structure.

3.1. The definition. We assume from now on that the topology on $X$ has a countable basis, and we restrict attention to separable $C^*$-algebras in the following sense:

**Definition 3.1.** A $C^*$-algebra $(A,\psi)$ over $X$ is called separable if $A$ is a separable $C^*$-algebra. Let $\mathcal{C}^*\text{sep}(X) \subseteq \mathcal{C}^*\text{alg}(X)$ be the full subcategory of separable $C^*$-algebras over $X$.

To describe the cycles for $KK^*_X(A;B)$ we recall that the usual Kasparov cycles for $KK^*_X(A;B)$ are of the form $(\varphi, \mathcal{H}_B, F, \gamma)$ in the even case (for $KK^0$) and $(\varphi, \mathcal{H}_B, F)$ in the odd case (for $KK^1$), where

- $\mathcal{H}_B$ is a right Hilbert $B$-module;
- $\varphi : A \to \mathcal{B}(\mathcal{H}_B)$ is a $^*$-representation;
- $F \in \mathcal{B}(\mathcal{H}_B)$;
- $\varphi(a)(F^2 - 1), \varphi(a)(F - F^*)$, and $[\varphi(a), F]$ are compact for all $a \in A$;
- in the even case, $\gamma$ is a $\mathbb{Z}/2$-grading on $\mathcal{H}_B$—that is, $\gamma^2 = 1$ and $\gamma = \gamma^*$—that commutes with $\varphi(A)$ and anti-commutes with $F$.

**Definition 3.2.** Let $A$ and $B$ be $C^*$-algebras over $X$. A Kasparov cycle $(\varphi, \mathcal{H}_B, F, \gamma)$ or $(\varphi, \mathcal{H}_B, F)$ for $KK^*_X(A;B)$ is called $X$-equivariant if

$$\varphi(A(U)) \cdot \mathcal{H}_B \subseteq \mathcal{H}_B \cdot B(U) \quad \text{for all } U \in \mathcal{O}(X).$$

Let $KK^*_X(A;B)$ be the group of homotopy classes of such $X$-equivariant Kasparov cycles for $KK^*_X(A;B)$; a homotopy is an $X$-equivariant cycle for $KK^*_X(A;C([0,1]) \otimes B)$, where we view $C([0,1]) \otimes B$ as a $C^*$-algebra over $X$ in the usual way (compare §2.4).

The subset $\mathcal{H}_B \cdot B(U) \subseteq \mathcal{H}_B$ is a closed linear subspace by the Cohen–Hewitt Factorisation Theorem.

If $X$ is Hausdorff, then the extra condition in Definition 3.2 is equivalent to $C_0(X)$-linearity of $\varphi$ (compare Proposition 2.11). Thus the above definition of $KK^*_X(A;B)$ agrees with the more familiar definition of $\mathcal{R}KK^*_X(A;B)$ in [6].

If $X = \ast$ is the one-point space, the $X$-equivariance condition is empty and we get the plain Kasparov theory $KK^*_X(A;B)$.

The same arguments as usual show that $KK^*_X(A;B)$ remains unchanged if we strengthen the conditions for Kasparov cycles by requiring $F = F^*$ and $F^2 = 1$.

3.2. Basic properties. The Kasparov theory defined above has all the properties that we can expect from a bivariant $K$-theory.

1. The groups $KK^*_X(A;B)$ define a bifunctor from $\mathcal{C}^*\text{sep}(X)$ to the category of $\mathbb{Z}/2$-graded Abelian groups, contravariant in the first and covariant in the second variable.

2. There is a natural, associative Kasparov composition product

$$KK^*_i(X;A,B) \times KK^*_j(X;B,C) \to KK^*_{i+j}(X;A,C)$$
if $A, B, C$ are $C^*$-algebras over $X$.

Furthermore, there is a natural exterior product
\[ \text{KK}_*(X; A, B) \times \text{KK}_*(Y; C, D) \to \text{KK}_{*+j}(X \times Y; A \otimes C, B \otimes D) \]
for two spaces $X$ and $Y$ and $C^*$-algebras $A, B$ over $X$ and $C, D$ over $Y$.

The existence and properties of the Kasparov composition product and the exterior product are verified in a more general context in [3 §3.2].

**Definition 3.3.** Let $\mathfrak{K}(X)$ be the category whose objects are the separable $C^*$-algebras over $X$ and whose morphism sets are $\text{KK}_0(X; A, B)$.

1. The zero $C^*$-algebra acts as a zero object in $\mathfrak{K}(X)$, that is,
   \[ \text{KK}_*(X; \{0\}, A) = 0 = \text{KK}_*(X; A, \{0\}) \quad \text{for all } A \in \mathfrak{K}(X). \]
2. The $C_0$-direct sum of a sequence of $C^*$-algebras behaves like a coproduct, that is,
   \[ \text{KK}_*(X; \bigoplus_{n \in \mathbb{N}} A_n, B) \cong \prod_{n \in \mathbb{N}} \text{KK}_*(X; A_n, B) \]
   if $A_n, B \in \mathfrak{K}(X)$ for all $n \in \mathbb{N}$.
3. The direct sum $A \oplus B$ of two separable $C^*$-algebras $A$ and $B$ over $X$ is a direct product in $\mathfrak{K}(X)$, that is,
   \[ \text{KK}(X; D, A \oplus B) \cong \text{KK}(X; D, A) \oplus \text{KK}(X; D, B) \]
   for all $D \in \mathfrak{K}(X)$.

Properties (3)–(5) are summarised as follows:

**Proposition 3.4.** The category $\mathfrak{K}(X)$ is additive and has countable coproducts.

1. The exterior product is compatible with the Kasparov product, $C_0$-direct sums, and addition, that is, it defines a countably additive bifunctor
   \[ \otimes : \mathfrak{K}(X) \otimes \mathfrak{K}(Y) \to \mathfrak{K}(X \times Y). \]
   This operation is evidently associative.
2. In particular, $\mathfrak{K}(X)$ is tensored over $\mathfrak{K}(\ast) \cong \mathfrak{K}$, that is, $\otimes$ provides an associative bifunctor
   \[ \otimes : \mathfrak{K}(X) \otimes \mathfrak{K}(Y) \to \mathfrak{K}(X). \]
3. The bifunctor $(A, B) \mapsto \text{KK}_*(X; A, B)$ satisfies Bott periodicity, homotopy invariance, and $C^*$-stability in each variable. This follows from the corresponding properties of $\mathfrak{K}$ using the tensor structure in (7).
   For instance, the Bott periodicity isomorphism $C_0(\mathbb{R}^2) \cong \mathbb{C}$ in $\mathfrak{K}$ yields $A \otimes C_0(\mathbb{R}^2) \cong A \otimes \mathbb{C} \cong A$ in $\mathfrak{K}(X)$ for all $A \in \mathfrak{K}(X)$.
4. The functor $f_* : \mathfrak{C}^*\text{-alg}(X) \to \mathfrak{C}^*\text{-alg}(Y)$ for a continuous map $f : X \to Y$ descends to a functor
   \[ f_* : \mathfrak{K}(X) \to \mathfrak{K}(Y). \]
   In particular, this covers the extension functors $i_X^Y$ for a subspace $X \subseteq Y$.
5. The restriction functor $r_X^Y$ for $Y \in \mathcal{L}(X)$ also descends to a functor
   \[ r_X^Y : \mathfrak{K}(X) \to \mathfrak{K}(Y). \]

**Definition 3.5.** A diagram $I \to E \to Q$ in $\mathfrak{C}^*\text{-alg}(X)$ is an extension if, for all $U \in \mathcal{O}(X)$, the diagrams $i(U) \to E(U) \to Q(U)$ are extensions of $C^*$-algebras. We write $I \to E \to Q$ to denote extensions.

An extension is called split if it splits by an $X$-equivariant $\ast$-homomorphism.
An extension is called semi-split if there is a completely positive contractive section \(Q \to E\) that is \(X\)-equivariant, that is, it restricts to sections \(Q(U) \to E(U)\) for all \(U \in \mathfrak{O}(X)\).

If \(I \to E \to Q\) is an extension of \(C^*\)-algebras over \(X\), then we get \(C^*\)-algebra extensions \(I(Y) \to E(Y) \to Q(Y)\) for all locally closed subsets \(Y \subseteq X\). If the original extension is semi-split, so are the extensions \(I(Y) \to E(Y) \to Q(Y)\) for \(Y \in \mathbb{L} \mathcal{C}(X)\). Even more, the functor \(r^*_X\colon \mathcal{C}^*\text{alg}(X) \to \mathcal{C}^*\text{alg}(Y)\) maps extensions in \(\mathcal{C}^*\text{alg}(X)\) to extensions in \(\mathcal{C}^*\text{alg}(Y)\), and similarly for split and semi-split extensions.

**Theorem 3.6.** Let \(I \to E \to Q\) be a semi-split extension in \(\mathcal{C}^*\text{sep}(X)\) and let \(B\) be a separable \(C^*\)-algebra over \(X\). There are six-term exact sequences

\[
\begin{array}{ccc}
\text{KK}_0(X;Q,B) & \longrightarrow & \text{KK}_0(X;E,B) \\
\downarrow \quad \partial & & \downarrow \quad \partial \\
\text{KK}_1(X;I,B) & \longleftarrow & \text{KK}_1(X;E,B) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{KK}_0(X;B,I) & \longrightarrow & \text{KK}_0(X;B,E) \\
\downarrow \quad \partial & & \downarrow \quad \partial \\
\text{KK}_1(X;B,Q) & \longleftarrow & \text{KK}_1(X;B,E) \\
\end{array}
\]

where the horizontal maps in both exact sequences are induced by the given maps \(I \to E \to Q\), and the vertical maps are, up to signs, Kasparov products with the class of our semi-split extension in \(\text{KK}_1(Q,I)\).

**Proof.** This follows from [3, Satz 3.3.10] or from [3, Korollar 5.6.13]. \(\square\)

**Theorem 3.7.** The canonical functor \(\mathcal{C}^*\text{sep}(X) \to \mathfrak{R}(X)\) is the universal split-exact \(C^*\)-stable (homotopy) functor.

**Proof.** This follows from [3, Satz 3.5.10], compare also [3, Korollar 5.6.13]. The homotopy invariance assumption is redundant because, by a deep theorem of Nigel Higson, a split-exact, \(C^*\)-stable functor is automatically homotopy invariant. This holds for \(\mathcal{C}^*\text{sep}\) itself and is inherited by \(\mathcal{C}^*\text{sep}(X)\) because of the tensor product operation \(\mathcal{C}^*\text{sep}(X) \times \mathcal{C}^*\text{sep} \to \mathcal{C}^*\text{sep}(X)\). \(\square\)

3.3. **Triangulated category structure.** We are going to turn \(\mathfrak{R}(X)\) into a triangulated category as in [11]. We have already remarked that \(\mathfrak{R}(X)\) is additive. The suspension functor is \(\Sigma(A) := C_0(\mathbb{R}, A) = C_0(\mathbb{R}) \otimes A\). This functor is an automorphism (up to natural isomorphisms) by Bott periodicity.

The mapping cone triangle

\[
\begin{array}{ccc}
A & \overset{\varphi}{\longrightarrow} & B \\
\downarrow & & \downarrow \quad \Sigma \varphi \\
C_\varphi & & \\
\end{array}
\]

of a morphism \(\varphi: A \to B\) in \(\mathcal{C}^*\text{sep}(X)\) is defined as in [11] and is a diagram in \(\mathfrak{R}(X)\). The circled arrow from \(B\) to \(C_\varphi\) means a \(^*\)-homomorphism \(\Sigma(B) \to C_\varphi\). A triangle in \(\mathfrak{R}(X)\) is called exact if it is isomorphic in \(\mathfrak{R}(X)\) to the mapping cone triangle of some morphism in \(\mathcal{C}^*\text{sep}(X)\).

As in [11], there is an equivalent description of the exact triangles using semi-split extensions in \(\mathcal{C}^*\text{sep}(X)\). An extension

\[
I \overset{i}{\to} E \overset{p}{\to} Q
\]
gives rise to a commuting diagram

\[
\begin{array}{ccc}
\Sigma Q & \xrightarrow{i} & E \xrightarrow{p} Q \\
\downarrow & & \downarrow \\
\Sigma Q & \xrightarrow{C_p} & E \xrightarrow{p} Q.
\end{array}
\]

**Definition 3.10.** We call the extension *admissible* if the map \( I \to C_p \) is invertible in \( \mathcal{A}(X) \).

The proof of the Excision Theorem [11] shows that this is the case if the extension is semi-split; but there are more admissible extensions than semi-split extensions. If the extension is admissible, then there is a unique map \( \Sigma Q \to I \) so that the top row becomes isomorphic to the bottom row as a triangle in \( \mathcal{A}(X) \). Thus any admissible extension in \( C^*\text{sep}(X) \) yields an exact triangle \( \Sigma Q \to I \to E \to Q \), called *extension triangle*.

Conversely, if \( \varphi: A \to B \) is a morphism in \( C^*\text{sep}(X) \), then its mapping cone triangle is isomorphic in \( \mathcal{A}(X) \) to the extension triangle for the canonically semi-split extension \( C_\varphi \to Z_\varphi \to B \), where \( Z_\varphi \) denotes the mapping cylinder of \( \varphi \), which is homotopy equivalent to \( A \). The above arguments work exactly as in the case of undecorated Kasparov theory discussed in [11].

As a result, a triangle in \( \mathcal{A}(X) \) is isomorphic to a mapping cone triangle of some morphism in \( C^*\text{sep}(X) \) if and only if it is isomorphic to the extension triangle of some semi-split extension in \( C^*\text{sep}(X) \).

**Proposition 3.11.** The category \( \mathcal{A}(X) \) with the suspension automorphism and extension triangles specified above is a triangulated category.

**Proof.** Most of the axioms amount to well-known properties of mapping cones and mapping cylinders, which are proven by translating corresponding arguments for the stable homotopy category of spaces, see [11].

The only axiom that requires a new argument in our case is (TR1), which asserts that any morphism in \( \mathcal{A}(X) \) is part of some exact triangle. The argument in [11] uses the description of Kasparov theory via the universal algebras \( qA \) by Joachim Cuntz. This approach can be made to work in \( \mathcal{A}(X) \), but it is rather unflexible because the primitive ideal space of \( qA \) is hard to control.

The following argument, which is inspired by [3], also applies to interesting subcategories of \( \mathcal{A}(X) \) like the subcategory of nuclear \( C^* \)-algebras over \( X \), which is studied in [3]. Hence this is a triangulated category as well.

Let \( f \in KK_0(X; A, B) \). We identify \( KK_0(X; A, B) \cong KK_1(X; A, \Sigma B) \). Represent the image of \( f \) in \( KK_1(X; A, \Sigma B) \) by a cycle \( (\varphi, \mathcal{H}, F) \). Adding a degenerate cycle, if necessary, we can achieve that the map \( \Phi: A \ni a \mapsto F^*\varphi(a)F \mod \mathbb{K}(\mathcal{H}) \) is an injection from \( A \) into the Calkin algebra \( \mathbb{K}(\mathcal{H})/\mathbb{K}(\mathcal{H}) \) of \( \mathcal{H} \) and that \( \mathcal{H} \) is full, so that \( \mathbb{K}(\mathcal{H}) \) is KK(\( X \))-equivalent to \( \Sigma B \). The properties of a Kasparov cycle mean that \( \Phi \) is the Busby invariant of a semi-split extension \( \mathbb{K}(\mathcal{H}) \to E \to A \) of \( C^* \)-algebras over \( X \). The composition product of the map \( \Sigma A \to \mathbb{K}(\mathcal{H}) \) in the associated extension triangle and the canonical KK(\( X \))-equivalence \( \mathbb{K}(\mathcal{H}) \cong \Sigma B \) is the suspension of \( f \in KK_0(X; A, B) \). Hence we can embed \( f \) in an exact triangle. \( \square \)

### 3.4. Adjointness relations.

**Proposition 3.12.** Let \( X \) be a topological space and let \( Y \in L\mathcal{C}(X) \).

If \( Y \subseteq X \) is open, then we have natural isomorphisms

\[
KK_*(X; i_Y^X(A), B) \cong KK_*(Y; A, r_Y^X(B))
\]
for all $A \in \mathcal{R}(Y)$, $B \in \mathcal{R}(X)$, that is, $i^Y_X$ is left adjoint to $r^Y_X$ as functors $\mathcal{R}(Y) \leftrightarrow \mathcal{R}(X)$.

If $Y \subseteq X$ is closed, then we have natural isomorphisms

$$\mathrm{KK}_*(Y; r^Y_X(A), B) \cong \mathrm{KK}_*(X; A, i^Y_X(B))$$

for all $A \in \mathcal{R}(X)$, $B \in \mathcal{R}(Y)$, that is, $i^Y_X$ is right adjoint to $r^Y_X$ as functors $\mathcal{R}(Y) \leftrightarrow \mathcal{R}(X)$.

**Proof.** Since both $i^Y_X$ and $r^Y_X$ descend to functors between $\mathcal{R}(X)$ and $\mathcal{R}(Y)$, this follows from the adjointness on the level of $C^\ast$-$\text{alg}(X)$ and $C^\ast$-$\text{alg}(Y)$ in Lemma 2.20.

An analogous assertion for induction and restriction functors for group actions on $C^\ast$-algebras is proven in \cite{9} §3.2. The point of the argument is that an adjointness relation is equivalent to the existence of certain natural transformations called unit and counit of the adjunction, subject to some conditions (see \cite{9}). These natural transformations already exist on the level of $^\ast$-homomorphisms, which induce morphisms in $\mathcal{R}(X)$ or $\mathcal{R}(Y)$. The necessary relations for unit and counit of adjunction hold in $\mathcal{R}(\ldots)$ because they already hold in $C^\ast$-$\text{alg}(\ldots)$. The unit and counit are natural in $\mathcal{R}(\ldots)$ and not just in $C^\ast$-$\text{alg}(\ldots)$ because of the uniqueness part of the universal property of $\mathcal{R}$.

**Proposition 3.13.** Let $X$ be a topological space and let $x \in X$. Then

$$\mathrm{KK}_*(X; A, i_x(B)) \cong \mathrm{KK}_*(A([x]), B)$$

for all $A \in C^\ast$-$\text{sep}(X)$, $B \in C^\ast$-$\text{sep}$. That is, the functor $i_x : \mathcal{R} \rightarrow \mathcal{R}(X)$ is right adjoint to the functor $A \mapsto A([x])$. Moreover,

$$\mathrm{KK}_*(X; i_x(A), B) \cong \mathrm{KK}_*(A, \bigcap_{U \in \mathcal{U}_x} B(U))$$

for all $A \in C^\ast$-$\text{sep}$, $B \in C^\ast$-$\text{sep}(X)$, where $\mathcal{U}_x$ denotes the open neighbourhood filter of $x$ in $X$. That is, the functor $i_x : \mathcal{R} \rightarrow \mathcal{R}(X)$ is left adjoint to the functor $B \mapsto \bigcap_{U \in \mathcal{U}_x} B(U)$. If $x$ has a minimal open neighbourhood $U_x$, then

$$\mathrm{KK}_*(X; i_x(A), B) \cong \mathrm{KK}_*(A, B(U_x))$$

**Proof.** This follows from Lemma 2.22 in the same way as Proposition 3.12. Notice that $B \mapsto \bigcap_{U \in \mathcal{U}_x} B(U)$ commutes with $C^\ast$-stabilisation and maps (semi)-split extensions in $C^\ast$-$\text{alg}(X)$ again to (semi)-split extensions in $C^\ast$-$\text{alg}$; therefore, it descends to a functor $\mathcal{R}(X) \rightarrow \mathcal{R}$.

## 4. The bootstrap class

Throughout this section, $X$ denotes a finite and sober topological space. Finiteness is crucial here. First we construct a canonical filtration on any $C^\ast$-algebra over $X$. We use this to study the analogue of the bootstrap class in $\mathcal{R}(X)$. Along the way, we also introduce the larger category of local $C^\ast$-algebras over $X$. Roughly speaking, this means that all the canonical $C^\ast$-algebra extensions that we get from $C^\ast$-algebras over $X$ are admissible. Objects in the $X$-equivariant bootstrap category have the additional property that their fibres belong to the usual bootstrap category.

### 4.1. The canonical filtration

We recursively construct a canonical increasing filtration

$$\emptyset = \mathfrak{F}_0 X \subset \mathfrak{F}_1 X \subset \cdots \subset \mathfrak{F}_\ell X = X$$

of $X$ by open subsets $\mathfrak{F}_j X$, such that the differences

$$X_j := \mathfrak{F}_j X \setminus \mathfrak{F}_{j-1} X$$
are discrete for all \( j = 1, \ldots, \ell \). In each step, we let \( X_j \) be the subset of all open points in \( X \setminus \mathcal{F}_{j-1} X \)—so that \( X_j \) is discrete—and put \( \mathcal{F}_j X = \mathcal{F}_{j-1} X \cup X_j \). Equivalently, \( X_j \) consists of all points of \( X \setminus \mathcal{F}_{j-1} X \) that are maximal for the specialisation preorder \( \prec \). Since \( X \) is finite, \( X_j \) is non-empty unless \( \mathcal{F}_{j-1} X = X \), and our recursion reaches \( X \) after finitely many steps.

**Definition 4.1.** The length \( \ell \) of \( X \) is the length of the longest chain \( x_1 \prec x_2 \prec \cdots \prec x_\ell \) in \( X \).

We assume \( X \) finite to ensure that the above filtration can be constructed. It is easy to extend our arguments to Alexandrov spaces of finite length; the only difference is that the discrete spaces \( X_j \) may be infinite in this case, so that we need infinite direct sums in some places, forcing us to reformulate Proposition 4.7. It should be possible to treat Alexandrov spaces of infinite length in a similar way. Since such techniques cannot work for non-Alexandrov spaces, anyway, we do not pursue these generalisations here.

**Definition 4.2.** We shall use the functors

\[
P_Y := i^Y_X \circ r^Y_X : \mathcal{C}^* \text{alg}(X) \to \mathcal{C}^* \text{alg}(X)
\]

for \( Y \in \mathcal{L}C(X) \). Thus \( (P_Y A)(Z) \cong A(Y \cap Z) \) for all \( Z \in \mathcal{L}C(X) \).

If \( Y \in \mathcal{L}C(X), U \subset \mathcal{O}(Y) \), then we get an extension

\[
(4.3) \quad P_U(A) \hookrightarrow P_Y(A) \hookrightarrow P_Y \setminus U(A)
\]

in \( \mathcal{C}^* \text{alg}(X) \) because we have extensions \( A(Z \cap U) \hookrightarrow A(Z \cap Y) \hookrightarrow A(Z \cap Y \setminus U) \) for all \( Z \in \mathcal{L}C(X) \).

Let \( A \) be a \( C^* \)-algebra over \( X \). We equip \( A \) with the canonical increasing filtration by the ideals

\[
\mathcal{F}_j A := P_{\mathcal{F}_j X}(A), \quad j = 0, \ldots, \ell,
\]

so that

\[
(4.4) \quad \mathcal{F}_j A(Y) = A(Y \cap \mathcal{F}_n X) = A(Y) \cap A(\mathcal{F}_n X) \quad \text{for all } Y \in \mathcal{L}C(X).
\]

Equation (4.3) shows that the subquotients of this filtration are

\[
(4.5) \quad \mathcal{F}_j A / \mathcal{F}_{j-1} A \cong P_{\mathcal{F}_j X \setminus \mathcal{F}_{j-1} X}(A) = P_{X_j}(A) \cong \bigoplus_{x \in X_j} \bigoplus_{i_X} i_x(A(x)).
\]

Here \( i_x \) is \( i^X_X \) for \( x \in X \) denotes the extension functor for the subset \( \{ x \} \subset X \):

\[
i_x : \mathcal{F} \mapsto \mathcal{F}(\{ x \}) \xrightarrow{i_x} \mathcal{F}(X), \quad (i_x B)(Y) = \begin{cases} B & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}
\]

**Example 4.6.** Consider the space \( X = \{ 1, 2 \} \) with the non-discrete topology described in Example 2.29. Here

\[
\mathcal{F}_0 X = \emptyset, \quad \mathcal{F}_1 X = \{ 1 \}, \quad \mathcal{F}_2 X = \{ 1, 2 \} = X, \quad X_1 = \{ 1 \}, \quad X_2 = \{ 2 \}.
\]

The filtration \( \mathcal{F}_j A \) on a \( C^* \)-algebra over \( X \) has one non-trivial layer \( \mathcal{F}_1 A \) because \( \mathcal{F}_0 A = \{ 0 \} \) and \( \mathcal{F}_2 A = A \). Recall that \( C^* \)-algebras over \( X \) correspond to extensions of \( C^* \)-algebras. For a \( C^* \)-algebra extension \( I \hookrightarrow A \twoheadrightarrow A/I \), the first filtration layer is simply the extension \( I \hookrightarrow I \twoheadrightarrow 0 \), so that the quotient \( A / \mathcal{F}_1 A \) is the extension \( 0 \hookrightarrow A/I \twoheadrightarrow A/I \). Our filtration decomposes \( I \hookrightarrow A \twoheadrightarrow A/I \) into an extension of \( C^* \)-algebra extensions as follows:

\[
(I \hookrightarrow I \twoheadrightarrow 0) \leftrightarrow (I \hookrightarrow A \twoheadrightarrow A/I) \leftrightarrow (0 \hookrightarrow A/I \twoheadrightarrow A/I).
\]

**Proposition 4.7.** The following are equivalent for a separable \( C^* \)-algebra \( A \) over \( X \):

[Provide the proposition statement here.]
(1) The extensions \( \mathcal{G}_{j-1}A \to \mathcal{G}_jA \to \mathcal{G}_jA/\mathcal{G}_{j-1}A \) in \( \mathcal{C}^*\text{sep}(X) \) are admissible for \( j = 1, \ldots, \ell \).

(2) \( A \in \mathcal{R}(X) \) belongs to the triangulated subcategory of \( \mathcal{R}(X) \) generated by objects of the form \( i_x(B) \) with \( x \in X, \ B \in \mathcal{R} \).

(3) \( A \in \mathcal{R}(X) \) belongs to the localising subcategory of \( \mathcal{R}(X) \) generated by objects of the form \( i_x(B) \) with \( x \in X, \ B \in \mathcal{R} \).

(4) For any \( Y \in \mathbb{L}C(X), \ U \in \mathbb{O}(Y) \), the extension

\[
P_U(A) \to P_Y(A) \to P_{Y \cup U}(A)
\]

in \( \mathcal{C}^*\text{sep}(X) \) described above is admissible.

Furthermore, if \( A \) satisfies these conditions, then it already belongs to the triangulated subcategory of \( \mathcal{R}(X) \) generated by \( i_x(A(x)) \) for \( x \in X \).

Recall that the localising subcategory generated by a family of objects in \( \mathcal{R}(X) \) is the smallest subcategory that contains the given objects and is triangulated and closed under countable direct sums.

**Proof.** (2)\( \Rightarrow \) (3) and (4)\( \Rightarrow \) (1) are trivial. We will prove (1)\( \Rightarrow \) (2) and (3)\( \Rightarrow \) (4).

(1)\( \Rightarrow \) (2): Since the extensions \( \mathcal{G}_{j-1}A \to \mathcal{G}_jA \to \mathcal{G}_jA/\mathcal{G}_{j-1}A \) are admissible, they yield extension triangles in \( \mathcal{R}(X) \). Thus \( \mathcal{G}_jA \) belongs to the triangulated subcategory of \( \mathcal{R}(X) \) generated by \( \mathcal{G}_{j-1}A \) and \( \mathcal{G}_jA/\mathcal{G}_{j-1}A \). Since \( \mathcal{G}_0A = 0 \), induction on \( j \) and (4.5) show that \( \mathcal{G}_jA \) belongs to the triangulated subcategory generated by \( i_xA(x) \) with \( x \in X \). Thus \( A = \mathcal{G}_jA \) belongs to the triangulated subcategory of \( \mathcal{R}(X) \) generated by \( i_x(A(x)) \) for \( x \in X \). This also yields the last statement in the proposition.

(3)\( \Rightarrow \) (4): It is clear that (4) holds for objects of the form \( i_x(B) \) because at least one of the three objects \( P_Ui_x(B) \), \( P_Yi_x(B) \), or \( P_{Y \cup U}i_x(B) \) vanishes. The property (4) is inherited by (countable) direct sums, suspensions, and mapping cones. To prove the latter, we use the definition of admissibility as an isomorphism statement in \( \mathcal{R}(X) \) and the Five Lemma in triangulated categories. Hence (4) holds for all objects of the localising subcategory generated by \( i_x(B) \) for \( x \in X, \ B \in \mathcal{R} \).

**Definition 4.8.** Let \( \mathcal{R}(X)_{\text{loc}} \subseteq \mathcal{R}(X) \) be the full subcategory of all objects that satisfy the equivalent conditions of Proposition 4.7.

The functor \( f_*: \mathcal{R}(X) \to \mathcal{R}(Y) \) for a continuous map \( f: X \to Y \) restricts to a functor \( \mathcal{R}(X)_{\text{loc}} \to \mathcal{R}(Y)_{\text{loc}} \) because \( f_* \circ i_x^Y = i_y^Y \) and \( f_* \) is an exact functor. Similarly, the restriction functor \( r_X^Y: \mathcal{R}(X) \to \mathcal{R}(Y) \) for a locally closed subset \( Y \subseteq X \) maps \( \mathcal{R}(X)_{\text{loc}} \) to \( \mathcal{R}(Y)_{\text{loc}} \) because \( r_X^Y \circ i_x^X = i_y^X \) is \( i_y^X \) for \( x \in Y \) and 0 otherwise.

**Proposition 4.9.** Let \( X \) be a finite topological space. Let \( A, B \in \mathcal{R}(X)_{\text{loc}} \) and let \( f \in \mathcal{K}_*(X; A, B) \). If \( f(x) \in \mathcal{K}_*(A(x), B(x)) \) is invertible for all \( x \in X \), then \( f \) is invertible in \( \mathcal{R}(X) \). In particular, if \( A(x) \cong 0 \) in \( \mathcal{R} \) for all \( x \in X \), then \( A \cong 0 \) in \( \mathcal{R}(X) \).

**Proof.** The second assertion follows immediately from the last sentence in Proposition 4.7. It implies the first one by a well-known trick: embed \( \alpha \) in an exact triangle by axiom (TR1) of a triangulated category, and use the long exact sequence to relate invertibility of \( \alpha \) to the vanishing of its mapping cone.

**Proposition 4.10.** Suppose that the extensions of \( C^*\)-algebras

\[
A(U_x \setminus \{x\}) \to A(U_x) \to A(x)
\]

are semi-split for all \( x \in X \). Then \( A \in \mathcal{R}(X)_{\text{loc}} \). In particular, this applies if the underlying \( C^*\)-algebra of \( A \in \mathcal{R}(X) \) is nuclear.
We can rewrite the long exact sequence above as an extension:
\[ A(X_j) \to A(\emptyset)_X \]
We only comment on this very briefly.

The boundary map is the diagonal map in the following commuting diagram:

where we have used Proposition 3.13. These groups comprise the other constructions, as explained in [11]. This includes admissible extensions, \( C^* \), of \( \mathcal{C} \)-algebras that contains \( \mathbb{C} \) and is closed under \( \mathcal{K} \)-equivalence, countable direct sums, suspensions, and the formation of mapping cones (see [3]).

\[ \xrightarrow{\delta} \]

It is not clear whether the mere admissibility in \( \mathcal{C} \)-sep of the extensions
\[ A(U_x \setminus \{ x \}) \to A(U_x) \to A(x) \]
suffices to conclude that \( A \in \mathcal{R}(\mathcal{X})_{\text{loc}} \). This condition is certainly necessary.

The constructions above can be used to generate spectral sequences as in [16]. We only comment on this very briefly.

Let \( A \in \mathcal{R}(\mathcal{X})_{\text{loc}} \). The admissible extensions \( \emptyset_j A \to \emptyset_j A \to \emptyset_j A/\emptyset_j A \) for \( j = 1, \ldots, \ell \) produce exact triangles in \( \mathcal{R}(\mathcal{X}) \). A homological or cohomological functor such as \( \mathcal{K} \mathcal{K}(X; D_{\ell}) \) or \( \mathcal{K} \mathcal{K}(X; 1, D) \) maps these exact triangles to a sequence of exact chain complexes. These can be arranged in an exact couple, which generates a spectral sequence (see [5]). This spectral sequence could, in principle, be used to compute \( \mathcal{K} \mathcal{K}_*(X; A, B) \) in terms of

\[ \mathcal{K} \mathcal{K}_*(X; \emptyset_j A/\emptyset_j A, B) \cong \prod_{x \in X_j} \mathcal{K} \mathcal{K}_*(X; i_x A(x), B) \cong \prod_{x \in X_j} \mathcal{K} \mathcal{K}_*(A(x), B(U_x)) \]

where we have used Proposition 3.13. These groups comprise the \( E_2 \)-terms of the spectral sequence that we get from our exact couple for the functor \( \mathcal{K} \mathcal{K}(X; 1, B) \).

For instance, consider again the situation of Example 4.13. Let \( I \triangleleft A \) and \( J \triangleleft B \) be \( C^* \)-algebras over \( X \), corresponding to \( C^* \)-algebra extensions \( I \to A \) and \( J \to B \). The above spectral sequence degenerates to a long exact sequence

\[ \mathcal{K} \mathcal{K}_0(A/I, B) \xrightarrow{\delta} \mathcal{K} \mathcal{K}_0(X; I \triangleleft A, J \triangleleft B) \xrightarrow{\delta} \mathcal{K} \mathcal{K}_0(I, J) \]

The boundary map is the diagonal map in the following commuting diagram:

\[ \mathcal{K} \mathcal{K}_0(I, J) \xrightarrow{\delta} \mathcal{K} \mathcal{K}_0(X; I \triangleleft A, J \triangleleft B) \xrightarrow{\delta} \mathcal{K} \mathcal{K}_0(I, J) \]

We can rewrite the long exact sequence above as an extension:

\[ \ker \delta \to \mathcal{K} \mathcal{K}_*(X; I \triangleleft A, J \triangleleft B) \to \ker \delta. \]

But we lack a description of \( \ker \delta \) and \( \text{coker} \delta \) as Hom- and Ext-groups. Therefore, the Universal Coefficient Theorem of Alexander Bonkat [3] seems more attractive.

4.2. The bootstrap class. The bootstrap class \( \mathcal{B} \) in \( \mathcal{R} \) is the localising subcategory generated by the single object \( \mathcal{C} \), that is, it is the smallest class of separable \( C^* \)-algebras that contains \( \mathbb{C} \) and is closed under \( \mathcal{K} \)-equivalence, countable direct sums, suspensions, and the formation of mapping cones (see [13]).

A localising subcategory of \( \mathcal{R}(\mathcal{X}) \) or \( \mathcal{R} \) is automatically closed under various other constructions, as explained in [11]. This includes admissible extensions,
admissible inductive limits (the appropriate notion of admissibility is explained in [11]), and crossed products by \( \mathbb{Z} \) and \( \mathbb{R} \) and, more generally, by actions of torsion-free amenable groups.

The latter result uses the reformulation of the (strong) Baum–Connes conjecture for such groups in [11]. This reformulation asserts that \( \mathbb{C} \) with the trivial representation of an amenable group \( G \) belongs to the localising subcategory of \( \mathfrak{R}(G) \) generated by \( C_0(G) \). Carrying this over to \( \mathfrak{R}(X) \), we conclude that \( A \times G \) for \( A \in \mathfrak{R}(X) \) belongs to the localising subcategory of \( \mathfrak{R}(X) \) generated by \( (A \otimes C_0(G)) \rtimes G \), which is Morita–Riefel equivalent to \( A \).

The following definition provides an analogue of Proposition 4.12.

**Definition 4.11.** Let \( B(X) \) be the localising subcategory of \( \mathfrak{R}(X) \) that is generated by \( i_x(\mathbb{C}) \) for \( x \in X \).

Notice that \( \{ i_x(\mathbb{C}) \mid x \in X \} \) lists all possible ways to turn \( \mathbb{C} \) into a \( C^* \)-algebra over \( X \).

**Proposition 4.12.** Let \( X \) be a finite topological space and let \( A \in \mathfrak{R}(X) \). The following conditions are equivalent:

1. \( A \in B(X) \);
2. \( A \in \mathfrak{R}(X)_{\text{loc}} \) and \( A(x) \in B \) for all \( x \in X \);
3. the extensions \( \mathfrak{g}_{j-1}A \to \mathfrak{g}_jA \to \mathfrak{g}_jA/(\mathfrak{g}_{j-1}A) \) are admissible for \( j = 1, \ldots, \ell \), and \( A(x) \in B \) for all \( x \in X \);

In addition, in this case \( A(Y) \in B \) for all \( Y \in \mathbb{LC}(X) \).

**Proof.** The equivalence of (2) and (3) is already contained in Proposition 4.7. Using the last remark in Proposition 4.7 we also get the implication (3) \( \Rightarrow \) (1) because \( i_x \) is exact and commutes with direct sums. The only assertion that is not yet contained in Proposition 4.7 is that \( A \in B(X) \) implies \( A(Y) \in B \) for all \( Y \in \mathbb{LC}(X) \). The reason is that the functor \( \mathfrak{R}(X) \to \mathfrak{R} \), \( A \mapsto A(Y) \), is exact, preserves countable direct sums, and maps the generators \( i_y(\mathbb{C}) \) for \( y \in X \) to either 0 or \( \mathbb{C} \) and hence into \( B \).

**Corollary 4.13.** If the underlying \( C^* \)-algebra of \( A \) is nuclear, then \( A \in B(X) \) if and only if \( A(x) \in B \) for all \( x \in X \).

**Proof.** Combine Propositions 4.10 and 4.12

**Example 4.14.** View a separable nuclear \( C^* \)-algebra \( A \) with only finitely many ideals as a \( C^* \)-algebra over \( \text{Prim}(A) \). Example 2.16 and Corollary 4.13 show that \( A \) belongs to \( B(\text{Prim} A) \) if and only if all its simple subquotients belong to the usual bootstrap class in \( \mathfrak{R} \).

**Proposition 4.15.** Let \( X \) be a finite topological space. Let \( A, B \in B(X) \) and let \( f \in \text{KK}_*(X; A, B) \). If \( f \) induces invertible maps \( K_*\big( A(x) \big) \to K_*\big( B(x) \big) \) for all \( x \in X \), then \( f \) is invertible in \( \mathfrak{R}(X) \). In particular, if \( K_*\big( A(x) \big) = 0 \) for all \( x \in X \), then \( A \cong 0 \in \mathfrak{R}(X) \).

**Proof.** As in the proof of Proposition 4.9, it suffices to show the second assertion. Since \( A(x) \in B \) for all \( x \in X \), vanishing of \( K_*\big( A(x) \big) \) implies vanishing of \( A(x) \) in \( \mathfrak{R} \), so that Proposition 4.9 yields the assertion.

**4.3. Complementary subcategories.** It is often useful to replace a given object of \( \mathfrak{R}(X) \) by one in the bootstrap class or \( \mathfrak{R}(X)_{\text{loc}} \) that is as close to the original as possible. This is achieved by functors

\[
L_B : \mathfrak{R}(X) \to B(X), \quad L : \mathfrak{R}(X) \to \mathfrak{R}(X)_{\text{loc}}
\]
Theorem 4.17. The pair of subcategories $I$ is trivial for $I$. Therefore, the ideal $F$ is a stable homological ideal that is compatible with direct sums (see \[10\]Proposition 3.37). The kernel of $F$ on objects is exactly $\mathcal{R}(X)^{\perp}$. Proposition 3.13 shows that the functor $F$ has a left adjoint, namely, the functor $F^+: (A_x)_{x \in X} \mapsto \bigoplus_{x \in X} i_x(A_x)$. Therefore, the ideal $I$ has enough projective objects by \[12\]Proposition 3.37; furthermore, the projective objects are retracts of direct sums of objects of the form $i_x(A_x)$. Hence the localising subcategory generated by the $I$-projective objects is $\mathcal{R}(X)_{\text{loc}}$ by Proposition 4.7. Finally, \[10\]Theorem 4.6 shows that the pair of subcategories $(\mathcal{R}(X)_{\text{loc}}, \mathcal{R}(X)^{\perp}_{\text{loc}})$ is complementary.

The argument for the bootstrap category is almost literally the same, but using the stable homological functor $K_* : \mathcal{R}(X) \to \prod_{x \in X} \mathcal{A}_{\mathbb{Z}/2}$ instead of $F$, where $\mathcal{A}_{\mathbb{Z}/2}$ denotes the category of countable $\mathbb{Z}/2$-graded Abelian groups. The adjoint of $K_* \circ F$ is defined on families of countable free Abelian groups, which is enough to conclude that $\ker(K_* \circ F)$ has enough projective objects. This time, the projective objects generate the category $\mathcal{B}(X)$, and the kernel of $F$ on objects is $\mathcal{B}(X)^{\perp}$. Hence \[10\]Theorem 4.6] shows that the pair of subcategories $(\mathcal{B}(X), \mathcal{B}(X)^{\perp})$ is complementary.

Lemma 4.18. The following are equivalent for $A \in \mathcal{R}(X)$:

1. $K_*(A(x)) = 0$ for all $x \in X$, that is, $A \in \mathcal{B}(X)^{\perp}$;
2. $K_*(A(Y)) = 0$ for all $Y \in \mathcal{L}(X)$;
3. $K_*(A(U)) = 0$ for all $U \in \mathcal{O}(X)$.

Proof. It is clear that (2) implies both (1) and (3). Conversely, (3) implies (2): write $Y \in \mathcal{L}(X)$ as $U \setminus V$ with $U, V \in \mathcal{O}(X)$, $V \subseteq U$, and use the K-theory long exact sequence for the extension $A(U) \to A(V) \to A(Y)$. It remains to check that (1) implies (2).

We prove by induction on $j$ that (1) implies $K_*(A(Y)) = 0$ for all $Y \in \mathcal{L}(\mathfrak{g}_j X)$. This is trivial for $j = 0$. If $Y \subseteq \mathfrak{g}_{j+1} \setminus X$, then $K_*(A(Y \cap \mathfrak{g}_j X)) = 0$ by induction assumption. The K-theory long exact sequence for the extension $A(Y \cap \mathfrak{g}_j X) \to A(Y) \to \bigoplus_{x \in X \setminus Y} A(x)$
yields $K_*(A(Y)) = 0$ as claimed. □

We can also apply the machinery of [10,12] to the ideal $I$ to generate a spectral sequence that computes $KK_*(X; A, B)$. This spectral sequence is somewhat more useful than the one we get from the canonical filtration because its second page involves derived functors. But this spectral sequence rarely degenerates to an exact sequence.

4.4. A definition for infinite spaces. The ideas in §4.3 suggest a definition of the bootstrap class for infinite spaces.

**Definition 4.19.** Let $X$ be a topological space. Let $B(X)^+ \subseteq \mathfrak{R}(X)$ consist of all separable $C^*$-algebras over $X$ with $K_*(A(U)) = 0$ for all $U \in \mathcal{O}(X)$.

Lemma 4.18 shows that this agrees with our previous definition for finite $X$. Furthermore, the same argument as in the proof of Lemma 4.18 yields $A \in B(X)^+$ if and only if $K_*(A(Y)) = 0$ for all $Y \in L\mathcal{C}(X)$. It is clear from the definition that $B(X)^+$ is a localising subcategory of $\mathfrak{R}(X)$.

**Definition 4.20.** Let $X$ be a topological space. We let $B(X)$ be the localisation of $\mathfrak{R}(X)$ at $B(X)^+$.

For finite $X$, we have seen that $B(X)^+$ is part of a complementary pair of localising subcategories, with partner $B(X)$. This shows that the localisation of $\mathfrak{R}(X)$ at $B(X)^+$ is canonically equivalent to $B(X)$. For infinite $X$, it is unclear whether $B(X)^+$ is part of a complementary pair. If it is, the partner must be

$$
D := \{ A \in \mathfrak{R}(X) \mid KK_*(X; A, B) = 0 \text{ for all } B \in \mathcal{B}(X)^+ \}.
$$

Since $B \in \mathcal{B}(X)^+$ implies nothing about the $K$-theory of $\bigcap_{U \in \mathcal{U}_x} B(U)$, in general, Proposition 3.13 shows that $i_x \mathcal{C}$ does not belong to $D$ in general.

If $X$ is Hausdorff, then $C_0(U) \in B(X)^+$ for all $U \in \mathcal{O}(X)$. Nevertheless, it is not clear whether $(D, B(X)^+)$ is complementary.

5. Making the fibres simple

**Definition 5.1.** A $C^*$-algebra $(A, \psi)$ over $X$ is called tight if $\psi: \text{Prim}(A) \to X$ is a homeomorphism.

Tightness implies that the fibres $A_x = A(x)$ for $x \in X$ are simple $C^*$-algebras. But the converse does not hold: the fibres are simple if and only if the map $\psi: \text{Prim}(A) \to X$ is bijective.

To equip $\mathfrak{R}(X)$ with a triangulated category structure, we must drop the tightness assumption because it is usually destroyed when we construct cylinders, mapping cones, or extensions of $C^*$-algebras over $X$. Nevertheless, we show below that we may reinstall tightness by passing to a $KK(X)$-equivalent object, at least in the nuclear case. The authors do not know whether nuclearity is really necessary here.

The special case where the space $X$ in question has only one point is already known:

**Theorem 5.2 ([15] Proposition 8.4.5).** Any separable nuclear $C^*$-algebra is $KK$-equivalent to a $C^*$-algebra that is separable, nuclear, purely infinite, $C^*$-stable and simple.

Stability is not part of the assertion in [15], but can be achieved by tensoring with the compact operators, without destroying the other properties. The main difficulty is to achieve simplicity. We are going to generalise this theorem as follows:
Theorem 5.3. Let $X$ be a finite topological space. Any separable nuclear $C^\ast$-algebra over $X$ is $KK(X)$-equivalent to a $C^\ast$-algebra over $X$ that is tight, separable, nuclear, purely infinite, and $C^\ast$-stable.

By the way, when we apply the following argument to the zero $C^\ast$-algebra, viewed as a $C^\ast$-algebra over $X$, then we reprove the known statement that there is a separable, nuclear, purely infinite, and stable $C^\ast$-algebra with spectrum $X$ for any finite topological space $X$.

Proof. We use the canonical filtration $\mathfrak{F}_j X$ of $X$ and the resulting filtration $\mathfrak{F}_j A$ introduced in (4.4). The subquotients

$$A_j := \mathfrak{F}_j A / \mathfrak{F}_{j-1} A$$

of the filtration are described in (4.5) in terms of the subquotients $A_x$ for $x \in X$.

Since $A$ is separable and nuclear, so are the subquotients $A_x$. Hence Theorem 5.2 provides simple, separable, nuclear, stable, purely infinite $C^\ast$-algebras $B_x$ and $KK$-equivalences $\phi_x : B_x \cong B_x$ for all $x \in X$.

We will recursively construct a sequence $B_x$ of $C^\ast$-algebras over $X$ that are supported on $\mathfrak{F}_j X$ and $KK(X)$-equivalent to $\mathfrak{F}_j A$ for $j = 0, \ldots, \ell$, such that $\mathfrak{F}_j B_x = B_x$ for $k \geq j$ and each $B_x$ is tight over $\mathfrak{F}_j X$, separable, nuclear, purely infinite, and stable. The last object $B_x$ in this series is $KK(X)$-equivalent to $\mathfrak{F}_j A = A$ and has all the required properties. Since $\mathfrak{F}_0 X = \emptyset$, the recursion must begin with $B_0 = A_0 = \{0\}$. We assume that $B_j$ has been constructed. Let

$$B_{j+1}^0 := \bigoplus_{x \in X_{j+1}} i_x(B_x).$$

We will construct $B_{j+1}$ as an extension of $B_j$ by $B_{j+1}^0$. This ensures that the fibres of $B_j$ are $B_x$ for $x \in \mathfrak{F}_j X$ and 0 for $x \in X \setminus \mathfrak{F}_j X$.

First we construct, for each $x \in X_{j+1}$, a suitable extension of $B_x$ by $B_j$. Let $U_x \subseteq \mathfrak{F}_{j+1} X$ be the minimal open subset containing $x$ and let $U'_x := U_x \setminus \{x\}$. Since $X_{j+1}$ is discrete, $U'_x$ is an open subset of $\mathfrak{F}_j X$. The extension

$$A(U'_x) \rightarrow A(U_x) \rightarrow A_x$$

is semi-split and thus provides a class $\delta^A_x$ in $KK_1(A_x, A(U'_x))$ because $A_x$ is nuclear. Since $B_x \cong A_x$ and $\mathfrak{F}_j A \cong B_j$ and $\mathfrak{F}_j A(U'_x) = A(U'_x)$, we can transform this class to $\delta^B_x$ in $KK_1(B_x, B_j(U'_x))$.

We abbreviate $B_{j+1} := B_j(U'_x)$ to simplify our notation. Represent $\delta^B_x$ by an odd Kasparov cycle $(H, \varphi, F)$, where $H$ is a Hilbert $B_j$-module, $\varphi : B_j \to \mathbb{B}(H)$ is a *-homomorphism, and $F \in \mathbb{B}(H)$ satisfies $F^2 = 1$, $F = F^*$, and $[F, \varphi(b)] \in \mathbb{K}(H)$ for all $b \in B_x$. Now we apply the familiar correspondence between odd KK-elements and $C^\ast$-algebra extensions. Let $P := \frac{1}{2}(1 + F)$, then

$$\psi : B_x \to \mathbb{B}(H)/\mathbb{K}(H), \quad b \mapsto P \varphi(x) P$$

is a *-homomorphism and hence the Busby invariant of an extension of $B_x$ by $\mathbb{K}(H)$. After adding a sufficiently big split extension, that is, a *-homomorphism $\psi_0 : B_x \to \mathbb{B}(H')$, the map $\psi : B_x \to \mathbb{B}(H)/\mathbb{K}(H)$ becomes injective and the ideal in $\mathbb{K}(H)$ generated by $\mathbb{K}(H)\psi(B_x)\mathbb{K}(H)$ is all of $\mathbb{K}(H)$. We assume these two extra properties from now on.

We also add to $\psi$ the trivial extension $B_j \rightarrow B_j \oplus B_x \rightarrow B_x$, whose Busby invariant is the zero map. This produces an extension of $B_x$ by $\mathbb{K}(B_j \oplus H) \cong B_j$; the last isomorphism holds because $B_j$ is stable, so that $B_j \oplus H \cong B_j$ for any Hilbert $B_j$-module $H'$. Since $\psi$ is injective, the extension we get is of the form $B_j \rightarrow E_{jx} \rightarrow B_x$. This extension is still semi-split, and its class in $KK_1(B_x, B_j)$
is the composite of $\delta^B_x$ with the embedding $B_{jx} \rightarrow B_j$. Our careful construction ensures that the ideal in $B_j$ generated by $B_j\psi(B_x)B_j$ is equal to $B(U'_x)$.

Now we combine these extensions for all $x \in X$ by taking their external direct sum. This is an extension of $\bigoplus_{x \in X_{j+1}} B_x = B^0_{j+1}$ by the $C^*$-algebra of compact operators on the Hilbert $B_j$-module $\bigoplus_{x \in X_{j+1}} B_j \cong B_j$, where we used the stability of $B_j$ once more. Thus we obtain an extension $B_j \rightarrow B_{j+1} \rightarrow B^0_{j+1}$. We claim that the primitive ideal space of $B_{j+1}$ identifies naturally with $\mathfrak{F}_{j+1}X$.

The extension $B_j \rightarrow B_{j+1} \rightarrow B^0_{j+1}$ decomposes $\text{Prim}(B^0_{j+1})$ into an open subset homeomorphic to $\text{Prim}(B_{j+1}) \cong \mathfrak{F}_jX$ and a closed subset homeomorphic to the discrete set $\text{Prim}(B^0_{j+1}) = X_{j+1}$. This provides a canonical bijection between $\text{Prim}(B_{j+1})$ and $\mathfrak{F}_{j+1}X$. We must check that it is a homeomorphism.

First let $U \subseteq \mathfrak{F}_{j+1}X$ be open in $\mathfrak{F}_{j+1}X$. Then $U \cap \mathfrak{F}_jX$ is open and contains $U'_x$ for each $x \in U \cap X_{j+1}$. Our construction ensures that $\psi(B_x) \subseteq B_{j+1}$ multiplies $B_j$ into $B_{jx} \subseteq B_j(U \cap \mathfrak{F}_jX)$. Hence $B_j(U \cap \mathfrak{F}_jX) + \sum_{x \in U \cap X_{j+1}} \psi(B_x)$ is an ideal in $B_{j+1}$. This shows that $U$ is open in $\text{Prim}(B_{j+1})$.

Now let $U \subseteq \mathfrak{F}_{j+1}X$ be open in $\text{Prim}(B_{j+1})$. Then $U \cap \mathfrak{F}_jX$ must be open in $\mathfrak{F}_jX \cong \text{Prim}(B_j)$. Furthermore, if $x \in U \cap X_{j+1}$, then the subset of $\text{Prim}(B_j)$ corresponding to the ideal in $B_j$ generated by $B_j\psi(B_x)B_j$ is contained in $U$. But our construction ensures that this subset is precisely $U'_x$. Hence

$$U = (U \cap \mathfrak{F}_jX) \cup \bigcup_{x \in U \cap X_{j+1}} U_x,$$

proving that $U$ is open in the topology of $\mathfrak{F}_{j+1}X$. This establishes that our canonical map between $\text{Prim}(B_{j+1})$ and $\mathfrak{F}_{j+1}X$ is a homeomorphism. Thus we may view $B_{j+1}$ as a $C^*$-algebra over $X$ supported in $\mathfrak{F}_{j+1}X$. It is clear from our construction that $B_j \rightarrow B_{j+1} \rightarrow B^0_{j+1}$ is an extension of $C^*$-algebras over $X$. Here we view $B^0_{j+1}$ as a $C^*$-algebra over $X$ in the obvious way, so that $B_x$ is its fibre over $x \in X_{j+1}$.

There is no reason to expect $B_{j+1}$ to be stable or purely infinite. But this is easily repaired by tensoring with $\mathbb{K} \otimes \mathcal{O}_\infty$. This does not change $B_j$ and $B^0_{j+1}$, up to isomorphism, because these are already stable and purely infinite, and it has no effect on the primitive ideal space, nuclearity or separability. Thus we may achieve that $B_j$ is stable and purely infinite.

By assumption, there is a KK$(X)$-equivalence $f_j \in \text{KK}_0(X; \mathfrak{F}_jA, B_j)$. Furthermore, our construction of $B^0_{j+1}$ ensures a KK$(X)$-equivalence $f'_{j+1}$ between $A^0_{j+1}$ and $B^0_{j+1}$. Due to the nuclearity of $A$, the arguments in §4.1 show that

$$\mathfrak{F}_jA \xrightarrow{f_j} \mathfrak{F}_{j+1}A \xrightarrow{f'_{j+1}} A^0_{j+1}$$

is a semi-split extension of $C^*$-algebras over $X$ and hence provides an exact triangle in $\text{KK}(X)$. The same argument provides an extension triangle for the extension $B_j \rightarrow B_{j+1} \rightarrow B^0_{j+1}$. Let $\delta^A_x$ and $\delta^B_x$ be the classes in $\text{KK}_1(X; A^0_{j+1}, \mathfrak{F}_jA)$ and $\text{KK}_1(B^0_{j+1}, B_j)$ associated to these extension; they appear in the exact triangles described above.

Both classes $\delta^A_x$ and $\delta^B_x$ are, essentially, the sum of the classes $\delta^A_x$ and $\delta^B_x$ for $x \in X_{j+1}$, respectively. More precisely, we have to compose each $\delta^A_x$ with the embedding $A(U'_x) \rightarrow \mathfrak{F}_jA$. Hence the solid square in the diagram

$$\begin{array}{ccc}
\Sigma A^0_{j+1} & \xrightarrow{\delta^A_x} & \mathfrak{F}_jA \\
\Sigma B^0_{j+1} & \xrightarrow{\delta^B_x} & B_j
\end{array}$$

$$\begin{array}{cccc}
\xrightarrow{f_j} & \xrightarrow{f'_{j+1}} & \xrightarrow{f^0_{j+1}} & \xrightarrow{f^0_{j+1}} \\
\cong & \cong & \cong & \cong
\end{array}$$

$$\begin{array}{ccc}
\Sigma A^0_{j+1} & \xrightarrow{\delta^A_x} & \mathfrak{F}_jA \\
\Sigma B^0_{j+1} & \xrightarrow{\delta^B_x} & B_j
\end{array}$$

is a semi-split extension of $C^*$-algebras over $X$ and hence provides an exact triangle in $\text{KK}(X)$.
commutes. By an axiom of triangulated categories, we can find the dotted arrow making the whole diagram commute. The Five Lemma for triangulated categories asserts that this arrow is invertible because $f_j$ and $f_j^{j+1}$ are. This completes the recursion step and shows that $B_{j+1}$ has all required properties. □

**Theorem 5.4.** Let $X$ be a finite topological space and let $A$ be a separable $C^*$-algebra over $X$. The following are equivalent:

- $A \in \mathfrak{R} (X)_{\text{loc}}$ and $A_x$ is KK-equivalent to a nuclear $C^*$-algebra for each $x \in X$;
- $A$ is KK$(X)$-equivalent to a $C^*$-algebra over $X$ that is tight, separable, nuclear, purely infinite, and $C^*$-stable.

**Proof.** The proof of Theorem 5.3 still works under the weaker assumption that $A \in \mathfrak{R} (X)_{\text{loc}}$ and $A_x$ is KK-equivalent to a nuclear $C^*$-algebra for each $x \in X$. The converse implication is trivial. □

**Corollary 5.5.** Let $X$ be a finite topological space and let $A$ be a separable $C^*$-algebra over $X$. The following are equivalent:

- $A \in \mathcal{B}(X)$;
- $A$ is KK$(X)$-equivalent to a $C^*$-algebra over $X$ that is tight, separable, nuclear, purely infinite, $C^*$-stable, and has fibres $A_x$ in the bootstrap class $\mathcal{B}$.

**Proof.** Combine Theorem 5.4 and Proposition 4.12. □

By a deep classification result by Eberhard Kirchberg (see [7]), two tight, separable, nuclear, purely infinite, stable $C^*$-algebras over $X$ are KK$(X)$-equivalent if and only if they are isomorphic as $C^*$-algebras over $X$. Therefore, the representatives found in Theorems 5.3 and 5.4 are unique up to isomorphism.

Let $\mathfrak{R}(X)_{\text{nuc}}$ be the subcategory of $\mathfrak{R}(X)$ whose objects are the separable nuclear $C^*$-algebras over $X$. This is a triangulated category as well because the basic constructions like suspensions, mapping cones, and extensions never leave this subcategory. The subcategory of $\mathfrak{R}(X)$ whose objects are the tight, separable, nuclear, purely infinite, stable $C^*$-algebras over $X$ is equivalent to $\mathfrak{R}(X)_{\text{nuc}}$ by Theorem 5.3 and hence inherits a triangulated category structure. It has the remarkable feature that isomorphisms in this triangulated category lift to $X$-equivariant $\ast$-isomorphisms.

### 6. Outlook

We have defined a bootstrap class $\mathcal{B}(X) \subseteq \mathfrak{R}(X)$ over a finite topological space $X$, which is the domain on which we should expect a Universal Coefficient Theorem to compute $\text{KK}^*_\ast(M;A,B)$. We have seen that any object of the bootstrap class is KK$(X)$-equivalent to a tight, purely infinite, stable, nuclear, separable $C^*$-algebra over $X$, for which Kirchberg’s classification results apply.

There are several spectral sequences that compute $\text{KK}^*_\ast(M;A,B)$, but applications to the classification programme require a short exact sequence. For some finite topological spaces, such a short exact sequence is constructed in [13] based on filtrated K-theory, so that filtrated K-theory is a complete invariant. This invariant comprises the K-theory $\text{K}_\ast(A(Y))$ of all locally closed subsets $Y$ of $X$ together with the action of all natural transformations between them. This is a consequence of a Universal Coefficient Theorem in this case. It is also shown in [13] that there are finite topological spaces for which filtrated K-theory is not yet a complete invariant.

At the moment, it is unclear whether there is a general, tractable complete invariant for objects of $\mathcal{B}(X)$.

Another issue is to treat infinite topological spaces. A promising approach is to approximate infinite spaces by finite non-Hausdorff spaces associated to open
coverings of the space in question. In good cases, there should be a \( \lim \leftarrow \) sequence that relates \( KK_\ast (X; A, B) \) to Kasparov groups over such finite approximations to \( X \), reducing computations from the infinite to the finite case. Such an exact sequence may be considerably easier for \( X \)-equivariant \( E \)-theory, where we do not have to worry about completely positive sections.

**References**


**Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstr. 3–5, 37073 Göttingen, Germany.**

**Københavns Universitets Institut for Matematiske Fag, Universitetsparken 5, 2100 København, Denmark**

**E-mail address**: rameyer@uni-math.gwdg.de

**E-mail address**: rnest@math.ku.dk