Universal coefficient theorems for Kirchberg’s bivariant K-theory II

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1 The bootstrap class

Locally closed subsets

Ideals for open subsets
Recall that the continuous map \( \text{Prim}(A) \to X \) that is part of a \( C^* \)-algebra over \( X \) yields a lattice morphism \( \mathcal{O}(X) \to \mathcal{O}(\text{Prim } A) \cong \mathbb{I}(A) \), mapping \( U \in \mathcal{O}(X) \) to the ideal \( A(U) \) of elements of \( A \) that vanish outside \( U \).

**Definition 1.** \( Y \subseteq X \) locally closed: \( Y = U \setminus V \) with \( U, V \in \mathcal{O}(X) \), \( V \subseteq U \)

**Definition and Lemma**
Let \( Y \subseteq X \) be locally closed, write \( Y = U \setminus V \) as above. Let \( A \) be a \( C^* \)-algebra over \( X \).

\[
A(Y) := A(U)/A(V)
\]

does not depend on the choice of \( U \) and \( V \).

**Proof.** Use that \( U \mapsto A(U) \) is a lattice morphism. \( \square \)

**Fibres of \( C^* \)-algebra bundles**

**Example 2** (Fibres of a \( C^* \)-algebra bundle). Let \( X \) be finite and let \( U_x \) for \( x \in X \) be the minimal open neighbourhood of \( x \). Then \( U_x \setminus \{x\} \) is open, so that \( \{x\} \) is locally closed.

View \( A(x) = A_x := A(\{x\}) \) as the fibre of \( A \) at \( x \).

**Example 3** (A space over which bundles have no fibres). Let \( X = [0, 1] \) with the topology \( \mathcal{O}(X) = \{(t, 1) \mid t \in [0, 1]\} \).

Then the locally closed subsets are of the form \( (a, b] \) with \( 0 \leq a \leq b \leq 1 \). Hence \( \{x\} \) is not locally closed for any \( x \in [0, 1] \).

\( C^* \)-algebra bundles over this space have no well-defined fibres.
1.1 The complement of the bootstrap class

**Definition and Lemma**
The following conditions for a $C^*$-algebra $A$ over $X$ are equivalent and define $\mathcal{B}(X)^{\perp}$:

- $K_*(A(Y)) = 0$ for all locally closed $Y \subseteq X$,
- $K_*(A(U)) = 0$ for all open $U \subseteq X$,
- each $x \in X$ has an open neighbourhood $U$ such that $K_*(A(V)) = 0$ for all open $V \subseteq U$.

If $X$ is finite, these are also equivalent to

- $K_*(A(U_x)) = 0$ for all $x \in X$;
- $K_*(A(x)) = 0$ for all $x \in X$.

**Important ingredient in the proof.** There is a filtration $\emptyset = \mathcal{F}_0 X \subset \mathcal{F}_1 X \subset \cdots \subset \mathcal{F}_\ell X = X$ by open subsets such that the differences $X_j := \mathcal{F}_j X \setminus \mathcal{F}_{j-1} X$ are all discrete. \hfill \Box

1.2 Generators of the bootstrap class

**Definition 4.** Given $x \in X$ and a $C^*$-algebra $A$, let $i_x(A)$ be $A$ together with the constant map $x: \text{Prim}(A) \to X$ with value $x$. This is a $C^*$-algebra over $X$ with

$$i_x(A)(U) = \begin{cases} A & \text{for } x \in U, \\ 0 & \text{for } x \notin U. \end{cases}$$

**Lemma 5.** Let $X$ be finite. Then $KK_*^X(i_x(A), B) \cong KK_*^X(A, B(U_x))$.

**Proof.** Since $i_x(A)(U) = A$ for $U \supseteq U_x$ open $i_x(A)(V) = 0$ for other open subsets, a cycle for $KK_*^X(i_x(A), B)$ determines one for $KK_*^X(i_x(A)(U_x), B(U_x)) = KK_*^X(A, B(U_x))$, and the latter does not have to satisfy any further conditions. \hfill \Box

The bootstrap class over finite spaces

**Lemma 6.** For finite $X$, the bootstrap class is (equivalent to) the localising subcategory of $KK(X)$ generated by $i_x\mathbb{C}$ for $x \in X$.

- More precisely, let $\mathcal{B}(X)$ be the smallest class of separable $C^*$-algebras that contains $i_x\mathbb{C}$ for $x \in X$ and is closed under suspensions, $KK^X$-equivalence, countable direct sums, and mapping cones.
- $KK^X_*^X(A, B) = 0$ if $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(X)^{\perp}$ because $KK^X_*^X(i_x\mathbb{C}, B) = 0$ for $B \in \mathcal{B}(X)^{\perp}$.
- $\mathcal{B}(X)$ and $\mathcal{B}(X)^{\perp}$ together “generate” all of $KK(X)$. (They form a complementary pair of localising subcategories.)
- The last statement is non-trivial, but it follows easily from some general machinery.
Characterisation of the bootstrap class

**Theorem 7.** Let $X$ be a finite $T_0$-space. For a nuclear separable $C^*$-algebra $A$ over $X$, the following are equivalent:

- $A$ belongs to the bootstrap class over $X$;
- $A(U_x)$ belongs to the bootstrap class for all $x \in X$;
- $A(x) := A(U_x)/A(U_x \setminus \{x\})$ belongs to the bootstrap class in $\text{KK}$ for all $x \in X$;
- $A(U)$ belongs to the bootstrap class for all $U \in \mathcal{O}(X)$.

### 1.3 Connection to the $C^*$-algebra classification programme

**Making the fibres simple**

**Theorem 8** (Kirchberg). Any exact $C^*$-algebra is $\text{KK}$-equivalent to a simple one. The latter can be assumed $\mathcal{O}_\infty \otimes \mathbb{K}$-stable, that is, purely infinite and $C^*$-stable.

**Theorem 9.** Let $X$ be finite. Any object of the bootstrap class $\mathcal{B}(X)$ is $\text{KK}^X$-equivalent to a separable, nuclear, $\mathcal{O}_\infty \otimes \mathbb{K}$-stable one for which the map $\text{Prim}(A) \to X$ is a homeomorphism. Then the $A(x)$ for $x \in X$ are the simple subquotients of $A$.

**Theorem 10** (Kirchberg). Let $A$ and $B$ be separable, nuclear, $\mathcal{O}_\infty \otimes \mathbb{K}$-stable $C^*$-algebras with homeomorphisms $\text{Prim}(A) \cong X \cong \text{Prim}(B)$. Then any invertible element in $\text{KK}^X(A,B)$ lifts to a $^*$-isomorphism $A \to B$ over $X$.

Hence the representative in the previous theorem is unique up to isomorphism over $X$.

### Summary from the classification viewpoint

Classifying separable, nuclear, purely infinite, stable $C^*$-algebras with finite primitive ideal space $X$ and simple subquotients in the bootstrap class up to isomorphism over $X$ $\iff$ classifying objects of $\mathcal{B}(X)$ up to $\text{KK}^X$-equivalence.

### Open question

Is there a K-theoretic complete invariant that achieves this classification?

## 2 The Universal Coefficient Theorem

**Question**

Can we compute $\text{KK}^*_s(A,B)$ from $K^*_s(A)$ and $K^*_s(B)$?

**Theorem 11.** Let $A$ and $B$ be separable $C^*$-algebras, suppose that $A$ belongs to the bootstrap class.

Then there is a natural short exact sequence of $\mathbb{Z}/2\mathbb{Z}$-graded Abelian groups

$$\text{Ext}\left(K^*_{s+1}(A), K^*_s(B)\right) \to \text{KK}^*_s(A,B) \to \text{Hom}\left(K^*_s(A), K^*_s(B)\right).$$

- $I := \ker K^*_s$
Naturality implies that the Kasparov product
\[ KK(A, B) \otimes \text{Ext}(K_*(C, A)) \to \text{Ext}(K_*(C, B)) \]
descends to \( \text{Hom}(K_*(A), K_*(B)) \), that is, \( \mathcal{I} \circ \mathcal{I} = 0 \) in the bootstrap class.

Some applications

Corollary 12. If both \( A \) and \( B \) belong to the bootstrap category, then

- \( \alpha \in KK_0(A, B) \) is a KK-equivalence if and only if \( K_*(\alpha): K_*(A) \to K_*(B) \) is invertible.
- Any grading preserving group homomorphism \( K_*(A) \to K_*(B) \) lifts to an element of \( KK_*(A, B) \).
- An isomorphism \( K_*(A) \cong K_*(B) \) lifts to a KK-equivalence \( A \cong B \).

Theorem 13. Any pair of countable Abelian groups is the K-theory of some \( C^* \)-algebra in the bootstrap class.

Summary

K-theory is a complete invariant for KK-equivalence classes of \( C^* \)-algebras in the bootstrap class.

2.1 Proof of the UCT

\( \mathcal{T} \) Kasparov theory, viewed as a category
\( \mathcal{C} \) category \( \mathbb{Ab}_{Z/2} \) of countable \( Z/2 \)-graded Abelian groups
\( F \) functor \( F := K_*: \mathcal{T} \to \mathcal{C} \)
\( A \in \mathcal{C} \) \( A \) is an object of the category \( \mathcal{C} \)

Basic Lemma

If \( A \in \mathcal{C} \) is projective (free), there is \( F^+(A) \in \mathcal{T} \) with
\[ \mathcal{T}(F^+(A), B) \cong \text{Hom}(A, F(B)) \]
Moreover, \( F \circ F^+(A) \cong A \).

Equivalently: the left adjoint of \( F \) is defined on all projective objects of \( \mathcal{C} \).

Proof.

- Since \( F^+ \) is compatible with suspensions and direct sums, it suffices to check this for \( A = \mathbb{Z} \).
- Take \( F^+(\mathbb{Z}) = \mathbb{C} \) and use
\[ \text{Hom}(\mathbb{Z}, K_*(B)) \cong K_0(B) \cong KK_0(\mathbb{C}, B) \]
and \( K_*(\mathbb{C}) = \mathbb{Z} \). \( \square \)
Exact chain complexes and projective objects

Definition 14. A chain complex \((C_n, d_n)\) with entries in \(\mathcal{T}\) is \(F\)-exact if the chain complex \(F(C_n), F(d_n)\) is exact.

Definition 15. A separable \(C^\ast\)-algebra \(A \in \mathcal{T}\) is \(F\)-projective if the chain complex 
\[
\cdots \rightarrow \mathcal{T}_n(A, C_n) \xrightarrow{\mathcal{T}_n(A, d_n)} \mathcal{T}_n(A, C_{n-1}) \rightarrow \cdots
\]
is exact for any \(F\)-exact chain complex \((C_n, d_n)\).

Lemma 16. Objects of the form \(F \triangleright (A)\) for projective \(A \in \mathcal{C}\) are \(F\)-projective.

Proof. \(\mathcal{T}(F^\ast(A), B) \cong \mathcal{C}(A, F(B))\) and \(\mathcal{C}(A, \cup)\) is exact. \(\square\)

Description of projective objects

Theorem 17. The functors \(F\) and \(F^\ast\) restrict to an equivalence of categories 
\[
\{F\text{-projective objects in } \mathcal{T}\} \rightleftharpoons \{\text{projective objects in } \mathcal{C}\}.
\]

Proof.

- \(F^\ast\) is a functor and \(F \circ F^\ast(A) \cong A\).
- Let \(B \in \mathcal{T}\) be \(F\)-projective, let \(\alpha : A \rightarrow F(B)\) be a quotient map with projective \(A\).
- \(\mathcal{C}(A, F(B)) \cong \mathcal{T}(F^\ast(A), B)\) maps \(\alpha \mapsto \alpha^\ast\)
- \(\alpha^\ast\) is split epimorphism because \(B\) is \(F\)-projective and \(F(\alpha^\ast)\) is surjective.
- \(B = F^\ast(eA)\) for an idempotent map \(e : A \rightarrow A\) because \(F^\ast\) is fully faithful.
- \(eA\) is again projective. \(\square\)

Lifting projective resolutions

Definition 18. An \(F\)-projective resolution of \(A \in \mathcal{T}\) is an \(F\)-exact chain complex 
\[
\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0 \rightarrow \cdots
\]
with \(F\)-projective \(P_n\) for \(n \in \mathbb{N}\).

Theorem 19 (Geometric resolutions). A projective resolution of \(F(A)\) in \(\mathcal{C}\) lifts uniquely up to isomorphism to an \(F\)-projective resolution in \(\mathcal{T}\).

Proof. Let \(P_n := F^\ast(B_n)\) for a projective resolution \((B_n)\), use \(\mathcal{C}(B_0, F(A)) \cong \mathcal{T}(P_0, A)\) to get the map \(P_0 \rightarrow A\). \(\square\)
Proof of the UCT for plain Kasparov theory

Crucial fact

Any object of \( \mathcal{C} = \mathbb{A}b_{\mathbb{Z}/2} \) has a free resolution of length 1.

- For \( A \in \mathfrak{T} \), choose a free resolution of \( F(A) \) of length 1 and lift it to an \( F\)-projective resolution \( 0 \to P_1 \xrightarrow{d_1} P_0 \to A \) in \( \mathfrak{T} \).

- Since \( \mathfrak{T} \) is triangulated, the map \( d_1 \) embeds in an exact triangle \( P_1 \xrightarrow{d_1} P_0 \to A' \to P_1[1] \).

To get this for \( \mathfrak{T} = \text{KK} \), view \( d_1 \in \text{KK}_1(\mathcal{C}_0(\mathbb{R}) \otimes P_1, P_0) \) as the equivalence class of a \( C^* \)-algebra extension \( P_0 \to A' \to \mathcal{C}_0(\mathbb{R}) \otimes P_1 \) with a completely positive contractive section.

- An analysis of the long exact sequence for this exact triangle yields a short exact sequence

\[
\text{Ext}(K_{s+1}(A), K_s(B)) \to \text{KK}_s(A', B) \to \text{Hom}(K_s(A), K_s(B)).
\]

because \( \text{KK}_s(P_0, B) \cong \text{Hom}(F(P_0), F(B)) \).

Proof of the UCT continued

- There is \( f \in \mathfrak{T}(A', A) \) making the following diagram commute:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & P_0 \\
\| & & \| \\
P_1 & \xrightarrow{d_1} & P_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & P_0 \\
\| & & \| \\
\| & & \| \\
A & \xrightarrow{f} & A' \\
\end{array}
\]

- \( f \) induces an isomorphism on K-theory.

- The class of \( X \in \mathfrak{T} \) for which \( f \) induces an isomorphism on \( \mathfrak{T}_s(X, \_ \_ \_) \) contains \( \mathbb{C} \) and is closed under isomorphism, suspensions, direct sums, and mapping cones.

- If \( A \) belongs to the bootstrap class, then \( f \) is invertible.

2.2 Abstraction

Most parts of the proof of the UCT work in all kinds of equivariant KK-theories.

Dramatis Personae

\( \mathfrak{T} \) a triangulated category

\( \mathcal{C} \) an Abelian category with a translation automorphism

\( F \) a homological functor \( \mathfrak{T} \to \mathcal{C} \) that intertwines the translation automorphisms
Definition 20. We call $F$ an invariant without hidden symmetries if

- $\mathcal{C}$ has enough projective objects;
- for $A \in \mathcal{C}$ projective, there is $F^+(A) \in \mathcal{T}$ with $\mathcal{T}(F^+(A), B) \cong \mathcal{C}(A, F(B));$
- $F \circ F^+(A) \cong A$ for all projective $A \in \mathcal{C}.$

An example of hidden symmetries

Dramatis Personæ

$\mathcal{T} \text{ KK}^G$ for some discrete group $G$

$\mathcal{C} \text{ Ab}_{\mathbb{Z}/2}^\mathbb{Z},$ the translation automorphism shifts the grading

$F \ F(A, \alpha) = K_*(A):$ forget group action and take K-theory

- $\text{KK}^*_*(\mathcal{C}_0(G, A), B) \cong \text{KK}_*(A, B)$ shows that $F^+(\mathbb{Z}) = \mathcal{C}_0(G)$ works.
- But $FF^+(\mathbb{Z}) = K_*(\mathcal{C}_0(G)) = \mathbb{Z}[G]$ is too big.
- A poor invariant like this can be refined uniquely to one without hidden symmetries:

Theorem 21. If there are enough $F$-projective objects, there is a unique stable homological functor $F'$ without hidden symmetries and $\ker F = \ker F'.$

The example rectified

Dramatis Personæ

$\mathcal{T} \text{ KK}^G$ for some discrete group $G$

$\mathcal{C} \text{ category of countable Z/2-graded Z[G]-modules}$

$F \ F(A, \alpha) = (K_*(A), K_*(\alpha)): \text{ take K-theory and remember the induced action of G on K}_*(A)$

- $\text{KK}^*_*(\mathcal{C}_0(G), B) \cong \text{KK}_*(\mathcal{C}, B) \cong \text{Hom}_G(\mathbb{Z}[G], K_*(B))$ shows that $F^+(\mathbb{Z}[G]) = \mathcal{C}_0(G)$ works.
- $FF^+(\mathbb{Z}[G]) = K_*(\mathcal{C}_0(G)) \cong \mathbb{Z}[G]$ shows that $F \cong \mathcal{C}_0(G)$ works.
- As above, this yields $F^+$ first for all free modules, then for all projective modules.
- This time the functor $F$ has no hidden symmetries.
Projective objects and abstract bootstrap class

Define $F$-exact chain complexes and $F$-projective objects in $\mathcal{T}$ as above.

Lemma 22. $F$ and $F^+$ restrict to an equivalence of categories

$$\{F\text{-projectives in } \mathcal{T}\} \cong \{\text{projectives in } \mathcal{C}\}.$$ 

Theorem 23. Projective resolutions of $F(A)$ lift uniquely up to isomorphism to $F$-projective resolutions of $A$.

Definition 24 (Abstract bootstrap category). Let $\mathcal{B}$ be the localising subcategory of $\mathcal{T}$ generated by the $F$-projective objects.

Let $\mathcal{B}^\perp$ consist of all $A$ with $F(A) = 0$.

Theorem 25. $\mathcal{B}$ and $\mathcal{B}^\perp$ are complementary if $F$ commutes with $\oplus$.

The abstract UCT

Theorem 26 (Universal Coefficient Theorem). If $F(A)$ has a projective resolution of length 1 and $A \in \mathcal{B}$, $B \in \mathcal{T}$, there is a UCT exact sequence

$$\text{Ext}^1_\mathcal{C}(F(A)[1], F(B)) \rightarrow \mathcal{T}(A, B) \rightarrow \text{Hom}_\mathcal{C}(F(A), F(B)).$$

Theorem 27. Any object of $\mathcal{C}$ with a projective resolution of length 1 is $F(A)$ for some $A \in \mathcal{B}$.

Open problem

The range of $F$ may be smaller than $\mathcal{C}$. Is it always an exact subcategory of $\mathcal{C}$? This would often allow to describe the range of $F$.

What happens for long projective resolutions?

- Without projective resolutions of length 1, the short exact sequence becomes a \textit{spectral sequence}.

- Its higher differentials provide \textit{obstructions} for lifting a map $F(A) \rightarrow F(B)$ back to $\mathcal{T}$.

- We should expect that there are $A, B \in \mathcal{B}$ with $F(A) \cong F(B)$ but $A \not\cong B$.

Theorem 28. Let $\alpha \in \mathcal{T}(A, B)$. If $F(\alpha)$ is invertible and $A, B \in \mathcal{B}$, then $\alpha$ is invertible.

Equivalently, $F$ restricts to a \textit{rigid} invariant on $\mathcal{B}$. 

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Summary: Homological algebra

- If our invariant $F$ has hidden symmetries, there is a unique way to make these explicit, replacing $F$ by an invariant without hidden symmetries.
- If $F: \mathcal{T} \to \mathcal{C}$ has no hidden symmetries, then the classification problem can be answered using homological algebra in $\mathcal{C}$.
- The bootstrap class and the object-kernel of $F$ are complementary if $F$ commutes with direct sums. Thus localisation at $F$ is easy to understand.
- Projective resolutions in $\mathcal{C}$ lift uniquely.
- If $F(A)$ has a projective resolution of length 1 and $A$ belongs to the bootstrap class for $F$, then $A$ satisfies a Universal Coefficient Theorem.
- Its ingredients are the derived functors in the target category of $F$.

3 An invariant for Kirchberg’s bivariant K-theory

Question
Which invariant should we use to study $KK^X$ for a (finite) topological space $X$?
Is there a complete invariant?
- $\bigoplus_{x \in X} K_*(A(U_x))$ is as rigid as possible: it gives the right bootstrap class.
- But it is never complete unless $X$ is discrete.
- $\bigoplus_{U \in \mathcal{O}(X)} K_*(A(U))$ is not better: it still fails to be complete for the non-Hausdorff two-point space.

Definition 29. Filtrated K-theory is the invariant without hidden symmetries associated to $F(A) := \bigoplus_{Y \in \mathcal{L}(X)} K_*(A(Y))$, where $\mathcal{L}(X)$ denotes the locally closed subsets of $X$.

Filtrated K-theory

Theorem 30. Filtrated K-theory is a complete invariant if $\mathcal{O}(X)$ is totally ordered.

Theorem 31. Filtrated K-theory is not complete for $Y_3 = \{0, 1, 2, 3\}$ with $0 < 1, 2, 3$ and no relation among $\{1, 2, 3\}$.
For this space, adding another K-theory group to $F$ refines filtrated K-theory to a complete invariant.

Question
Can we always refine filtrated K-theory to a manageable and complete invariant?

What I expect
This should become impossible, in a sense to be made precise, for sufficiently complicated spaces, maybe already for $Y_n$, defined like $Y_3$ with $n \geq 7$. 9
Where problems could come from

- In the examples I know, the K-theory functors that are used to build a complete invariant correspond to irreducible modules over the quiver algebra of the partial order \(\prec\) on \(X\).
- In this algebraic context, there are three cases:
  - **finite** there are only finitely many irreducible modules
  - **tame** there are infinitely many irreducible modules, but they can still be classified
  - **wild** no classification of irreducible modules is possible
- The examples where things work all correspond to the finite case.

How to find the hidden symmetries?

**Problem**
Find the functor without hidden symmetries attached to \(F(A) := \bigoplus_{Y \in LC(X)} K_*(A(Y))!\)

- Find a representing object for this functor or, equivalently, for the functors \(A \mapsto K_*(A(Y))\), that is,
  \[ KK_*^X(R_Y, A) \cong K_*(A(Y)) \]
- Then \(R := \bigoplus_{Y \in LC(X)} R_Y\) satisfies \(KK_*^X(R, A) \cong F(A)\).
- The graded ring \(N^{T_*} := KK_*(R, R)\) acts naturally on \(F(A)\) by Kasparov product.

**Theorem 32.** Representing objects \(R_Y\) exist and can be constructed explicitly, and filtrated K-theory is the functor \(A \mapsto \bigoplus_{Y \in LC(X)} K_*(A(Y))\), viewed as a functor to the category \(\mathcal{C}\) of countable \((\mathbb{Z}/2\text{-graded})\) \(N^{T_*}\)-modules.