Universal coefficient theorems for Kirchberg’s bivariant K-theory I

Ralf Meyer

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1 Non-commutative topology

1.1 The role of Kasparov theory and K-theory

Definition 1. A non-commutative homology theory is a functor on a category of (separable) \( \mathbb{C}^* \)-algebras (with extra structure) that is

- \( \mathbb{C}^* \)-stable (Morita invariant)
- split-exact
- homotopy invariant
- has Puppe exact sequence for mapping cones

Example 2. K-theory is a non-commutative homology theory for \( \mathbb{C}^* \)-algebras. It maps separable \( \mathbb{C}^* \)-algebras to the category \( \text{Ab}_{\mathbb{Z}/2} \) of \( \mathbb{Z}/2 \)-graded countable Abelian groups.

Example 3. \( \text{KK}^G \) is a non-commutative homology theory for \( \mathbb{C}^* \)-algebras with a \( G \)-action.

Theorem 4 (Joachim Cuntz and Nigel Higson). Bivariant KK-theory is the universal \( \mathbb{C}^* \)-stable, split-exact functor on the category of separable \( \mathbb{C}^* \)-algebras.

That is, a functor from the category of separable \( \mathbb{C}^* \)-algebras to some additive category factors through KK if and only if it is \( \mathbb{C}^* \)-stable and split-exact, and this factorisation is unique if it exists.

Equivariant versions of KK are characterised by analogous universal properties.

Corollary 5. \( \mathbb{C}^* \)-stability and split-exactness \( \implies \) homotopy invariance, Bott periodicity, Connes–Thom Isomorphism, \( \ldots \)
The central role of $K$-theory

**Theorem 6.** Let $X$ and $Y$ be locally compact spaces with

$$K_\ast(X) \cong K_\ast(Y).$$

Then $C_0(X)$ and $C_0(Y)$ are $KK$-equivalent.

Thus $F(C_0(X)) \cong F(C_0(Y))$ if $F$ is $C^\ast$-stable split-exact.

**Proof.** This follows from the Universal Coefficient Theorem. 

**Bad news**
All the intricacies of the stable homotopy category disappear in the non-commutative setting.

**Good news**
All the intricacies of the stable homotopy category disappear in the non-commutative setting.

**Equivariant Kasparov theory**

- KK has become a tool instead of an object of study.
- You can prove theorems with KK, but not about KK.
- For each additional structure $C^\ast$-algebras can carry, there is an appropriate version of KK:
  - group actions of locally compact groups
  - groupoid actions of locally compact groupoids
  - coactions of locally compact quantum groups
  - $C(X)$-algebras
  - $C^\ast$-algebra bundles over non-Hausdorff spaces
  
  This situation generates Kirchberg’s bivariant $K$-theory.

- These equivariant bivariant $K$-theories are more intricate—you may also prove theorems about them.

**1.2 Commutative versus non-commutative topology**

**Why is non-commutative topology so effective?**

**Question**
Some results in topology can, so far, be proved only with $C^\ast$-algebra methods.

*How can that be?*

**Possible answer**
Applications of $C^\ast$-algebras in topology usually involve spaces with an extra structure like a group action.

The interaction of the extra structure with the topology of the space becomes much simpler in the non-commutative setting.
A hopeless task

Exercise
Classify simplicial complexes up to (stable) homotopy equivalence! — Forget it!

Theorem 7. K-theory is a complete invariant up to KK-equivalence for \( C^* \)-algebras in the bootstrap class.
Two \( C^* \)-algebras in the bootstrap class are KK-equivalent if and only if they have isomorphic K-theory.

Any pair of countable Abelian groups is the K-theory of some \( C^* \)-algebra in the bootstrap class.

A ridiculous question

Question
If a \( G \)-map \( f : A \to B \) is a (stable) homotopy equivalence, is it automatically a \( G \)-equivariant (stable) homotopy equivalence? — Forget it!

Theorem 8 (Extra strong Baum–Connes property). Let \( G \) be a locally compact group that acts properly by affine isometries on a Hilbert space. If \( \alpha \in \text{KK}^G(A, B) \) becomes invertible in \( \text{KK}^H \) for each compact subgroup \( H \subseteq G \), then \( \alpha \) is invertible in \( \text{KK}^G \).

Corollary 9 (Homotopic actions are equivalent). Let \( G \) be a torsion-free group with the extra strong Baum–Connes property. Then homotopic \( G \)-actions on a separable \( C^* \)-algebra \( A \) are KK\(^G\)-equivalent.

Example 10. Non-commutative tori of the same dimension are KK-equivalent.

A general line of inquiry

• We want to study equivariant homology theories on a category of \( C^* \)-algebras with some extra structure.

• First we need a bivariant K-theory in this setting.

• Secondly, we need an invariant that we use to probe this equivariant KK-category.

Goal
Compute the equivariant KK-theory and other equivariant homology theories using the chosen invariant.

1.3 The rigidity question

What can we know?

Rigidity question
If the invariant \( F(A) = 0 \) vanishes, does it follow that \( A \) is equivariantly KK-equivalent to \( 0 \)?

Equivalently, if a morphism \( \alpha \) induces an isomorphism on the invariant \( F \), is it already invertible?
Definition 11. We call the invariant $F$ rigid if $F(A) = 0 \iff A \cong 0$.

Example 12 (Extra strong Baum–Connes property). Let $G$ be a locally compact group, let $F$ be the family of restriction functors $\text{KK}^G \to \text{KK}^H$ for $H \subseteq G$ compact.

This invariant is rigid if and only if the group $G$ has the extra strong Baum–Connes property.

Rigid invariants

Example 13. Let $G$ be a connected Lie group with torsion-free fundamental group.

• Let $T \subseteq G$ be a maximal torus. The forgetful functor $\text{KK}^G \to \text{KK}^T$ is a rigid invariant.

• The crossed product functor (descent)

$$\text{KK}^G \to \text{KK}, \quad A \mapsto A \rtimes G$$

is rigid.

Example 14 (Bootstrap category). There exist separable $C^*$-algebras with $K_*(A) = 0$ but $\text{KK}_0(A, A) \neq 0$.

Localisation at the invariant

• We want to focus on the part of our bivariant K-theory that our invariant can detect.

• There is a general process to do this: localisation of triangulated categories

• Often it works as follows.

Theorem 15. Let $\mathcal{T}$ be a triangulated category and let $F$ be a stable homological functor.

$$\mathcal{N} := \{ A \mid F(A) = 0 \}$$

$$\mathcal{N}^\perp := \{ A \mid \mathcal{T}(A, B) = 0 \quad \forall B \in \mathcal{N} \}$$

If $\mathcal{N} \cup \mathcal{N}^\perp$ generates $\mathcal{T}$, then $\mathcal{T}/\mathcal{N}$ is equivalent to $\mathcal{N}^\perp$.

Example 16. This works for the K-theory functor on $\text{KK}$, where $\mathcal{N}^\perp$ is the bootstrap class or UCT class.

1.4 The classification question

Lemma 17. After localisation, the invariant $F$ becomes rigid, that is, it detects zero objects and isomorphisms: $\alpha$ invertible $\iff F(\alpha)$ invertible

Classification question

If $F(A) \cong F(B)$, does it follow that $A \cong B$?

If yes, can you also describe the range of $F$?

Definition 18. We call the invariant $F$ complete if the answer to both questions is “Yes” after localisation.

Example 19. K-theory is a complete invariant on $\text{KK}$.

Example 20. The forgetful functor $\text{KK}^\mathbb{Z} \to \text{KK}$ is both rigid and complete.
Necessity of a Universal Coefficient Theorem

For most equivariant situations, we do not expect a manageable complete invariant to exist.

(The identity invariant is always complete and rigid.)

Open question

Is there a manageable complete invariant for $\mathbb{Z}^2$-actions?

**Theorem 21** (No-Go Theorem). Let $\mathcal{I} := \ker F$ be the ideal of morphisms in $\mathfrak{T}$ annihilated by the invariant $F$. Assume enough $\mathcal{I}$-projective objects.

$\mathcal{I} \circ \mathcal{I} \neq 0 \implies$ there are $A, B$ with $F(A) \cong F(B)$ but $A \ncong B$.

**Corollary 22.** Let $F$ be a complete, rigid invariant on $\mathfrak{T}$. Then there is a natural exact sequence

$$\mathcal{I}/\mathcal{I}^2(\mathcal{A}, \mathcal{B}) \rightarrow \mathfrak{T}(\mathcal{A}, \mathcal{B}) \rightarrow \mathfrak{T}/\mathcal{I}(\mathcal{A}, \mathcal{B}).$$

Homological algebra in triangulated categories

Questions

When is $\mathcal{I}^2 = \mathcal{I} \circ \mathcal{I} = 0$?

Are $\mathfrak{T}/\mathcal{I}$ and $\mathcal{I}/\mathcal{I}^2$ derived functors?

- There is a general machinery for homological algebra in triangulated categories using an ideal such as $\ker F$.

- It yields an Abelian approximation $\mathfrak{C}$ to our triangulated category $\mathfrak{T}$ and a homological functor $F' : \mathfrak{T} \rightarrow \mathfrak{C}$ such that homological algebra in $\mathfrak{C}$ lifts back to $\mathfrak{T}$.

- A universal coefficient exact sequence exists for $A$ if $F'(A)$ has a projective resolution of length 1.

- This need not be the case and may be hard to check.

Difficult problem

Refine an incomplete invariant to make it (more) complete!

2 \hspace{1em} $C^*$-algebra bundles over non-Hausdorff spaces

- We will first reformulate the notion of a $C(X)$-algebra in a way suitable for non-Hausdorff spaces.

- This leads to a definition of a $C^*$-algebra (bundle) over a non-Hausdorff space, which we illustrate by some examples.

- Then we introduce Kirchberg’s bivariant KK-theory for such $C^*$-algebra bundles.
2.1 Equivalent definitions in the Hausdorff case

**Theorem 23.** Let $X$ be a Hausdorff topological space and $A$ a $C^*$-algebra. The following additional structures on $A$ are equivalent:

- a non-degenerate $^*$-homomorphism from $C_0(X)$ to the centre of the multiplier algebra of $A$
- a non-degenerate $^*$-homomorphism $C_0(X, A) \to A$; it has a class in $\text{KK}(C_0(X, A), A)$
- a continuous map $\text{Prim}(A) \to X$, where $\text{Prim}(A)$ is the primitive ideal space of $A$
- a map from the lattice of open subsets of $X$ to the lattice of ideals in $A$ that commutes with arbitrary suprema and finite infima

The last two conditions make sense for non-Hausdorff spaces.

2.2 Definition and Examples

**Definition 24.** Let $X$ be a topological space. A $C^*$-algebra over $X$ is a $C^*$-algebra $A$ together with a continuous map $\psi: \text{Prim}(A) \to X$.

**Theorem 25.** The lattice $\mathcal{O}(\text{Prim}A)$ of open subsets of $\text{Prim}(A)$ is isomorphic to the lattice $\mathbb{I}(A)$ of ideals in $A$.

**Lemma 26.** A continuous map $\psi: \text{Prim}(A) \to X$ is equivalent to a lattice morphism $\mathcal{O}(X) \to \mathcal{O}(\text{Prim}A) \cong \mathbb{I}(A)$ that preserves arbitrary suprema (provided $X$ is sober).

Finite $T_0$-spaces and partial orders

- We will concentrate on finite topological $T_0$-spaces here.
- Let $x \preceq y$ if $\overline{\{x\}} \subseteq \overline{\{y\}}$, this is a partial order on $X$.

**Lemma 27.** A subset $S \subseteq X$ is closed $\iff$ $x \prec y \in S$ implies $x \in S$.
A subset $S \subseteq X$ is open $\iff$ $x \succ y \in S$ implies $x \in S$.

**Important observation**

$T_0$-topologies and partial orders on finite sets are equivalent.

**Example 28.** Consider $X_n = \{1, \ldots, n\}$ with the total order $1 \succ 2 \succ \cdots \succ n$.

- The open subsets are $U_k := \{1, \ldots, k\}$ for $k = 0, \ldots, n$.
- A $C^*$-algebra over $X_n$ is a $C^*$-algebra $A$ together with an increasing chain of ideals
  $$0 = A(U_0) \triangleleft A(U_1) \triangleleft \cdots \triangleleft A(U_{n-1}) \triangleleft A(U_n) = A.$$  
- For $n = 2$, we get a chain $I \triangleleft A$, so that we study $C^*$-algebra extensions.
Example 29. Consider $Y_n = \{0, 1, \ldots, n\}$ with the partial order $0 < 1, 2, \ldots, n$ and no further relation between $1, \ldots, n$.

- The open subsets of $Y_n$ are all subsets of $\{1, \ldots, n\}$ and $Y_n$ itself.
- The lattice of open subsets is already generated by the singletons $U_j := \{j\}$ for $j = 1, \ldots, n$ and $U_0 := Y_n$.
- A $C^*$-algebra over $Y_n$ is a $C^*$-algebra $A$ together with $n$ orthogonal ideals $A(U_j)$, $j = 1, \ldots, n$, that is, $A(U_j) \cap A(U_k) = \{0\}$.
- This is equivalent to a $C^*$-algebra extension $I \hookrightarrow A \twoheadrightarrow A/I$ with a direct sum decomposition $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$.

2.3 Kirchberg’s bivariant K-theory

Definition 30. A morphism between two $C^*$-algebras over $X$ is a $^*$-homomorphism $f : A \to B$ that maps $A(U)$ to $B(U)$ for all open subsets $U \subseteq X$.

Theorem 31. If $X$ is Hausdorff and $A$ and $B$ are $C^*$-algebras over $X$ (equivalently, $C_0(X)$-algebras), then the morphisms are the $C_0(X)$-linear $^*$-homomorphisms.

Definition 32 (Eberhard Kirchberg). Let $A$ and $B$ be $C^*$-algebras over $X$ and let $(E, F)$ be a Kasparov cycle for $KK(A, B)$. We call it a Kasparov cycle over $X$ if $A(U) \cdot E \subseteq E \cdot B(U)$ for all $U \in \mathcal{O}(X)$.

$KK^X(A, B)$ is the group of homotopy classes of such Kasparov cycles over $X$.

Formal properties

- Since we only impose restrictions on the Hilbert module, the Kasparov product works as usual for Kasparov cycles over $X$.
- There is an exterior product

$$KK^X(A, B) \otimes KK(D, E) \to KK^X(A \otimes D, B \otimes E).$$

This allows us to carry over properties like $C^*$-stability and Bott periodicity from $KK$ to $KK^X$.

- There are long exact sequences in both variables for an extension $I \hookrightarrow E \twoheadrightarrow Q$ of $C^*$-algebras over $X$ with a completely positive contractive section over $X$, that is, the section maps $Q(U)$ to $E(U)$ for all $U \in \mathcal{O}(X)$.

In particular, $KK^X$ is split-exact in both variables.

Theorem 33 (Universal property). Kirchberg’s bivariant K-theory is the universal split-exact $C^*$-stable functor on the category of separable $C^*$-algebras over $X$. 

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3 The bootstrap class

3.1 The classical case

**Definition 34.** The bootstrap class $\mathcal{B}$ in KK is the smallest class of $C^*$-algebras containing the generator $\mathbb{C}$ and closed under the following operations:

- suspensions $A \mapsto C_0(\mathbb{R}, A)$
- countable direct sums
- KK-equivalence
- mapping cones: if $f : A \to B$ is a $^*$-homomorphism and $A$ and $B$ belong to the bootstrap class, so does cone$(f)$.

**Lemma 35.** The bootstrap class $\mathcal{B}$ is also closed under

- extensions with completely positive section
- inductive limits with completely positive approximations
- fibred products of $A \to B \leftarrow A'$ if $p$ is surjective with a completely positive section

**Theorem 36.** Let $G$ be a locally compact group with the extra strong Baum-Connes property and let $A$ be a $G$-$C^*$-algebra.

$A \rtimes H$ in $\mathcal{B}$ for all compact subgroups $H \subseteq G \implies A \rtimes G$ in $\mathcal{B}$

**Theorem 37.** Let $G$ be a connected Lie group with torsion-free fundamental group, let $T \subseteq G$ be a maximal torus, and let $A$ be a $G$-$C^*$-algebra.

$A \rtimes G$ in $\mathcal{B} \iff A \rtimes T$ in $\mathcal{B} \implies A$ in $\mathcal{B}$

Further properties of the bootstrap class

**Question**

Are all separable nuclear $C^*$-algebras in the bootstrap class?

**Example 38** (Georges Skandalis). If $G$ is a cocompact lattice in $\text{Sp}(n,1)$, then $C^*_r(G)$ does not belong to the bootstrap class.

**Reason:** The image of $\gamma \in \text{KK}_0^G(\mathbb{C}, \mathbb{C})$ in $\text{KK}_0(C^*_rG, C^*_rG)$ acts identically on $K_* (C^*_rG)$, but it is not invertible.

**Exercise**

The bootstrap class is not closed under crossed products by actions of $\mathbb{Z}/2$.

**Hint (Chris Phillips)**

There is an action of $\mathbb{Z}/2$ on a contractible, commutative $C^*$-algebra with $K_*(\mathbb{Z}/2 \rtimes A) \neq 0$. Tensor this with Skandalis’ counterexample.
3.2 Bootstrap class over a space

Localisation at K-theory over a space $X$

$X$: a topological $T_0$-space.

$F(A): \bigoplus_{U \in \mathcal{O}(X)} K_*(A(U))$ as functor on $KK(X)$

$\mathcal{B}(X)^\perp: \{ A \in KK(X) \mid F(A) = 0 \}$

- If $A \in \mathcal{B}(X)^\perp$, then any computation that is based on K-theoretic invariants of $A$ must give zero.

- The bootstrap class is supposed to be the localisation of $KK(X)$ at the subcategory $\mathcal{B}(X)^\perp$: it is a “quotient” of $KK(X)$ where
  - objects of $\mathcal{B}(X)^\perp$ become zero;
  - an element in $KK^X_0(A,B)$ becomes invertible if it induces an isomorphism $F(A) \cong F(B)$.

**Lemma 39.** Let $\mathcal{U}$ be an open covering of $X$.
If $K_*(A(U)) = 0$ whenever $U \in \mathcal{O}(X)$ is contained in some $V \in \mathcal{U}$, then $A \in \mathcal{B}(X)^\perp$.

Small open subsets versus fibres

- If $X$ is a Hausdorff space, then a $C^*$-algebra over $X$ has fibres $A(x) := A/A(X \setminus \{x\})$ for all $x \in X$.

- For continuous bundles of nuclear $C^*$-algebras, Marius Dadarlat’s lecture showed that we can detect invertibility of a $KK^X$-morphism by its restrictions to these fibres.

- This suggests that a $KK^X$-morphism should become invertible in $\mathcal{B}(X)$ once its restrictions to the fibres in $KK_*(A(x), B(x))$ are invertible for all $x \in X$.
I do not know how to prove this.

- Such a reduction to fibres is impossible for general non-Hausdorff spaces because bundles over such spaces need not have any “fibres” at all.