Asymptotic Unitary Equivalence and Classification of Simple $C^*$-algebras

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U.S.A.
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Let \( A \in \mathcal{N} \) be a unital simple \( C^* \)-algebra. Define

\[
\text{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \gamma_A),
\]

where \( \gamma_A : T(A) \to S(K_0(A)) \) is defined by \( \gamma_A(\tau)([p]) = \tau(p) \) for all projections in \( A \otimes \mathcal{K} \).
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Let $A$ and $B$ be two unital simple $C^*$-algebras in $\mathcal{N}$. We say

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$$\gamma_A((\lambda^T)^{-1}(\tau))(x) = \gamma_B(\tau)(\lambda_0(x))$$

for $x \in K_0(A)$ and $\tau \in T(B)$. 

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where \( A_n = \bigoplus_{i=1}^{k(n)} P_{(i,n)} M_{R(i,n)}(C(X_{(i,n)})) P_{(i,n)}, \)

and \( P_{(i,n)} \in M_{R(i,n)}(C(X_n)) \) is a projection

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\[ \star \text{. If } A \text{ is simple, we say } A \text{ has slow dimension growth if} \]

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\( A \) is said to have no dimension growth, if there is an integer \( m > 0 \) such that

\[ \dim X(i,n) \leq m \]

for all \( i \) and \( n \).
Theorem

(Elliott, Gong and Li, Invent. Math. 168 (2007), 249-320)

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Let $A$ and $B$ be two unital simple AH-algebras with no dimension growth. Then $A \cong B$ if and only if

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Definition

Let $A$ be a unital simple $C^*$-algebra. Then $A$ has tracial rank no more than one and we will write $TR(A) \leq 1$ if the following holds:

For any $\epsilon > 0$, and any finite subset $F \subset A$ containing a nonzero element $a \in A^+$, there is a $C^*$-subalgebra $C$ in $A$ where $C = \bigoplus_{k=1}^{\infty} M_{n_k}(C(X_i))$, where each $X_i$ is a finite CW complex with dimension no more than one

such that

(i) $\|px - xp\| < \epsilon$ for $x \in F$,

(ii) $pxp \in \epsilon C$ for $x \in F$, and

(iii) $1 - p$ is equivalent to a projection in $aAa$.

In the above definition, if $C$ can be chosen to be a finite dimensional $C^*$-subalgebra then $TR(A) = 0$.

If $TR(A) \leq 1$ but $TR(A) \neq 0$, then we will write $TR(A) = 1$. 

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In the above definition, if $C$ can be chosen to be a finite dimensional $C^*$-subalgebra then $TR(A) = 0$. If $TR(A) \leq 1$ but $TR(A) \neq 0$ then we will write $TR(A) = 1$. 

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where \( q = (1 - p) \).

\[
\begin{pmatrix}
qAq \\
\end{pmatrix}
\]

\[
B
\]

\[TR(A) = 0, \text{ if } \dim B < \infty, \quad TR(A) \leq 1, \text{ if } B = \bigoplus_{i=1}^{n} M_{R_i}(C(X_i))) \text{ with } \dim X_i \leq 1.\]
Theorem

(L–2001) Let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$. Then

- $A$ is quasidiagonal;
- $A$ has real rank zero, or real rank one;
- $A$ has stable rank one;
- $K_0(A)$ is weakly unperforated and with Riesz interpolation property;
- $A$ has the fundamental comparison property: if $p, q \in A$ are two projections and $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \sim q'$ with $q' \leq q$. 

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Theorem

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(G. Gong) Every unital simple AH-algebra with no dimension growth has tracial rank no more than one.

In fact, every unital simple AH-algebra with (very) slow dimension growth has tracial rank no more than one.
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For a unital simple AH-algebra, the following are equivalent:

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(iv) \( A \) has real rank zero, stable rank one and has weakly unperforated \( K_0(A) \).

That (iii) \( \rightarrow \) (i) was proved by Elliott and Gong (at least for the no dimension growth case).
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Theorem

(L-2003) Let $A$ be a unital simple $C^*$-algebra which is locally type I.

Suppose that $A$ has a unique tracial state, real rank zero, stable rank one and weakly unperforated $K_0(A)$. Then $\text{TR}(A) = 0$.
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(N. C. Phillips and L - 2004) Let $X$ be an infinite compact metric space with finite covering dimension and let $\alpha: X \to X$ be a minimal homeomorphism. Denote $A_\alpha = C(X) \rtimes \mathbb{Z}$. Then $\text{TR}(A_\alpha) = 0$ if and only if $\rho(K_0(A_\alpha))$ is dense in $\text{Aff}(T(A_\alpha))$. 

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Theorem

(L—2007)

Let $A, B \in \mathcal{N}$ be two unital simple $C^*$-algebras with $TR(A) \leq 1$ and $TR(B) \leq 1$. Then $A \sim B$ if and only if $(K_0(A), K_0(A) + 1, T(A)) \sim (K_0(B), K_0(B) + 1, T(B))$. 

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Let $A, B \in \mathcal{N}$ be two unital simple $C^*$-algebras with $TR(A) \leq 1$ and $TR(B) \leq 1$. Then $A \cong B$ if and only if

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Asymptotic Unitary Equivalence and Classification of Simple $C^*$-algebras

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\( \mathcal{Z} \) (the Jiang-Su algebra) is a unital simple separable amenable \( C^* \)-algebra (an ASH-algebra) in \( \mathcal{N} \) such that

\[
\text{Ell}(\mathcal{Z}) = \text{Ell}(\mathbb{C}).
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\* $\mathcal{Z}$ (the Jiang-Su algebra) is a unital simple separable amenable C*-algebra (an ASH-algebra) in $\mathcal{N}$ such that

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Let $p$ and $q$ be a pair of relatively prime supernatural numbers of infinite type.

Let

$$\mathcal{Z}_{p,q} = \{ f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1_{M_q}, f(1) \in 1_{M_p} \otimes M_q \}$$

One has $\mathcal{Z} = \lim_{n \to \infty} (\mathcal{Z}_{p,q})$. So $A \otimes \mathcal{Z} = \lim (A \otimes \mathcal{Z}_{p,q}, \text{id}_A \otimes \alpha)$. 

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One has

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\mathcal{Z} = \lim_{n \to \infty} (\mathcal{Z}_{p,q}, \alpha).
\]

So

\[
A \otimes \mathcal{Z} = \lim(A \otimes \mathcal{Z}_{p,q}, id_A \otimes \alpha).
\]
Let \( \varphi : A \otimes \mathcal{Z}_{p,q} \to B \otimes \mathcal{Z}_{r,s} \) be a homomorphism. Denote by \( \Gamma(\varphi) : \) \( (K_0(A), K_0(A) + \mathbb{Z}, \mathbb{Z}[A], K_1(A), T(A)) \to (K_0(B), K_0(B) + \mathbb{Z}, \mathbb{Z}[B], K_1(B), T(B)) \) be the map induced by \( \varphi \) which is the triple: 
\[
(\varphi^*_0, \varphi^*_1, \varphi^*_T)
\]
where \( \varphi^*_T : T(B) \to T(A) \) defined by \( \varphi^*_T(\tau)(a) = \tau \circ \varphi(a) \) for all \( a \in A \).

Let \( \Lambda : \operatorname{Ell}(A) \cong \operatorname{Ell}(B) \). Then \( \Lambda \) induces \( \Lambda^p : (K_0(A \otimes M_p), K_0(A \otimes M_p) + \mathbb{Z}[A \otimes M_p], K_1(A \otimes M_p), T(A \otimes M_p)) \to (K_0(B \otimes M_p), K_0(B \otimes M_p) + \mathbb{Z}[B \otimes M_p], K_1(B \otimes M_p), T(B \otimes M_p)) \).
Suppose that there are isomorphisms \( \phi : A \otimes M_p \to B \otimes M_p \) and \( \psi : A \otimes M_q \to B \otimes M_q \).
\[ A \otimes \mathcal{Z}_{p,q} : \]

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Suppose that there are isomorphisms \( \phi : A \otimes M_p \to B \otimes M_p \) and \( \psi : A \otimes M_q \to B \otimes M_q \).

Then one has

\[
\begin{align*}
(A \otimes M_p) \otimes M_q & \quad \sim \quad (A \otimes M_p) \otimes M_p \\
\downarrow_{\phi \otimes \text{id}_{M_q}} & \quad \quad \quad \quad \quad \downarrow_{\psi \otimes \text{id}_p} \\
B \otimes M_p \otimes M_q & \quad \sim \quad B \otimes M_q \otimes M_p
\end{align*}
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Suppose that there are isomorphisms $\phi : A \otimes M_p \to B \otimes M_p$ and $\psi : A \otimes M_q \to B \otimes M_q$. Then one has

\[
\begin{align*}
(A \otimes M_p) \otimes M_q & \sim (A \otimes M_p) \otimes M_p \\
\downarrow \phi \otimes \text{id}_{M_q} & \downarrow \psi \otimes \text{id}_p \\
B \otimes M_p \otimes M_q & \sim B \otimes M_q \otimes M_p
\end{align*}
\]

Let $\phi : A \to B$ be a homomorphism. Denote by $\Gamma(\phi) : (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \to (K_0(B), K_0(B), [1_B], K_1(B), T(B))$ be the map induced by $\phi$ which is the triple:

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where $\phi_T : T(B) \to T(A)$ defined by $\phi_T(\tau)(a) = \tau \circ \phi(a)$ for all $a \in A$. 

\[A \otimes \mathcal{Z}_{p,q} : \]

\[
A \otimes M_p \sim A \otimes M_q
\]
Suppose that there are isomorphisms $\phi : A \otimes M_p \to B \otimes M_p$ and $\psi : A \otimes M_q \to B \otimes M_q$.

Then one has

$$(A \otimes M_p) \otimes M_q \sim (A \otimes M_p) \otimes M_p$$

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Let $\Lambda : Ell(A) \cong Ell(B)$. Then $\Lambda$ induces

$$\Lambda_p : (K_0(A \otimes M_p), K_0(A \otimes M_p)_+, [1_{A \otimes M_p}], K_1(A \otimes M_p), T(A \otimes M_p))$$
(Winter 2007) Let $A$ and $B$ be two unital simple $C^*$-algebras in $\mathcal{N}$.
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$$\Lambda : \text{Ell}(A) \cong \text{Ell}(B).$$

Then $A \otimes \mathbb{Z} \cong B \otimes \mathbb{Z}$. 
Theorem

(Winter 2007) Let $A$ and $B$ be two unital simple $C^*$-algebras in $\mathcal{N}$. Suppose that

$$\Lambda : \text{Ell}(A) \cong \text{Ell}(B).$$

Suppose also that there are relatively prime supernatural numbers of infinite type $p$ and $q$ such that there exist isomorphisms

$$\varphi : A \otimes M_p \rightarrow B \otimes M_p$$

and

$$\psi : A \otimes M_q \rightarrow B \otimes M_q$$

with $\Gamma(\varphi) = \Lambda_p$ and $\Gamma(\psi) = \Lambda_q$. Then $A \otimes \mathbb{Z} \cong B \otimes \mathbb{Z}$. 

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Asymptotic Unitary Equivalence and Classification of Simple $C^*$-algebras
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and there exists a continuous path of unitaries $\{u_t : t \in [0,1)\} \subset B \otimes M_p \otimes M_q$ such that

$$u_t(0) = 1, \quad \lim_{t \to 1} \text{ad}u_t \circ \phi \otimes \text{id}_{M_q}(a) = \psi \otimes \text{id}_{M_p}(a)$$

for all $a \in A \otimes M_p \otimes M_q$. Then $A \otimes Z \cong B \otimes Z$. 

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Theorem

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Asymptotic Unitary Equivalence and Classification of Simple $C^*$-algebras
Theorem

(Uniqueness Theorem A—L 2005) Let $C$ be a unital AH-algebra and $A$ be a unital separable simple $C^*$-algebra with $TR(A) = 0$. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Then there exists a sequence of unitaries $\{u_n\} \subset A$ such that
$$\lim_{n \to \infty} \text{ad} u_n \circ \varphi(c) = \psi(c)$$
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for all $\tau \in T(A)$.
Suppose that

\[ u_n^* \phi_1(a) u_n \approx \frac{1}{2^n} \phi_2(a) \]
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Suppose that there is a continuous path of unitaries \( \{ u(n, t) : t \in [0, 1] \} \)
such that \( u(n, 0) = u_n \) and \( u(n, 1) = u_{n+1} \).
Suppose that

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But \( u(n, t)^* \phi_1(a) u(n, t) \) may be very different from \( u_n^* \phi_1(a) u_n \).
Suppose that
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Consider \( w = u_n^* u_{n+1} \).
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\[ w^* \phi_2(a) w \approx \frac{1}{2^n} \quad u_{n+1}^* \phi_1(a) u_{n+1} \]
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\[
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    w^* \phi_2(a) w & \approx \frac{1}{2^n} \quad u_{n+1}^* \phi_1(a) u_{n+1} \\
    & \approx \frac{1}{2^{n+1}} \phi_2(a). \quad (e0.1)
\end{align*}
\]

So \( w \) almost commutes with \( \phi_2(a) \).
If we have a continuous path of unitaries \( \{ w(t) : t \in [0, 1] \} \) such that

\[
\begin{align*}
\text{Then} & \quad u(n, t) \approx 1/2 n w(t) \\
\text{and} & \quad w(t) \approx \phi_2(a) w(t)
\end{align*}
\]

We may then use this \( u(n, t) \).
If we have a continuous path of unitaries \( \{ w(t) : t \in [0, 1] \} \) such that \( w(0) = 1, \ w(1) = w \)

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If we have a continuous path of unitaries \( \{ w(t) : t \in [0, 1] \} \) such that

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If we have a continuous path of unitaries \( \{w(t) : t \in [0, 1]\} \) such that
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and \( w(t)^* \phi_2(a) w(t) \) is almost the same as \( \phi_2(a) \).
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Question 5  For any $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C$, there exist a $\delta > 0$ and a finite subset $G \subset C$ such that, for any unital separable simple $C^*$-algebra $A$ with tracial rank zero and any unitary $u \in A$ with

$$
\| \phi(a)u - u\phi(a) \| < \delta
$$

for all $a \in G$ does there exist a continuous path of unitaries $\{u(t) : t \in [0, 1]\}$ such that

$$
\| \phi(a)u(t) - u(t)\phi(a) \| < \epsilon
$$

for all $f \in F$, $u(0) = u$ and $u(1) = 1$?

Consider the case that $C = C(T)$. It turns out that the answer is negative even $A$ is assumed to be finite dimensional.
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$$\|\phi(a)u - u\phi(a)\| < \delta$$

for all $a \in \mathcal{G}$.

Consider the case that $C = C(T)$. It turns out that the answer is negative even $A$ is assumed to be finite dimensional. The Bott element appears to be the obstacle in this case.
**Question 5** For any $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C$, is there a $\delta > 0$ and a finite subset $\mathcal{G} \subset C$ such that, for any unital separable simple C*-algebra $A$ with tracial rank zero and any unitary $u \in A$ with

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$$\|[u, v(t)]\| < \epsilon \text{ and } \text{Length}(\{v(t)\}) \leq 4\pi + 1.$$
Theorem (L–07) Let $X$ be a finite CW complex with dimension 1.
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Suppose that $A \in \mathcal{B}$, suppose that $h : C(X) \to A$ is a unital monomorphism and suppose that there is a unitary $u \in A$ with $[u] = 0$ such that for all $a \in G$,

$$\|h(a), u\| < \delta$$

and

$$\text{bott}_1(h, u) = 0.$$ \hspace{1cm} (e 0.2)

Then there exists a rectifiable continuous path of unitaries $\{u_t : t \in [0, 1]\}$ of $A$ such that

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for all $a \in F$ and all $t \in [0, 1]$.

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Suppose that there is a unitary $\|h(a), u\| < \delta$ for all $f \in G$, $\text{Bott}(h, u)|_P = 0$.

Then there exists a continuous path of unitaries $\{u_t : t \in [0, 1]\}$ such that $u_0 = u$, $u_1 = u$, $\|h(a), v_t\| < \epsilon$ for all $f \in \mathcal{F}$ and for all $t \in [0, 1]$ and $\text{Length}(\{u_t\}) \leq 2\pi + \epsilon$. 

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Asymptotic Unitary Equivalence and Classification of Simple $C^*$-algebras
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Definition

Let $C$ be a separable $C^*$-algebra and $A$ is unital $C^*$-algebra. Suppose that $\phi, \psi : C \to A$ are two homomorphisms. We say $\phi$ and $\psi$ are asymptotically unitarily equivalent if there exists a continuous path of unitaries $\{u(t) : t \in [0, 1)\} \subset A$ such that $\lim_{t \to 1} \text{ad} u(t) \circ \phi(c) = \psi(c)$ for all $c \in C$. 

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Mapping torus. Let $C$ and $A$ be a unital $C^*$-algebras and let $\phi_1, \phi_2 : C \to A$ be two unital monomorphisms. Set

$$M_{\phi_1, \phi_2} = \{ f \in C([0, 1], A) : f(0) = \phi_1(c), f(1) = \phi_2(c) \text{ for some } c \in C \}.$$
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We obtain a short exact sequence:

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**Definition**

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Define $\theta : C \to M_{\phi_1, \phi_2}$ by

$$\theta(c)(t) = u(t)^* \phi_1(c) u(t) \text{ for all } t \in [0,1)$$

and

$$\theta(c)(1) = \phi_2(c).$$
Thus if $\phi_1$ and $\phi_2$ are asymptotically unitarily equivalent, one must have $[\phi_1] = [\phi_2]$ in $KK(C, A)$. 
Thus if $\phi_1$ and $\phi_2$ are asymptotically unitarily equivalent, one must have $[\phi_1] = [\phi_2]$ in $KK(C, A)$. Furthermore,

$$\tau \circ \phi_1 = \tau \circ \phi_2.$$
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Let $u \in M_{\phi_1,\phi_2}$ be a unitary such that $t \mapsto u(t)$ is piecewise $C^1$. 

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Let $u \in M_{\phi_1, \phi_2}$ be a unitary such that $t \mapsto u(t)$ is piecewise $C^1$. For $\tau \in T(A)$, we define

$$\rho_\tau(u) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{du(t)}{dt}u(t)^*\right)dt.$$
Thus if $\phi_1$ and $\phi_2$ are asymptotically unitarily equivalent, one must have $[\phi_1] = [\phi_2]$ in $KK(C, A)$. Furthermore,

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This gives a homomorphism $\rho_\tau : K_1(M_{\phi_1, \phi_2}) \to \mathbb{R}$. 

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This gives a homomorphism $\rho_\tau : K_1(M_{\phi_1, \phi_2}) \rightarrow \mathbb{R}$. Consequently, we obtain a homomorphism $R_{\phi_1, \phi_2} : K_1(M_{\phi_1, \phi_2}) \rightarrow \text{Aff}(T(A)).$ It is called the rotation map.
Consider

\[ 0 \to K_0(A) \to K_1(M_{\phi_1,\phi_2}) \to K_1(C) \to 0. \]

It splits.
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\[ u(t) = e^{2\pi it} p + (1 - p). \]
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We have the following commutative diagram:

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\begin{array}{ccc}
K_0(A) & \xrightarrow{\iota_*} & K_1(M_{\phi_1,\phi_2}) \\
\rho_A & \downarrow & \quad \swarrow R_{\phi_1,\phi_2} \\
& \text{Aff}(T(A)), & 
\end{array}
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& Aff(T(A)), & \\
\end{array}
\]
where \( \rho_A([p])(\tau) = \tau(p) \) for each \( \tau \in T(A). \)
Consider

$$0 \to K_0(A) \to K_1(M_{\phi_1,\phi_2}) \to K_1(C) \to 0.$$  

It splits. If $p \in A$ is a projection, $\iota_*([p]) = [u]$ can be defined by

$$u(t) = e^{2\pi i t} p + (1 - p).$$

We have the following commutative diagram:

$$
\begin{array}{ccc}
K_0(A) & \xrightarrow{\iota_*} & K_1(M_{\phi_1,\phi_2}) \\
\rho_A & \searrow & \nearrow R_{\phi_1,\phi_2} \\
& Aff(\mathcal{T}(A)), &
\end{array}
$$

where $\rho_A([p])(\tau) = \tau(p)$ for each $\tau \in \mathcal{T}(A)$. Moreover, $R_{\phi_1,\phi_2}$ extends $\rho_A$. 
Suppose that there is $\theta \in \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(M_{\phi,\psi}))$ such that $[\pi_0] \circ \theta = [\text{id}_A]$. In particular, one has a monomorphism $\theta|_{K_1(A)} : K_1(A) \to K_1(M_{\phi,\psi})$ such that $[\pi_0] \circ \theta|_{K_1(A)} = (\text{id}_A)_1$. 
Suppose that there is $\theta \in Hom_{\Lambda}(K(A), K(M_{\phi,\psi}))$ such that $[\pi_0] \circ \theta = [id_A]$. In particular, one has a monomorphism $\theta|_{K_1(A)} : K_1(A) \to K_1(M_{\phi,\psi})$ such that $[\pi_0] \circ \theta|_{K_1(A)} = (id_A)_* \chi_1$. Thus, one may write

$$K_1(M_{\phi,\psi}) = K_0(B) \oplus K_1(A).$$  \hspace{1cm} (e 0.5)
Suppose that there is $\theta \in \text{Hom}_\Lambda(K(A), K(M_{\phi, \psi}))$ such that $[\pi_0] \circ \theta = [\text{id}_A]$. In particular, one has a monomorphism $\theta|_{K_1(A)} : K_1(A) \rightarrow K_1(M_{\phi, \psi})$ such that $[\pi_0] \circ \theta|_{K_1(A)} = (\text{id}_A)_*1$. Thus, one may write

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Suppose also that $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(A)$. 

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\[
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Suppose also that \( \tau \circ \phi = \tau \circ \psi \) for all \( \tau \in T(A) \). Then one obtains the homomorphism
\[
R_{\phi, \psi} \circ \theta|_{K_1(A)} : K_1(A) \to \text{Aff}(T(B)).
\]
Suppose that there is $\theta \in \text{Hom}_\Lambda(K(A), K(M_\phi, \psi))$ such that $[\pi_0] \circ \theta = [\text{id}_A]$. In particular, one has a monomorphism $\theta|_{K_1(A)} : K_1(A) \to K_1(M_\phi, \psi)$ such that $[\pi_0] \circ \theta|_{K_1(A)} = (\text{id}_A)_* 1$. Thus, one may write

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$$R_{\phi, \psi} \circ \theta|_{K_1(A)} : K_1(A) \to \text{Aff}(T(B)).$$

We say a rotation related map vanishes, if there exists a such splitting map $\theta$ such that

$$R_{\phi, \psi} \circ \theta|_{K_1(A)} = 0.$$
Suppose that there is \( \theta \in \text{Hom}_\Lambda(K(A), K(M_{\phi, \psi})) \) such that 
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\[
K_1(M_{\phi, \psi}) = K_0(B) \oplus K_1(A). \tag{e 0.5}
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\[
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\]

Denote by \( \mathcal{R}_0 \) the set of those homomorphisms
\( \lambda \in \text{Hom}(K_1(A), \text{Aff}(T(B))) \) for which there is a homomorphism 
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Suppose that there is $\theta \in \text{Hom}_\Lambda(K(A), K(M_\phi, \psi))$ such that $[\pi_0] \circ \theta = [\text{id}_A]$. In particular, one has a monomorphism $\theta|_{K_1(A)} : K_1(A) \rightarrow K_1(M_\phi, \psi)$ such that $[\pi_0] \circ \theta|_{K_1(A)} = (\text{id}_A)_* 1$. Thus, one may write

$$K_1(M_\phi, \psi) = K_0(B) \oplus K_1(A).$$

Suppose also that $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(A)$. Then one obtains the homomorphism

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Denote by $\mathcal{R}_0$ the set of those homomorphisms $\lambda \in \text{Hom}(K_1(A), \text{Aff}(T(B)))$ for which there is a homomorphism $h : K_1(A) \rightarrow K_0(B)$ such that $\lambda = \rho_A \circ h$. 
Suppose that there is $\theta \in \text{Hom}_\Lambda(K(A), K(M_\phi, \psi))$ such that $[\pi_0] \circ \theta = [\text{id}_A]$. In particular, one has a monomorphism $\theta|_{K_1(A)} : K_1(A) \to K_1(M_\phi, \psi)$ such that $[\pi_0] \circ \theta|_{K_1(A)} = (\text{id}_A)_*1$. Thus, one may write

$$K_1(M_\phi, \psi) = K_0(B) \oplus K_1(A).$$

(S0.5)

Suppose also that $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(A)$. Then one obtains the homomorphism

$$R_{\phi, \psi} \circ \theta|_{K_1(A)} : K_1(A) \to \text{Aff}(T(B)).$$

We say a rotation related map vanishes, if there exists a such splitting map $\theta$ such that

$$R_{\phi, \psi} \circ \theta|_{K_1(A)} = 0.$$ 

Denote by $R_0$ the set of those homomorphisms $\lambda \in \text{Hom}(K_1(A), \text{Aff}(T(B)))$ for which there is a homomorphism $h : K_1(A) \to K_0(B)$ such that $\lambda = \rho_A \circ h$. It is a subgroup of $\text{Hom}(K_1(A), \text{Aff}(T(B)))$. 

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Let $\theta, \theta' \in \text{Hom}_\Lambda(K(A), K(M_{\phi, \psi}))$ such that $[\pi_0] \circ [\theta] = [\text{id}_C] = [\pi_0] \circ [\theta']$. Then $(\theta - \theta')|_{K_1(A)} \subset K_0(B)$. In other words, $R_{\phi, \psi} \circ (\theta - \theta')|_{K_1(A)} \in R_0$. Thus, we obtain a well-defined element $R_{\phi, \psi} \in \text{Hom}(K_1(A), \text{Aff}(T(B)))$ which does not depend on the choices of $\theta$.
Let $\theta, \theta' \in \text{Hom}_{\Lambda}(K(A), K(M_{\phi, \psi}))$ such that $[\pi_0] \circ [\theta] = [\text{id}_C] = [\pi_0] \circ [\theta']$. Then $(\theta - \theta')(K_1(A)) \subset K_0(B)$.
Let $\theta, \theta' \in Hom_\Lambda(K(A), K(M_{\phi,\psi}))$ such that $[\pi_0] \circ [\theta] = [\text{id}_C] = [\pi_0] \circ [\theta']$. Then $(\theta - \theta')(K_1(A)) \subset K_0(B)$. In other words,

$$R_{\phi,\psi} \circ (\theta - \theta')|_{K_1(A)} \in \mathcal{R}_0.$$
Let $\theta, \theta' \in Hom_{\Lambda}(K(A), K(M_{\phi,\psi}))$ such that $[\pi_0] \circ [\theta] = [\text{id}_C] = [\pi_0] \circ [\theta']$. Then $(\theta - \theta')(K_1(A)) \subset K_0(B)$. In other words,

$$R_{\phi,\psi} \circ (\theta - \theta')|_{K_1(A)} \in \mathcal{R}_0.$$

Thus, we obtain a well-defined element

$$\overline{R}_{\phi,\psi} \in Hom(K_1(A), \text{Aff}(T(B)))/\mathcal{R}_0$$

(which does not depend on the choices of $\theta$).
In this case, if there is a homomorphism \( \theta_1' : K_1(A) \to K_1(M_{\phi, \psi}) \) such that 
\[
(\pi_0)_* \circ \theta_1' = \text{id}_{K_1(A)} \quad \text{and} 
\]
\[
R_{\phi, \psi} \circ \theta_1' \in \mathcal{R}_0, 
\]
In this case, if there is a homomorphism \( \theta'_1 : K_1(A) \to K_1(M_{\phi,\psi}) \) such that 
\[ (\pi_0)_1 \circ \theta'_1 = \text{id}_{K_1(A)} \] and

\[ R_{\phi,\psi} \circ \theta'_1 \in R_0, \]

then there is \( \Theta \in \text{Hom}_\Lambda(K(A), K(M_{\phi,\psi})) \) such that
In this case, if there is a homomorphism \( \theta'_1 : K_1(A) \to K_1(M_{\phi,\psi}) \) such that 
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and 
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then there is \( \Theta \in Hom_{\Lambda}(K(A), K(M_{\phi,\psi})) \) such that 
\[ [\pi_0] \circ \Theta = [id_A] \text{ and } R_{\phi,\psi} \circ \Theta = 0. \]

When \( \overline{R}_{\phi,\psi} = 0 \), \( \theta(K_1(A)) \in \ker R_{\phi,\psi} \) for some \( \theta \) so that \( [\pi_0] \circ \theta = id_{K_1(C)}. \)
In this case, if there is a homomorphism $\theta'_1 : K_1(A) \to K_1(M_{\phi,\psi})$ such that 
$(\pi_0)_* 1 \circ \theta'_1 = \text{id}_{K_1(A)}$ and

$$R_{\phi,\psi} \circ \theta'_1 \in R_0,$$

then there is $\Theta \in \text{Hom}_\Lambda(K(A), K(M_{\phi,\psi}))$ such that

$$[\pi_0] \circ \Theta = [\text{id}_A] \quad \text{and} \quad R_{\phi,\psi} \circ \Theta = 0.$$

When $\bar{R}_{\phi,\psi} = 0$, $\theta(K_1(A)) \in \ker R_{\phi,\psi}$ for some $\theta$ so that $[\pi_0] \circ \theta = \text{id}_{K_1(C)}$. Thus $\theta$ also gives the following:

$$\ker R_{\phi,\psi} = \ker \rho_B \oplus K_1(A).$$
Theorem

(Uniqueness Theorem B—L 2007) Let $C$ be a unital AH-algebra and $A$ be a unital separable simple $C^*$-algebra with $TR(A) = 0$. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms. Then there exists a continuous path of unitaries $\{u_t\} \subset A$ such that
$$\lim_{t \to \infty} \text{ad} u_t \circ \phi(c) = \psi(c)$$
for all $c \in C$ if and only if $[\phi] = [\psi]$ in $KK(C, A)$ and $R_{\phi, \psi} = 0$. 

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\]

for all \( c \in C \) if and only if

\[
[\phi] = [\psi] \quad \text{in} \quad KK(C, A)
\]

\[
\tau \circ \phi = \tau \circ \psi \quad \text{for all} \quad \tau \in T(A) \quad \text{and} \quad R_{\phi, \psi} = 0.
\]
Theorem

(2007 Winer –with an appendix by L) Let $A$ and $B$ be two unital $C^*$-algebras in $\mathcal{N}$. Suppose that $\Ell(A) \cong \Ell(B)$ and suppose that $\text{TR}(A \otimes M_p) = \text{TR}(B \otimes M_q) = 0$ for any supernatural number $p$ of infinite type. Then $A \otimes \mathbb{Z} \cong B \otimes \mathbb{Z}$, provided that $K^*(A)$ and $K^*(B)$ are finitely generated.
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Theorem

(2007 Winer –with an appendix by L) **Let A and B be two unital C*-algebras in \( \mathcal{N} \). Suppose that**

\[
\text{Ell}(A) \cong \text{Ell}(B)
\]

**and suppose that** \( \text{TR}(A \otimes M_p) = \text{TR}(B \otimes M_q) = 0 \) **for any supernatural number** \( p \) **of infinite type. Then**

\[
A \otimes \mathbb{Z} \cong B \otimes \mathbb{Z},
\]

**provided that** \( K_*(A) \) **and** \( K_*(B) \) **are finitely generated.**
**Definition**

Denote by $KK_e(C, A)^{++}$ the set of those $\kappa \in KK(C, A)$ for which $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$ such that $\kappa([1_C]) = [1_A]$. 

Theorem (L and Z. Niu 2008) (Existence Theorem B (part I))

Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $TR(A) = 0$. Then for any $\kappa \in KK_e(C, A)^{++}$ and for any $\lambda : T(A) \to T_f(C)$ which is compatible with $\kappa$, there exists a unital monomorphism $\phi : C \to A$ such that $\phi = \kappa$ and $\lambda(\tau)(c) = \tau \circ \phi(c)$ for all $c \in C$. 

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Denote by $T_f(C)$ the set of those tracial states $t \in T(C)$ such that $t(c) > 0$ for any $c \in C_+ \setminus \{0\}$.
Definition

Denote by $KK_e(C, A)^{++}$ the set of those $\kappa \in KK(C, A)$ for which $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$ such that $\kappa([1_C]) = [1_A]$.

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Definition

Denote by $KK_e(C, A)^{++}$ the set of those $\kappa \in KK(C, A)$ for which $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$ such that $\kappa([1_C]) = [1_A]$.

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Theorem

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Theorem

( L and Z. Niu 2008) (Existence Theorem B (part I)) Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $TR(A) = 0$. 

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**Definition**

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**Theorem**

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**Definition**

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**Theorem**

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$$[\phi] = \kappa \text{ and } \lambda(\tau)(c) = \tau \circ \phi(c)$$

for all $c \in C$. 
Definition

Denote by $KKT(C, A)^{++}$ the set of pairs $(\kappa, \lambda)$, where $\kappa \in KK_e(C, A)^{++}$ and $\lambda : T(A) \rightarrow T_f(C)$ is an affine continuous map which is compatible with $\kappa$. 
Definition

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Definition
Let $A$ be a unital $C^*$-algebra, and let $C$ be a unital separable $C^*$-algebra. Denote by $\text{Mon}_{asu}^e(C, A)$ the set of asymptotically unitary equivalence classes of unital monomorphisms.
**Definition**

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**Definition**

Let $A$ be a unital $\text{C}^*$-algebra, and let $C$ be a unital separable $\text{C}^*$-algebra. Denote by $\text{Mon}_{asu}^e(C, A)$ the set of asymptotically unitary equivalence classes of unital monomorphisms.
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Definition

Let $A$ be a unital C*-algebra, and let $C$ be a unital separable C*-algebra. Denote by $\text{Mon}_{asu}(C, A)$ the set of asymptotically unitary equivalence classes of unital monomorphisms. Denote by $\mathcal{K}$ the map from $\text{Mon}_{asu}^e(C, A)$ into $KKT(C, A)^{++}$ defined by

$$\phi \mapsto ([\phi], \phi T)$$

for all $\phi \in \text{Mon}_{asu}^e(C, A)$. 
Definition

Denote by $K KT(C, A)^{++}$ the set of pairs $(\kappa, \lambda)$, where $\kappa \in KK_e(C, A)^{++}$ and $\lambda : T(A) \to T_f(C)$ is an affine continuous map which is compatible with $\kappa$.

Definition

Let $A$ be a unital $C^*$-algebra, and let $C$ be a unital separable $C^*$-algebra. Denote by $Mon^e_{asu}(C, A)$ the set of asymptotically unitary equivalence classes of unital monomorphisms. Denote by $\mathfrak{A}$ the map from $Mon^e_{asu}(C, A)$ into $K KT(C, A)^{++}$ defined by

$$\phi \mapsto ([\phi], \phi_T) \text{ for all } \phi \in Mon^e_{asu}(C, A).$$

Denote by $\langle \kappa, \lambda \rangle$ the classes of $\phi \in Mon^e_{asu}(C, A)$ such that $\mathfrak{A}(\phi) = (\kappa, \lambda)$. 
Theorem

(L and Z. Niu 2008) (*Existence Theorem B (part II)*)

Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $\text{TR}(A) = 0$. Then the map $K: \text{Mon}((C, A)) \to \text{KKT}(C, A)^{++}$ is surjective. Moreover, for each $(\kappa, \lambda) \in \text{KKT}(C, A)^{++}$, there exists a bijection $\eta: \langle \kappa, \lambda \rangle \to \text{Hom}(K_1(C), \text{Aff}(T(A)))/R_0$. 

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$$\eta : \langle \kappa, \lambda \rangle \to \text{Hom}(K_1(C), \text{Aff}(T(A))) / \mathcal{R}_0.$$

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**Theorem**

(L and Z. Niu) *Let $A$ and $B$ be unital simple $C^*$-algebras in $\mathcal{N}$ such that $\Ell(A) = \Ell(B)$. Then $A \otimes Z \cong B \otimes Z$.***
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Using Winter’s theorem, one has the following:

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(L–2008) (Uniqueness Theorem C) Let $C$ and $A$ be two unital simple $C^*$-algebras with $\text{TR}(A)$, $\text{TR}(C) \leq 1$.

Suppose that $C \in N$ and $\varphi, \psi : C \to A$ are two unital homomorphisms. Then there exists a continuous path of unitaries $\{u_t : t \in [0, \infty)\} \subset A$ such that

$$\lim_{t \to \infty} \text{ad} u_t \circ \varphi(c) = \psi(c)$$

for all $c \in C$ if and only if $[\varphi] = [\psi]$ in $\text{KK}(C, A)$

$\varphi_T = \psi_T$, $\varphi^\perp = \psi^\perp$ and $R_{\varphi, \psi} = 0$.

Here $\varphi^\perp : \text{U}(C)/\text{CU}(C) \to \text{U}(A)/\text{CU}(A)$ is the induced homomorphism by $\varphi$ and $\text{CU}(C)$ (and $\text{CU}(A)$) is the closure of the commutator group of $\text{U}(C)$ (of $\text{U}(A)$).
Theorem

(L–2008) (Uniqueness Theorem C) Let $C$ and $A$ be two unital simple $C^*$-algebras with $\text{TR}(A), \text{TR}(C) \leq 1$. Suppose that $C \in \mathcal{N}$ and $\phi, \psi : C \to A$ are two unital homomorphisms. Then there exists a continuous path of unitaries $\{u_t : t \in [0, \infty)\} \subset A$ such that

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for all $c \in C$ if and only if $[\phi] = [\psi]$ in $KK(C, A)$.

$\phi^T = \psi^T, \phi^\perp = \psi^\perp$ and $R_{\phi, \psi} = 0$.

In particular, $\phi$ and $\psi$ are asymptotically unitarily equivalent if and only if $[\phi] = [\psi]$ in $KK(C, A)$.

Here $\phi^\perp : U(C)/CU(C) \to U(A)/CU(A)$ is the induced homomorphism by $\phi$ and $CU(C)$ (and $CU(A)$) is the closure of the commutator group of $U(C)$ (of $U(A)$).
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**Definition**

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(1) If $A \in \mathcal{A}$, then for any projection $p \in A$, $pAp \in \mathcal{A}$;
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(8) Every unital simple AH-algebra is in $\mathcal{A}$. 
Theorem

(L–2008) Let $A$ and $B$ be two $C^*$-algebras in $A$. Then $A \otimes \mathbb{Z} \cong B \otimes \mathbb{Z}$ if and only if

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Corollary

Let $A$ be a unital simple AH-algebra. Then $A \otimes \mathbb{Z}$ is a unital simple unital AH-algebra with no dimension growth.
Corollary

Suppose that $A$ is a unital simple AH-algebra. Then the following are equivalent:

1. $\text{TR}(A) \leq 1$,
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3. $A$ is approximately divisible,
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