Towards an analogue of the Baum-Connes conjecture for quantum groups

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Vanderbilt
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The Baum-Connes conjecture

Let $G$ be a second countable locally compact group. The Baum-Connes conjecture asserts that the assembly map

$$
\mu : K^{\text{top}}_* (G) \rightarrow K_* (C^*_\text{red} (G))
$$

is an isomorphism. Here $E_G$ is the universal proper $G$-space.

More generally, the Baum-Connes conjecture with coefficients states that

$$
\mu : K^{\text{top}}_* (G; A) \rightarrow K_* (G \ltimes^R A)
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is an isomorphism for every $G$-$C^*$-algebra $A$.

Here $G \ltimes^R A$ is the reduced crossed product of $A$ by $G$.

What happens if $G$ is a locally compact quantum group?

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What happens if $G$ is a locally compact \textit{quantum} group?
Basic definitions

Let $G$ be a locally compact group. A $G$-$C^*$-algebra is a $C^*$-algebra $A$ with a strongly continuous action of $G$ by $*$-automorphisms.

A $*$-homomorphism $f : A \to B$ is equivariant if $f(t \cdot a) = t \cdot f(a)$ for all $t \in G$, $a \in A$.

Let $H \subset G$ be a closed subgroup.

If $A$ is a $G$-$C^*$-algebra then $A$ becomes an $H$-$C^*$-algebra by restriction of the action.

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$$\text{ind}^G_H(B) = \{ f \in C_b(G, B) | f(xh) = h^{-1} \cdot f(x), \ xH \mapsto \| f(x) \| \in C_0(G/H) \}$$
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Reformulation of the conjecture by Meyer-Nest

Equivariant Kasparov theory yields a category $\text{KK}$

- objects in $\text{KK}$ are all separable $G$-$C^*$-algebras.
- morphism sets are the bivariant Kasparov $K$-groups $\text{KK}_G(A, B)$, and composition of morphisms is given by the Kasparov product.

In fact, the category $\text{KK}_G$ is triangulated — this allows to do homological algebra.

A basic example of a triangulated category to have in mind is the homotopy category of chain complexes $\text{CH}(R)$ of $R$-modules over a ring $R$. 

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- The (inverse of the) suspension $\Sigma(A) = C_0(\mathbb{R}) \otimes A$ yields the translation functor.
- Distinguished *-triangles are all triangles isomorphic to mapping cone triangles
  \[
  \Sigma(B) \to C_f \to A \to B
  \]
  for equivariant *-homomorphisms $f : A \to B$. 
A $G\ast C$-algebra is called compactly induced if it is of the form $\text{ind}_{GH}(B)$ for a compact subgroup $H \subset G$.

(corresponds to a projective chain complex in CH(R))

A $G\ast C$-algebra is called weakly contractible if $\text{res}_{GH}(A) \sim 0 \in \text{KK}_{H}$ for every compact subgroup $H \subset G$.

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A $\langle CI \rangle$-simplicial approximation of a $G$-$C^*$-algebra $A$ is a weak equivalence $\tilde{A} \to A$ with $\tilde{A} \in \langle CI \rangle$. 

Theorem
For every $A \in KK^G$ there exists a $\langle CI \rangle$-simplicial approximation $\tilde{A}$ which is unique up to isomorphism.

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For the functor $F(A) = K^*(G \ltimes \text{red} A)$ the transformation $L F(A) = K^*(G \ltimes \text{red} \tilde{A}) \to K^*(G \ltimes \text{red} A) = F(A)$ is isomorphic to the Baum-Connes assembly map.
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is isomorphic to the Baum-Connes assembly map.

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Locally compact quantum groups

A Hopf-$C^*$-algebra is a $C^*$-algebra $\mathcal{H}$ together with a nondegenerate injective $^*$-homomorphism $\Delta : \mathcal{H} \to M(\mathcal{H} \otimes \mathcal{H})$ such that $\mathcal{H} \Delta \to \Delta \to \mathcal{H} \otimes \mathcal{H}$ is commutative and $\Delta(\mathcal{H})(1 \otimes \mathcal{H})$ and $(\mathcal{H} \otimes 1)\Delta(\mathcal{H})$ are dense subspaces of $\mathcal{H} \otimes \mathcal{H}$.

A locally compact quantum group is given by a Hopf-$C^*$-algebra $\mathcal{H}$ together with left and right Haar integrals.
Definition

A Hopf-$C^*$-algebra is a $C^*$-algebra $H$ together with a nondegenerate injective $*$-homomorphism $\Delta : H \to M(H \otimes H)$ such that

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\begin{array}{ccc}
H & \xrightarrow{\Delta} & M(H \otimes H) \\
\downarrow & & \downarrow \text{id} \otimes \Delta \\
M(H \otimes H) & \xrightarrow{\Delta \otimes \text{id}} & M(H \otimes H \otimes H)
\end{array}
\]

is commutative and $\Delta(H)(1 \otimes H)$ and $(H \otimes 1)\Delta(H)$ are dense subspaces of $H \otimes H$. 
Definition

A *-Hopf-C*-algebra is a C*-algebra $H$ together with a nondegenerate injective *-homomorphism $\Delta : H \to M(H \otimes H)$ such that

$$
\begin{array}{ccc}
H & \xrightarrow{\Delta} & M(H \otimes H) \\
\downarrow & & \downarrow \text{id} \otimes \Delta \\
M(H \otimes H) & \xrightarrow{\Delta \otimes \text{id}} & M(H \otimes H \otimes H)
\end{array}
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is commutative and $\Delta(H)(1 \otimes H)$ and $(H \otimes 1)\Delta(H)$ are dense subspaces of $H \otimes H$.

A locally compact quantum group is given by a Hopf-C*-algebra $H$ together with left and right Haar integrals.
If $G$ is a locally compact group then $H = C_0(G)$ defines a locally compact quantum group. The comultiplication $\Delta : C_0(G) \to C^b(G \times G)$ is given by $\Delta(f)(s,t) = f(st)$, and the integrals are given by left/right Haar measure.
Examples
If $G$ is a locally compact group then $H = C_0(G)$ defines a locally compact quantum group.
The comultiplication $\Delta : C_0(G) \to C_b(G \times G)$ is given by

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In general we will write $H = C_{\text{red}}^0(G)$ for a locally compact quantum group. We think of $C_{\text{red}}^0(G)$ as "the algebra of functions" on the (imaginary) quantum group $G$. For every locally compact quantum group $G$ there exists a dual locally compact quantum group $\hat{G}$ given by $\hat{H} = C^*_{\text{red}}(\hat{G}) = C_{\text{red}}^0(\hat{G})$ and the Pontrjagin duality theorem holds. Every locally compact quantum group comes equipped with a Hilbert space $H_G$ (the GNS-space of the left Haar weight) and a multiplicative unitary $W \in \mathcal{M}(C_{\text{red}}^0(G) \otimes C^*_{\text{red}}(G))$. In the sequel all locally compact quantum groups are assumed to be strongly regular.
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We think of $C^\text{red}_0(G)$ as "the algebra of functions" on the (imaginary) quantum group $G$.

For every locally compact quantum group $G$ there exists a \textit{dual locally compact quantum group} $\hat{G}$ given by

$$\hat{H} = C^*_\text{red}(G) = C^\text{red}_0(\hat{G})$$

and the \textit{Pontrjagin duality theorem} holds.
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In general we will write $H = C_{0}^{\text{red}}(G)$ for a locally compact quantum group.

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For every locally compact quantum group $G$ there exists a dual locally compact quantum group $\hat{G}$ given by

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Every locally compact quantum group comes equipped with a Hilbert space $\mathbb{H}_{G}$ (the GNS-space of the left Haar weight) and a *multiplicative unitary* $W \in M(C_{0}^{\text{red}}(G) \otimes C_{\text{red}}^{*}(G))$.

In the sequel all locally compact quantum groups are assumed to be strongly regular.
Actions

Definition

A left coaction of a Hopf $C^*$-algebra $H$ on a $C^*$-algebra $A$ is an injective nondegenerate $\ast$-homomorphism $\alpha : A \to M(H \otimes A)$ such that the diagram

\[ \begin{array}{ccc}
A & \xrightarrow{\alpha} & A \\
\downarrow & & \downarrow \\
M(H \otimes A) & \xrightarrow{\Delta \otimes \text{id}} & M(H \otimes H \otimes A)
\end{array} \]

is commutative and $\alpha(A)(H \otimes 1) \subset H \otimes A$ is dense.

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A (left) coaction of a Hopf $C^*$-algebra $H$ on a $C^*$-algebra $A$ is an injective nondegenerate $*$-homomorphism $\alpha : A \to M(H \otimes A)$ such that the diagram

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The Kasparov category

Let $G$ be a locally compact quantum group. A $G$-C*-algebra is a C*-algebra $A$ with a coaction of $C_{red}^0(G)$. For locally compact groups this recovers the usual definition. Baaj and Skandalis defined $KK_G$ for quantum groups. As in the group case one obtains a triangulated category with objects the separable $G$-C*-algebras and morphisms given by equivariant Kasparov groups.
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Braided tensor products

Definition

Let $G$ be a locally compact quantum group and $H = C^*_red(G)$ and let $\hat{H} = C^*_\alpha(G)$. A $G$-Yetter-Drinfeld algebra is a $C^*$-algebra $A$ equipped with a coaction $\alpha$ of $H$ and a coaction $\lambda$ of $\hat{H}$ such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & M(\hat{H} \otimes A) \\
\downarrow & & \downarrow \\
\sigma \otimes id & & id \otimes \lambda \\
\end{array}
\]

\[
\begin{array}{ccc}
M(H \otimes A) & \xrightarrow{id \otimes \alpha} & M(H \otimes \hat{H} \otimes A) \\
\downarrow & & \downarrow \\
M(H \otimes \hat{H} \otimes A) & \xrightarrow{\text{ad}(W)} & M(H \otimes \hat{H} \otimes A)
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is commutative.

Here $\sigma: \hat{H} \otimes H \rightarrow H \otimes \hat{H}$ is the flip map.

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Here $\sigma : \hat{H} \otimes H \to H \otimes \hat{H}$ is the flip map.
Braided tensor products

Examples

- Ordinary locally compact group $G$ - then every $G$-C$^*$-algebra $A$ is a $G$-YD-algebra with the trivial coaction $\lambda: A \to M(C^*_{\text{red}}(G) \otimes A)$ given by $\lambda(a) = 1 \otimes a$.

- Discrete group $G$ - then coactions of $C^*_{\text{red}}(G)$ correspond to Fell bundles. A YD-structure is equivalent to having a $G$-equivariant Fell bundle.

- In general - if $H \subset G$ is a quantum subgroup then the induced algebra $\text{ind}_G^H(A)$ of a $H$-YD-algebra $A$ is a $G$-YD-algebra.
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▶ $G$ ordinary locally compact group - then every $G$-$C^*$-algebra $A$ is a $G$-YD-algebra with the trivial coaction $\lambda : A \to M(C^\text{red}_G \otimes A)$ given by

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▶ $G$ discrete group - then coactions of $C^\text{red}_G$ correspond to Fell bundles. A YD-structure is equivalent to having a $G$-equivariant Fell bundle.

▶ in general - if $H \subset G$ is a quantum subgroup then the induced algebra $\text{ind}^G_H(A)$ of a $H$-YD-algebra $A$ is a $G$-YD-algebra.
Braided tensor products and the Drinfeld double

Definition

Let $G$ be a locally compact quantum group. The Drinfeld double $D(G)$ is the locally compact quantum group given by

$$\text{C}_{\text{red}}^0(D(G)) = \text{C}_{\text{red}}^0(G) \otimes \text{C}^*_\text{red}(G)$$

with the comultiplication

$$\Delta_{D(G)} = (\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(W) \otimes \text{id})(\Delta \otimes \hat{\Delta})$$

where $\text{ad}(W)(x) = WxW^*$ for $x \in \text{C}_{\text{red}}^0(G) \otimes \text{C}^*_\text{red}(G)$.

Proposition

A $G$-Yetter-Drinfeld algebra is the same thing as a $D(G)$-$\text{C}^*$-algebra.

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Definition

Let $G$ be a locally compact quantum group. The *Drinfeld double* $D(G)$ is the locally compact quantum group given by $C^\text{red}_0(D(G)) = C^\text{red}_0(G) \otimes C^*_\text{red}(G)$ with the comultiplication $\Delta_{D(G)} = \text{id} \otimes \sigma \otimes \text{id} \cdot \text{id} \otimes \text{ad}(W) \otimes \text{id} \cdot (\Delta \otimes \hat{\Delta})$ where $\text{ad}(W)(x) = WxW^*$ for $x \in C^\text{red}_0(G) \otimes C^*_\text{red}(G)$. 
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Proposition
A $G$-Yetter-Drinfeld algebra is the same thing as a $D(G)$-$C^*$-algebra.
Braided tensor products

Definition

Let \( A \) be a \( G \)-YD-algebra and let \( B \) be a \( G \)-algebra. The braided tensor product is

\[
A \triangledown B = \big[ \lambda(\mathcal{A}) \big]_{12} \beta(\mathcal{B})_{13} \subset \mathcal{L}(H^G \otimes \mathcal{A} \otimes \mathcal{B}).
\]

\( A \triangledown B \) is a \( C^\ast \)-algebra, and \( \lambda \) (resp. \( \beta \)) define injective \( \ast \)-homomorphisms \( \iota_A : A \to M(\mathcal{A} \triangledown \mathcal{B}) \) (resp. \( \iota_B : B \to M(\mathcal{A} \triangledown \mathcal{B}) \)).

There is a coaction \( A \triangledown B \to M(C_{red}^0(G) \otimes (\mathcal{A} \triangledown \mathcal{B})) \) such that \( \iota_A \) and \( \iota_B \) are equivariant.

If \( B \) is a YD-algebra then \( A \triangledown B \) is a YD-algebra and \( (A \triangledown B) \triangledown C \sim = A \triangledown (B \triangledown C) \) for all \( G \)-algebras \( C \).
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Let $A$ be a $G$-YD-algebra and let $B$ be a $G$-algebra. The braided tensor product is

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$A \boxtimes B$ is a $C^*$-algebra, and $\lambda$ (resp. $\beta$) define injective $*$-homomorphisms $\iota_A : A \to M(A \boxtimes B)$ (resp. $\iota_B : B \to M(A \boxtimes B)$).
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- There is a coaction $A \boxtimes B \to M(C_0^{\text{red}}(G) \otimes (A \boxtimes B))$ such that $\iota_A$ and $\iota_B$ are equivariant.
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**Definition**
Let $A$ be a $G$-YD-algebra and let $B$ be a $G$-algebra. The *braided tensor product* is

$$A oxtimes B = [\lambda(A)_{12} \beta(B)_{13}] \subset \mathbb{L}(H_G \otimes A \otimes B).$$

- $A \boxtimes B$ is a $C^*$-algebra, and $\lambda$ (resp. $\beta$) define injective $*$-homomorphisms $\iota_A : A \to M(A \boxtimes B)$ (resp. $\iota_B : B \to M(A \boxtimes B)$).

- There is a coaction $A \boxtimes B \to M(C^*_0(G) \otimes (A \boxtimes B))$ such that $\iota_A$ and $\iota_B$ are equivariant.

- If $B$ is a YD-algebra then $A \boxtimes B$ is a YD-algebra and

$$ (A \boxtimes B) \boxtimes C \cong A \boxtimes (B \boxtimes C) $$

for all $G$-algebras $C$. 
Theorem

Let $A_1$, $B_1$ and $D$ be $G$-$YD$ algebras and let $A_2$, $B_2$ be $G$-algebras. There is an exterior Kasparov product

$$\text{KK}^D(G) \ast (A_1 \boxtimes D) \times \text{KK}^G(D \boxtimes A_2, B_2) \rightarrow \text{KK}^G(A_1 \boxtimes A_2, B_1 \boxtimes B_2)$$

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**Theorem**

Let $A_1$, $B_1$ and $D$ be $G$-YD algebras and let $A_2$, $B_2$ be $G$-algebras. There is an exterior Kasparov product

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The quantum group $SU_q(2)$

**Definition**

Fix $q \in (0,1]$. The unital $\ast$-algebra $O(SU_q(2))$ (over $\mathbb{C}$) is generated by elements $\alpha$ and $\gamma$ satisfying the relations:

\[
\begin{align*}
\alpha \gamma &= q \gamma \alpha, \\
\alpha \gamma^\ast &= q \gamma^\ast \alpha, \\
\gamma \gamma^\ast &= \gamma^\ast \gamma, \\
\alpha^\ast \alpha + \gamma^\ast \gamma &= 1, \\
\alpha \alpha^\ast + q^2 \gamma \gamma^\ast &= 1.
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These relations are equivalent to saying that the fundamental matrix $(\alpha - q \gamma^\ast \gamma \alpha^\ast)$ is unitary.
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These relations are equivalent to saying that the fundamental matrix

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\begin{pmatrix}
\alpha & -q \gamma^* \\
\gamma & \alpha^*
\end{pmatrix}
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is unitary.
The comultiplication \( \Delta : \mathcal{O}(SU_q(2)) \to \mathcal{O}(SU_q(2)) \otimes \mathcal{O}(SU_q(2)) \) is defined by

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\Delta \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix}
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In fact, $\mathcal{O}(SU_q(2))$ is a Hopf-*-algebra.
The quantum group $SU_q(2)$

The $\ast$-algebra $\mathcal{O}(SU_q(2))$ can be completed uniquely to a $C^*$-algebra $C(SU_q(2))$. This yields a (locally) compact quantum group.
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The $\ast$-algebra $\mathcal{O}(SU_q(2))$ can be completed uniquely to a $C^\ast$-algebra $C(SU_q(2))$. This yields a (locally) compact quantum group.

For $q = 1$ one obtains in this way the algebras $\mathcal{O}(SU(2))$ and $C(SU(2))$ of polynomial and continuous functions on $SU(2)$, respectively.
The Podleś sphere

The maximal torus $T = S^1 \subset SU_q(2)$ is given by the projection

$$\pi: C(SU_q(2)) \to C(T) \supset C[z, z^{-1}]$$

given by

$$\pi(\alpha - q \gamma^* \gamma^* \alpha) = (z^0, 0, z^{-1})$$

The (standard) Podleś sphere is the homogeneous space $SU_q(2)/T$ given by the algebra of coinvariants $C(SU_q(2)/T) = \{x \in C(SU_q(2)) | (id \otimes \pi) \Delta(x) = x \otimes 1\}$ under right translations.

We remark that for $q \in (0, 1)$ one has $C(SU_q(2)/T) \cong K^+$. There is an algebraic version $O(SU_q(2)/T)$ as well.
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The Baum-Connes conjecture

In the sequel we let $q \in (0, 1]$ and write $G = SU_q(2)$ as well as $\hat{G}$ for its dual. We shall formulate and prove an analogue of the Baum-Connes conjecture for the dual quantum group $\hat{G}$ of $SU_q(2)$. What is the Baum-Connes conjecture in this situation?
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The Baum-Connes conjecture

The discrete quantum group $\hat{G}$ is torsion-free.

The proper homogeneous $\hat{G}$-algebra corresponding to the trivial subgroup is $C^*(G) = C_0(\hat{G})$.

We write $\langle CI \rangle$ for the localizing subcategory of $KK(\hat{G})$ generated by algebras of the form $C^*(G) \otimes A$ where $A$ is some $C^*$-algebra and the coaction is inherited from $C^*(G)$.

Theorem

One has $\langle CI \rangle = KK(\hat{G})$. 

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**Theorem**

*One has* $\langle CI \rangle = KK\hat{G}$. 
Outline of the proof

Let us concentrate on the following part of the argument.

Theorem

We have $C \in \langle CI \rangle \subset \text{KK} \hat{G}$.

Theorem (Baaj-Skandalis)

The reduced crossed product functor $\text{KK} \hat{G} \to \text{KK} G$ is an equivalence of categories.

As a consequence, in order to prove $C \in \langle CI \rangle$ it suffices to show $C(G) \in \langle C \rangle \in \text{KK} G$. 

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Outline of the proof

We have \( C(G) \in \langle C(G/T) \rangle \) in \( KK \). This follows from (the validity of) the Baum-Connes conjecture for \( \hat{T} \) and induction.

Hence it suffices to show

\[
\text{Theorem:}\quad C(G/T) \sim C \oplus C \quad \text{in} \quad KK.
\]

In the case \( q = 1 \) this is a consequence of equivariant Poincaré duality for \( G/T \).

We need some information about the equivariant \( K \)-theory and \( K \)-homology of the Podleś sphere \( G/T \).
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We need some information about the equivariant $K$-theory and $K$-homology of the Podleś sphere $G/T$. 
$K$-theory of the Podleś sphere

The quantum group $G$ acts on the homogenous space $G/T$ from the left. Natural elements in $K^G_0(C(G/T))$ are given by the finitely generated projective $\mathcal{O}(G/T)$-modules $\Gamma(G \times T \mathbb{C}^k) = \{ x \in \mathcal{O}(SU_q(2)) | (id \otimes \pi) \Delta(x) = x \otimes z^k \}$ for $k \in \mathbb{Z}$.

Geometrically, $\Gamma(G \times T \mathbb{C}^k)$ corresponds to an induced bundle on $G/T$. 

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Natural elements in $K_0^G(C(G/T))$ are given by the finitely generated projective $O(G/T)$-modules

$$\Gamma(G \times_T \mathbb{C}_k) = \{x \in O(SU_q(2)) | (\text{id} \otimes \pi) \Delta(x) = x \otimes z^{-k}\}$$

for $k \in \mathbb{Z}$.

Geometrically, $\Gamma(G \times_T \mathbb{C}_k)$ corresponds to an *induced bundle* on $G/T$. 
The induced bundles yield elements

\[ [\Gamma(G \times_T \mathbb{C}_k)] \in KK^G(\mathbb{C}, C(G/T)). \]
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Taking into account the left action of \( C(G/T) \) by multiplication we obtain in fact elements

\[ [[\Gamma(G \times_T \mathbb{C}_k)]] \in KK^G(C(G/T), C(G/T)) \]

such that

\[ [\Gamma(G \times_T \mathbb{C}_k)] = [1] \cdot [[\Gamma(G \times_T \mathbb{C}_k)]] \]

where \( [1] \in KK^G(\mathbb{C}, C(G/T)) \) is the class of the unit homomorphism.
Dabrowski and Sitarz have constructed a spectral triple \((\mathcal{O}(G/T), L^2(G \times T \mathbb{S}), D)\) for the Podleś sphere representing the Dirac operator on \(G/T\).

The Hilbert space \(L^2(G \times T \mathbb{S})\) is the completion of \(\Gamma(G \times T \mathbb{C}^1) \oplus \Gamma(G \times T \mathbb{C}^{-1})\) for the scalar product induced from \(L^2(G)\).

Representation of \(\mathcal{O}(G/T)\) by left multiplication.

\[
D = (0 D - D 0, D^\pm |l, m\rangle \pm_{\pm} = [l + 1/2] q |l, m\rangle
\]

This defines an element \(D\) \(\in KK(G(C(G/T)), C)\).
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The Dirac operator for the Podleś sphere

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This defines an element \([D] \in KK^G(C(G/T), \mathbb{C})\).
Define elements $\alpha \in KK^G_0(C(G/T), C^2)$ by

$$\alpha = \left[ D \right] \oplus \left[ \Gamma(G \times T \cdot C^{-1}) \right] \cdot \left[ D \right]$$

and $\beta \in KK^G_0(C^2, C(G/T))$ by

$$\beta = \Gamma(G \times T \cdot C^1) \oplus - \Gamma(G \times T \cdot C^0).$$

Proposition

We have $\beta \circ \alpha = 1$, $\alpha \circ \beta = 1$ and hence $C(G/T) \cong C^2$ in $KK^G$.

This finishes the proof of the theorem.
Define elements $\alpha \in KK_0^G(C(G/T), \mathbb{C}^2)$ by

$$\alpha = [D] \oplus [[\Gamma(G \times T \mathbb{C}_{-1})]] \cdot [D]$$

and $\beta \in KK_0^G(\mathbb{C}^2, C(G/T))$ by

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This finishes the proof of the theorem.
Equivariant Poincaré duality

Let us call two $G$-YD-algebras $P$ and $Q$ equivariantly Poincaré dual to each other if there exists a natural isomorphism

$$\text{KK}_G(P \otimes A, B) \cong \text{KK}_G(A, Q \otimes B)$$

for all $G$-algebras $A$ and $B$.

Given Poincaré dual algebras $P$ and $Q$ we have natural elements

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Theorem

Let $G = SU_q(2)$. The Podleś sphere $C(G/T)$ is equivariantly Poincaré dual to itself. In fact, $KK_D(G)(C(G/T) \boxtimes A, B) \cong KK_D(G)(A, C(G/T) \boxtimes B)$ for all $G$-YD-algebras $A$ and $B$.

The element $\alpha \in KK_D(G)(C(G/T) \boxtimes C(G/T), C)$ implementing this duality is given by the Dirac operator $D$ acting on $L^2(G \times T \Sigma)$.

The representation $\phi$ of $C(G/T) \boxtimes C(G/T)$ on $L^2(G \times T \Sigma)$ is

$$\phi(f \boxtimes g)(h) = fgh.$$
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