Equivariant Spectral Geometry. II

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Some experimental findings; equivariant spectral triples on:

toric noncommutative geometry
(including and generalizing nc tori)

the manifold of quantum SU(2)

families of quantum two spheres

higher dimensional quantum orthogonal spheres

quantum projective spaces

The guiding principle:

equivariance with respect to a ‘quantum symmetry’

equivariance will build all the geometry from scratch
A recent general strategy for isospectral Dirac operators on c.q.gs. via a Drinfel’d twist
Neshveyev-Tuset, OA/0703161 ;
Beggs-Majid, QA/0506450

the quantum Dirac operator is of the form

\[ D_q = \mathcal{F} D \mathcal{F}^{-1}, \quad \Rightarrow \quad \text{Spec}(D_q) = \text{Spec}(D) \]
“There exists formulas for $q$-analogues of the Dirac operator on quantum groups ...... ; let us call $Q$ these “naive” Dirac operators. Now the fundamental equation to define the thought for true Dirac operator $D$ which we used above implicitly on the deformed 3-sphere (after suspension to the 4-sphere and for deformation parameters which are complex of modulus one) is,

$$[D]_q^2 = Q.$$ 

where the symbol $[x]_q$ has the usual meaning in $q$-analogues, ... 
The main point is that it is only by virtue of this equation that the commutators $[D, a]$ will be bounded ..... ”

A. Connes, G. L., Noncommutative Manifolds, the Instanton Algebra and Isospectral Deformations, CMP (2001).
After some initial skepticism, this programme was completed in

L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom, J.C. Varilly
The Dirac operator on $SU_q(2)$, CMP (2005).

An isospectral nc geometry on the ‘manifold underlying’ $SU_q(2)$

The guiding principle:

equivariance with respect to a ‘quantum symmetry’

equivariance will build all the geometry from scratch
\((A, \mathcal{H}, D), \quad \gamma\)
A real structure on \((\mathcal{A}, \mathcal{H}, D)\)

A \textit{real structure} \(J\) for the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is given by an antilinear isometry \(J\) on \(\mathcal{H}\) such that

\[
[\pi(a), J\pi(b)J^{-1}] \in \mathcal{I}, \quad [[D, \pi(a)], J\pi(b)J^{-1}] \in \mathcal{I}, \quad a, b \in \mathcal{A},
\]

with \(\mathcal{I}\) an operator ideal of \textit{‘infinitesimals’}

On \(\mathcal{A}(\text{SU}_q(2))\) and \(\mathcal{A}(S^2_{qt})\) one can takes \(\mathcal{I} = \mathcal{O} \mathcal{P}^{-\infty}\)
Symmetries

implemented by the action of a Hopf ∗-algebra \( \mathcal{U}_q(g) \), a quantum universal enveloping algebra;

Let \( \mathcal{U} = (\mathcal{U}, \Delta, S, \varepsilon) \) be a Hopf ∗-algebra and \( \mathcal{A} \) be a left \( \mathcal{U} \)-module ∗-algebra, i.e., there is a left action \( \triangleright \) of \( \mathcal{U} \) on \( \mathcal{A} \),

\[
h \triangleright xy = (h_{(1)} \triangleright x)(h_{(2)} \triangleright y),
\]

\[
h \triangleright 1 = \varepsilon(h)1, \quad (h \triangleright x)^* = S(h)^* \triangleright x^*,
\]

notation \( \Delta(h) = h_{(1)} \otimes h_{(2)} \).
A \ast\text{-representation } \pi \text{ of } \mathcal{A} \text{ on } \mathcal{V} \text{ (a dense subspace of } \mathcal{H}) \text{ is called } \mathcal{U}\text{-equivariant if there is a } \ast\text{-representation } \lambda \text{ of } \mathcal{U} \text{ on } \mathcal{V} \text{ s. t.}

\lambda(h) \pi(x) \xi = \pi(h(1) \triangleright x) \lambda(h(2)) \xi,

for all \( h \in \mathcal{U}, \ x \in \mathcal{A} \text{ and } \xi \in \mathcal{V}. \)

This is the same as a \ast\text{-representation of the left crossed product \ast\text{-algebra } } \mathcal{A} \ltimes \mathcal{U} \text{ defined as the \ast\text{-algebra generated by the two \ast\text{-subalgebras } } \mathcal{A} \text{ and } \mathcal{U} \text{ with crossed commutation relations}

\[ hx = (h(1) \triangleright x)h(2), \quad h \in \mathcal{U}, \ x \in \mathcal{A} \]

A linear operator \( D \) on \( \mathcal{V} \) is \textbf{equivariant} if it commutes with \( \lambda(h) \), for all \( h \in \mathcal{U} \) and \( \xi \in \mathcal{V} \)

\[ D\lambda(h) \xi = \lambda(h)D \xi \]
An antiunitary operator $J$ is **equivariant** if it leaves $\mathcal{V}$ invariant and if it is the antiunitary part in the polar decomposition of an antilinear (closed) operator $T$ that satisfies the condition

$$T\lambda(h)\xi = \lambda(S(h)^*)T\xi,$$

for all $h \in \mathcal{U}$ and $\xi \in \mathcal{V}$, where $S$ is the antipode of $\mathcal{U}$.

A (real graded) spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ is $\mathcal{U}$-equivariant if the representation of $\mathcal{A}$ and the operators $D$ and $J$ are equivariant (and $\gamma$ commutes with the equivariant representation).
The nc geometry of $A(SU_q(2))$

The algebra:

With $0 < q < 1$, let $A = A(SU_q(2))$ be the $*$-algebra generated by $a$ and $b$, with relations:

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b,$$

$$a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1$$

these state that the defining matrix is unitary

$$U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$$
\( \mathcal{A}(SU_q(2)) \) is a Hopf \(*\)-algebra (a quantum group) with

- **coproduct:**
  \[
  \Delta \left( \begin{array}{cc} a & b \\ -qb^* & a^* \end{array} \right) := \left( \begin{array}{cc} a & b \\ -qb^* & a^* \end{array} \right) \otimes \left( \begin{array}{cc} a & b \\ -qb^* & a^* \end{array} \right)
  \]

- **counit:**
  \[
  \varepsilon \left( \begin{array}{cc} a & b \\ -qb^* & a^* \end{array} \right) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
  \]

- **antipode:**
  \[
  S \left( \begin{array}{cc} a & b \\ -qb^* & a^* \end{array} \right) = \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix}
  \]
The symmetry:

The quantum universal enveloping algebra $\mathcal{U} = \mathcal{U}_q(\text{su}(2))$ is the $\ast$-algebra generated by $e, f, k$, with $k$ invertible, and relations

\[
ek = qke, \quad kf = qfk, \]

\[
k^2 - k^{-2} = (q - q^{-1})(fe - ef),
\]

the $\ast$-structure is simply

\[
k^\ast = k, \quad f^\ast = e, \quad e^\ast = f
\]
The Hopf $*$-algebra structure

- coproduct:
  \[ \Delta k = k \otimes k, \quad \Delta f = f \otimes k + k^{-1} \otimes f, \quad \Delta e = e \otimes k + k^{-1} \otimes e \]

- counit:
  \[ \epsilon(k) = 1, \quad \epsilon(f) = 0, \quad \epsilon(e) = 0 \]

- antipode:
  \[ Sk = k^{-1}, \quad Sf = -qf, \quad Se = -q^{-1}e \]
The action of $\mathcal{U}$ on $\mathcal{A}$

A natural bilinear pairing between $\mathcal{U}$ and $\mathcal{A}$,

$$\langle k, a \rangle = q^{\frac{1}{2}}, \quad \langle k, a^* \rangle = q^{-\frac{1}{2}}, \quad \langle e, -qb^* \rangle = \langle f, b \rangle = 1$$

gives canonical left and right $\mathcal{U}$-module algebra structures on $\mathcal{A}$ (they mutually commute):

$$h \triangleright x := x_1 \langle h, x_2 \rangle, \quad x \triangleleft h := \langle h, x_1 \rangle x_2$$

we use the notation $\Delta(x) = x_1 \otimes x_2$
The invertible antipode transforms the right action \( \triangleleft \) into a second left action of \( \mathcal{U} \) on \( \mathcal{A} \), commuting with the first

\[
h \cdot x \coloneqq x \triangleleft S^{-1}(\vartheta(h)),
\]

with automorphism \( \vartheta \) of \( \mathcal{U}_q(\mathfrak{su}(2)) \) given by

\[
\vartheta(k) \coloneqq k^{-1}, \quad \vartheta(f) \coloneqq -e, \quad \vartheta(e) \coloneqq -f
\]
The representation theory of $\mathcal{U}_q(\text{su}(2))$

The irreducible finite dim representations $\sigma_l$ of $\mathcal{U}_q(\text{su}(2))$ are labelled by nonnegative half-integers (the spin) $l \in \frac{1}{2}\mathbb{N}_0$:

$$\sigma_l(k) |lm\rangle = q^m |lm\rangle,$$

$$\sigma_l(f) |lm\rangle = \sqrt{[l - m][l + m + 1]} |l, m + 1\rangle,$$

$$\sigma_l(e) |lm\rangle = \sqrt{[l - m + 1][l + m]} |l, m - 1\rangle,$$

on the irreducible $\mathcal{U}$-module $\mathcal{V}_l = \text{span}\{ |lm\rangle, m = -l, \ldots, l \}$

$\sigma_l$ is a $\ast$-representation, with respect to the hermitian scalar product on $\mathcal{V}_l$ for which the vectors $|lm\rangle$ are orthonormal.

the brackets denote $q$-integers

$$[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{provided} \quad q \neq 1$$
Left regular representation of $\mathcal{A}(SU_q(2))$

The left regular representation of $\mathcal{A}$ as an equivariant representation with respect to the two left actions $\triangleright$ and $\cdot$ of $\mathcal{U}$

With $\lambda$ and $\rho$ mutually commuting representations of $\mathcal{U}$ on $\mathcal{V}$, a representation $\pi$ of the $\ast$-algebra $\mathcal{A}$ on $\mathcal{V}$ is $(\lambda, \rho)$-equivariant if

$$\lambda(h) \pi(x) \xi = \pi(h_1 \cdot x) \lambda(h_2) \xi,$$

$$\rho(h) \pi(x) \xi = \pi(h_1 \triangleright x) \rho(h_2) \xi, \quad \forall \ h \in \mathcal{U}, \ x \in \mathcal{A}, \ \xi \in \mathcal{V}$$

As a representation space the preHilbert space

$$\mathcal{V} := \bigoplus_{2l=0}^{\infty} \mathcal{V}_l \otimes \mathcal{V}_l, \quad |lmn\rangle := |lm\rangle \otimes |ln\rangle, \quad m, n = -l, \ldots, l$$
Two copies of $\mathcal{U}_q(\text{su}(2))$ act via the irreps $\sigma_l$:

$$\lambda(h) = \sigma_l(h) \otimes \text{id}, \quad \rho(h) = \text{id} \otimes \sigma_l(h)$$

A $(\lambda, \rho)$-equivariant $*$-repr $\pi$ of $A(\text{SU}_q(2))$ on the Hilbert space $\mathcal{V}$ is the left regular representation. It has the form:

$$\pi(a) |lmn\rangle = A_{lmn}^+ |l^+ m^+ n^+\rangle + A_{lmn}^- |l^- m^+ n^+\rangle,$$

$$\pi(b) |lmn\rangle = B_{lmn}^+ |l^+ m^+ n^-\rangle + B_{lmn}^- |l^- m^+ n^-\rangle,$$

with suitable explicit constants $A_{lmn}^\pm, B_{lmn}^\pm$.

Here $l^\pm := l \pm \frac{1}{2}, \quad m^\pm := m \pm \frac{1}{2}, \quad n^\pm := n \pm \frac{1}{2}$
Spin representation

We amplify the left regular representation $\pi$ of $A$ to the spinor representation $\pi' = \pi \otimes \text{id}$ on

$$W := \mathcal{V} \otimes \mathbb{C}^2 = \mathcal{V} \otimes \mathcal{V}_{\frac{1}{2}}$$

We also set $\rho' = \rho \otimes \text{id}$ on $W$.

But, we replace $\lambda$ on $\mathcal{V}$ by its tensor product with $\sigma_{\frac{1}{2}}$ on $\mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)})$$

The spinor representation $\pi'$ of $A$ on $W$ is $(\lambda', \rho')$-equivariant.

A basis $\{|j\mu n\uparrow\rangle, |j\mu n\downarrow\rangle\}$ for $W$ from its $q$-Clebsch-Gordan decomp.
Equivariant Dirac operator

Any self-adjoint operator on $\mathcal{H} = (V \otimes C^2)^{cl}$, that commutes with both actions $\rho'$ and $\lambda'$ of $\mathcal{U}_q(su(2))$ is of the form

$$D|j\mu n\uparrow\rangle = d_j^\uparrow|j\mu n\uparrow\rangle, \quad D|j\mu n\downarrow\rangle = d_j^\downarrow|j\mu n\downarrow\rangle,$$

with $d_j^\uparrow$ and $d_j^\downarrow$ real eigenvalues of $D$; depend only on $j$;

Restrictions on eigenvalues comes by requiring boundedness of the commutators $[D, \pi'(x)]$ for $x \in \mathcal{A}$

Unbounded commutators are obtained with the “naive $q$-Dirac”

$$d_j^\uparrow = \frac{2[2j + 1]}{q + q^{-1}}, \quad d_j^\downarrow = -d_j^\uparrow \quad [\text{BibikovKulish}]$$

$q$-analogues of the classical eigenvalues of $\mathcal{H} - \frac{1}{2}$;
$\mathcal{D}$ is the classical (‘round’) Dirac operator on the sphere $S^3$;

An operator $D$ with spectrum the one of the classical $\mathcal{D}$ [CL]

With eigenvalues “linear in $j$”:

$$d_j^\uparrow = c_1^\uparrow j + c_2^\uparrow, \quad d_j^\downarrow = c_1^\downarrow j + c_2^\downarrow,$$

with $c_1^\downarrow c_1^\uparrow < 0$ (for a nontrivial sign), all commutators $[D, \pi'(x)]$ for $x \in \mathcal{A}$ are bounded operators.

up to irrelevant scaling factors the choice of $c_j^\uparrow, c_j^\downarrow$ is immaterial; with $d_j^\uparrow = 2j + \frac{3}{2}, \quad d_j^\downarrow = -2j - \frac{1}{2},$

the spectrum of $D$ (with multiplicity) is the classical one

Essentially the only possibility for a Dirac operator satisfying a (modified) first-order condition
The real structure

For the l.r.r. $\mathcal{H}_\psi = L^2(\text{SU}_q(2), \psi)$, $\psi$ the Haar state; the Tomita operator $T_\psi =: J_\psi \Delta_\psi^{1/2}$ defines the positive “modular operator” $\Delta_\psi$ and the antiunitary “modular conjugation” $J_\psi$. On the basis:

$$T_\psi |lmn\rangle = (-1)^{2l+m+n} q^{m+n} |l, -m, -n\rangle.$$

With respect to the representations $\lambda$ and $\rho$ of $\mathcal{U}_q(\text{su}(2))$:

$$T_\psi \lambda(h) T_\psi^{-1} = \lambda(Sh)^*, \quad T_\psi \rho(h) T_\psi^{-1} = \rho(Sh)^*$$

i.e. the Tomita operator for the Haar state of the Hopf $*$-algebra $\mathcal{A}$ implements the antilinear involutory automorphism $h \mapsto (Sh)^*$ of the (dual) Hopf $*$-algebra $\mathcal{U}$ (cf. [Masuda-Nakagami-Woronowicz])
The commuting ∗-antirepresentation of $A$ on $\mathcal{H}_\psi$

$$\pi^\circ(x) := J_\psi \pi(x^*) J_\psi^{-1}$$

is a ∗-representation of the opposite algebra $A(SU_{1/q}(2))$.

It is equivariant:

$$\lambda(h) \pi^\circ(x) \xi = \pi^\circ(\tilde{h}_2 \cdot x) \lambda(h_{(1)}) \xi,$$
$$\rho(h) \pi^\circ(x) \xi = \pi^\circ(\tilde{h}_2 \triangleright x) \rho(h_{(1)}) \xi,$$

for all $h \in \mathcal{U}$, $x \in A$ and $\xi \in V$.

$h \mapsto \tilde{h}$ is the automorphism of $\mathcal{U}$:

$$\tilde{k} := k, \quad \tilde{f} := q^{-1} f, \quad \tilde{e} := q e.$$
The real structure for spinors is not the modular conjugation $J_\psi \oplus J_\psi$ for the spinor representation of $\mathcal{A}(\text{SU}_q(2))$

conjugation of the spinor rep. $\pi'(\mathcal{A}(\text{SU}_q(2)))$ by the modular operator yields a representation of the opposite algebra $\mathcal{A}(\text{SU}_{1/q}(2))$

this force the Dirac operator $D$ to be equivariant under the corresponding symmetry of $U_{1/q}(\text{su}(2))$ as well

this extra equivariance condition force $D$ to be a scalar operator:

no equivariant $3^+$-summable real spectral triple on $\mathcal{A}(\text{SU}_q(2))$ with the modular conjugation operator
The remedy is to modify $J$ to a non-Tomita conjugation operator while keeping with the symmetry of the spinor representation. However, the expected properties of real spectral triples do hold only “up to compact perturbations”

The strategy:

Construct “the right representation” of the algebra $\mathcal{A}$ on spinors by its symmetry alone, requiring $(\lambda', \rho')$-equivariance

Deduce the conjugation operator $J$ on spinors as the intertwiner of the left and right spinor representations

NB. These representations do not commute any longer
The conjugation operator $J$ is the antilinear operator on $\mathcal{H}$ defined explicitly on the orthonormal spinor basis by

$$J |j\mu n\uparrow\rangle := i^2(2j+\mu+n) |j, -\mu, -n, \uparrow\rangle,$$
$$J |j\mu n\downarrow\rangle := i^2(2j-\mu-n) |j, -\mu, -n, \downarrow\rangle.$$

$J$ is antiunitary and $J^2 = -1$. $J$ is equivariant.

Also, $[J, D] = 0$ with the isospectral $D$

The commutant and the first-order conditions are satisfied only up to infinitesimals of arbitrary high order:

$$[\pi'(x), J\pi'(y)J^{-1}] \in \mathcal{I},$$
$$[\pi'(x), [D, J\pi'(y)J^{-1}]] \in \mathcal{I}, \quad \mathcal{I} \subset \mathbb{OP}^{-\infty}$$
Proposition 1. The spectral triple \((\mathcal{A}(SU_q(2)), \mathcal{H}, D)\) is regular and \(3^+\)-summable. It has simple dimension spectrum given by \(\Sigma = \{1, 2, 3\}\).

Its KO-dimension is 3.

The dimension spectrum is worked out by constructing a symbol map from order zero pseudodifferential operators on \(\mathcal{A}(SU_q(2))\) to a noncommutative version of the cosphere bundle.
For all cases:
an interesting pseudo-differential calculus

Formulæ for Connes-Moscovici local index thm

Started by A. Connes, Cyclic cohomology, quantum group symmetries and the local index formula for $SU_q(2)$, (2004)

for the ‘singular’ (no limit when $q \rightarrow 1$) spectral triple in


A lot of experimental evidence for a general theory
The cosphere bundle and the dimension spectrum for $\text{SU}_q(2)$

On $\ell^2(\mathbb{N})$ with o.n. basis $\{\varepsilon_x : x \in \mathbb{N}\}$, two representation of $\mathcal{A}(\text{SU}_q(2))$:

$$\pi_{\pm}(a) \varepsilon_x := \sqrt{1 - q^{2x+2}\varepsilon_{x+1}}, \quad \pi_{\pm}(b) \varepsilon_x := \pm q^x \varepsilon_x.$$ 

not faithful: $b - b^* \in \ker \pi_{\pm}$. The quotients $\mathcal{A}(D^2_{q_{\pm}})$

$$0 \to \ker \pi_{\pm} \to \mathcal{A}(\text{SU}_q(2)) \xrightarrow{r_{\pm}} \mathcal{A}(D^2_{q_{\pm}}) \to 0,$$

are the algebras of two quantum disks $D^2_{q_{\pm}}$; the two hemispheres of the equatorial Podleś sphere $\mathbb{S}_q^2 = \mathbb{S}_{qt=0}^2$. 
The subalgebra $\mathcal{B}$ of $\Psi^0(A)$ generated by all $\delta^k(A)$ for $k \geq 0$ is indeed generated by a set of diagonal operators

$$\tilde{a}_\pm := \pm \delta(a_\pm) = Pa_\pm P + (1 - P)a_\pm (1 - P),$$
$$\tilde{b}_\pm := \pm \delta(b_\pm) = Pb_\pm P + (1 - P)b_\pm (1 - P),$$

together with the off-diagonal operators

$$a_\slash := (1 - P)a_+ P + Pa_- (1 - P), \quad b_\slash := (1 - P)b_+ P + Pb_- (1 - P),$$

with a natural decomposition of the spinor rep.

$$\pi'(a) := a_+ + a_-, \quad \pi'(b) := b_+ + b_-$$

$\pm$ map the label $j$ to $j \pm \frac{1}{2}$ of the spinor basis

and $P := \frac{1}{2} (1 + F)$, $F = \text{Sign } D$
There is a ∗-homomorphism
\[ \rho : B \to \mathcal{A}(D_{q+}^2) \otimes \mathcal{A}(D_{q-}^2) \otimes \mathcal{A}(S^1) \]

\[
\rho(\tilde{a}_+) := r_+(a) \otimes r_-(a) \otimes u, \quad \rho(\tilde{a}_-) := -q r_+(b) \otimes r_-(b^*) \otimes u^*, \\
\rho(\tilde{b}_+) := -r_+(a) \otimes r_-(b) \otimes u, \quad \rho(\tilde{b}_-) := -r_+(b) \otimes r_-(a^*) \otimes u^*. \\
\rho(a/) := \rho(b/) = 0. 
\]

\( u \) generates \( \mathcal{A}(S^1) \)

**Definition 2.** The cosphere bundle on \( SU_q(2) \), denoted by \( \mathcal{A}(S^*_q) \), is defined as the range of the map \( \rho \) in \( \mathcal{A}(D_{q+}^2) \otimes \mathcal{A}(D_{q-}^2) \otimes \mathcal{A}(S^1) \).

Element \( x \in B \) are determined by \( \rho(x) \in \mathcal{A}(S^*_q) \) up to smoothing operators (these drop out from relevant computations)
A symbol map $\sigma: A(D^2\pm) \to A(S^1)$; on generators

$$\sigma(r_{\pm}(a)) := u, \quad \sigma(r_{\pm}(b)) := 0,$$

maps $D^2\pm$ to their common boundary $S^1$

On the algebras $A(D^2\pm)$ three linear functionals $\tau_1$ and $\tau_0^{\uparrow}$:

$$\tau_1(x) := \frac{1}{2\pi} \int_{S^1} \sigma(x),$$

$$\tau_0^{\uparrow}(x) := \lim_{N \to \infty} \text{Tr}_N \pi'(x) - (N + \frac{3}{2})\tau_1(x),$$

$$\tau_0^{\downarrow}(x) := \lim_{N \to \infty} \text{Tr}_N \pi'(x) - (N + \frac{1}{2})\tau_1(x),$$

for $x \in A(D^2\pm)$; and $\text{Tr}_N(T) := \sum_{k=0}^{N} \langle \epsilon_k | T \epsilon_k \rangle$. 
With the natural restriction map,

\[ r : A(S_q^*) \to A(D_{q^+}^2) \otimes A(D_{q^-}^2). \]

**Proposition 3.** The spectral triple \((A(SU_q(2)), \mathcal{H}, D)\) has simple dimension spectrum by \(\{1, 2, 3\}\). The corresponding residues are

\[
\int T |D|^{-3} = 2(\tau_1 \otimes \tau_1)(r\rho(T)^0),
\]
\[
\int T |D|^{-2} = (\tau_1 \otimes (\tau_0^\uparrow + \tau_0^\downarrow) + (\tau_0^\uparrow + \tau_0^\downarrow) \otimes \tau_1)(r\rho(T)^0),
\]
\[
\int T |D|^{-1} = (\tau_0^\uparrow \otimes \tau_0^\downarrow + \tau_0^\downarrow \otimes \tau_0^\uparrow)(r\rho(T)^0),
\]

with \(T \in \mathcal{B}\).

\(\rho(T)^0\) the degree-zero part with respect to the \(\mathbb{Z}\)-grading on \(A(S_q^*)\) induced (in its representation) by the one-parameter group of automorphisms \(\gamma(t)\) generated by \(|D|\).
The local index formula for 3-dimensional geometries

A Fredholm index of the operator $D$, $\varphi : K_1(A) \to \mathbb{Z}$.

$$\varphi([u]) := \text{Index}(PuP) = \dim \ker PUP - \dim \ker PU^*P.$$ 

With $F = \text{Sign } D$ and $P = \frac{1}{2}(1 + F)$.

Computed by pairing $K_1(A)$ with “a nonlocal” cyclic cocycle

$$\chi_1(a_0, a_1) = \text{Tr}(a_0 [F, a_1]) ;$$

the Fredholm module $(\mathcal{H}, F')$ over $A = \mathcal{A}(\text{SU}_q(2))$ is 1-summable since all commutators $[F, \pi(x)]$ are trace-class.
The C–M local index theorem expresses the index map in terms of a local cocycle \( \phi_{\text{odd}} = (\phi_1, \phi_2) \)

\[
\phi_1(a_0, a_1) := \int a_0 [D, a_1] |D|^{-1} - \frac{1}{4} \int a_0 \nabla([D, a_1]) |D|^{-3} \\
+ \frac{1}{8} \int a_0 \nabla^2([D, a_1]) |D|^{-5},
\]

\[
\phi_3(a_0, a_1, a_2, a_3) := \frac{1}{12} \int a_0 [D, a_1] [D, a_2] [D, a_3] |D|^{-3},
\]

\( \nabla(T) := [D^2, T] \)

With \([F, a]\) traceclass for each \(a \in \mathcal{A}\),

\[
\phi_1(a_0, a_1) = \int a_0 \delta(a_1) F |D|^{-1} - \frac{1}{2} \int a_0 \delta^2(a_1) F |D|^{-2} \\
+ \frac{1}{4} \int a_0 \delta^3(a_1) F |D|^{-3},
\]

\[
\phi_3(a_0, a_1, a_2, a_3) = \frac{1}{12} \int a_0 \delta(a_1) \delta(a_2) \delta(a_3) F |D|^{-3}.
\]
With the additional use of a simple dimension spectrum not containing 0 and bounded above by 3, the Chern character $\chi_1$ is equal to $\phi_{\text{odd}} - (b + B)\phi_{\text{ev}}$ where the $\eta$-cochain $\phi_{\text{ev}} = (\phi_0, \phi_2)$ is

$$
\phi_0(a) := \left. \text{Tr}(Fa |D|^{-z}) \right|_{z=0},
$$

$$
\phi_2(a_0, a_1, a_2) := \frac{1}{24} \int a_0 \delta(a_1) \delta^2(a_2) F|D|^{-3};
$$

$$
\phi_1 = \chi_1 + b\phi_0 + B\phi_2, \quad \phi_3 = b\phi_2.
$$

With the same conditions on the dimension spectrum and commutators $[F, a]$, the local Chern character $\phi_{\text{odd}} = \psi_1 - (b + B)\phi'_{\text{ev}},$

$$
\psi_1(a_0, a_1) := \int a_0 \delta(a_1) P|D|^{-1} - \int a_0 \delta^2(a_1) P|D|^{-2}
+ \frac{2}{3} \int a_0 \delta^3(a_1) P|D|^{-3},
$$
and $\phi_{ev}' = (\phi'_0, \phi'_2)$,

\[
\phi'_0(a) := \left. \text{Tr}(a |D|^{-z}) \right|_{z=0},
\]

\[
\phi'_2(a_0, a_1, a_2) := -\frac{1}{24} \int a_0 \delta(a_1) \delta^2(a_2) F|D|^{-3}.
\]

The term in $P|D|^{-3}$ would vanish if the latter were traceclass [Connes] (this is the statement that $P$ has metric dimension 2)

Summing up, up to coboundaries, the cyclic 1-cocycles $\chi_1$ can be given by means of one single $(b, B)$-cocycle $\psi_1$:

\[
\chi_1 = \psi_1 - b\beta, \quad \text{where} \quad \beta(a) = 2 \left. \text{Tr}(Pa |D|^{-z}) \right|_{z=0}
\]
Compute $\text{Index}(PUP)$ when $U$ is the defining unitary

$$U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix},$$

acting $\mathcal{H} \otimes \mathbb{C}^2$ via the representation $\pi' \otimes \mathbf{1}_2$:

kernel of $PU^*P$ is trivial

kernel of $PUP$ contains only elements proportional to the vector

$$\begin{pmatrix} |0, 0, -\frac{1}{2}, \uparrow\rangle \\ -q^{-1}|0, 0, \frac{1}{2}, \uparrow\rangle \end{pmatrix},$$

leading to $\varphi([U]) = \text{Index}(PUP) = 1$.

Calculating $\psi_1(U^{-1}, U)$ (with the local cyclic cocycle extended by $\text{Tr}_{\mathbb{C}^2}$), up to an overall $-\frac{1}{2}$ factor gives

$$2 \int U_{kl}^* \delta(U_{lk}) P|D|^{-1} - \int U_{kl}^* \delta^2(U_{lk}) P|D|^{-2} + \frac{2}{3} \int U_{kl}^* \delta^3(U_{lk}) P|D|^{-3},$$
(summation over \( k, l = 0, 1 \))

Since \( \rho(\delta^2(U_{kl})) = \rho(U_{kl}) \), one is left to compute the degree 0 part of \( \rho(U_{kl}^* \delta(U_{lk})) \), which using the relations of \( A(D_{q\pm}^2) \) is

\[
\rho(U_{kl}^* \delta(U_{lk}))^0 = 2(1 - q^2) 1 \otimes r_-(b)^2
\]

Then,

\[
\psi_1(U^{-1}, U) = 2(1 - q^2)(2\tau_0^\uparrow \otimes \tau_0^\downarrow + \frac{2}{3}\tau_1 \otimes \tau_1)(1 \otimes r_-(b)^2) \\
- (\tau_1 \otimes \tau_0^\downarrow + \tau_0^\uparrow \otimes \tau_1)(1 \otimes 1)
\]

\[= -2.\]

With the proper coefficient:

\[\text{Index}(PUP) = -\frac{1}{2}\psi_1(U^{-1}, U) = 1.\]