A Atiyah-Singer-type index theorem for manifolds with corners

Bertrand Monthubert
Joint work with V. Nistor

Université Paul Sabatier,
118 Route de Narbonne, 31062 Toulouse Cedex, France
Email: bertrand@monthubert.net
http://bertrand.monthubert.net
**Introduction**

In Noncommutative Geometry: **pseudodifferential calculus ↔ groupoid**

**Background**: index theorem on foliations (Connes, Skandalis)

**Problem**: apply Connes’ approach to singular manifolds: understand the index theory on singular manifolds in terms of operators algebras, **using groupoids methods**.

**Scheme**: to define a pseudodifferential calculus, define a groupoid and use the general tools developed for the pseudodifferential calculus on a groupoid.

**Collaborators**: R. Lauter, P.Y. Le Gall, V. Nistor, F. Pierrot
Atiyah-Singer’s Index Theorem for closed manifolds: embed $M$ in $\mathbb{R}^n$, then

$$\mathbb{Z} \xrightarrow{=\text{}} \mathbb{Z}$$

$$\text{ind}_a^M \uparrow \cong \text{ind}_a^{\mathbb{R}^n}$$

$$K^0(TM) \xrightarrow{i!} K^0(T\mathbb{R}^n)$$

**Index theorem:** $\text{ind}_a^M = \text{ind}_a^{\mathbb{R}^n} \circ i!$

For manifolds with corners: what is the analytic index?
$G$ is a Lie groupoid (more generally a continuous family groupoid) $\rightsquigarrow$ algebra of pseudodifferential operators $\Psi^\infty(G)$

Pseudodifferential operator on $G$: $G$-equivariant continuous family of pseudodifferential operators on the fibers of $G$

Example:

If $M$ is a manifold without boundary, and $G = M \times M$, $\Psi^\infty(G)$ is the algebra of pseudodifferential operators on $M$.

If $M$ is a manifold with corners, there exists a groupoid $G(M)$ such that $\Psi^\infty(G(M))$ is the $b$-calculus of Melrose.

$$G(M) = \{(x, y, \lambda) \in M \times M \times \mathbb{R}^*_+, \rho(x) = \lambda \rho(y)\}$$

$\rho$: defining function of $\partial M$. 

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Atiyah-Singer exact sequence

\[ 0 \to C^*(G') \to \Psi^0(G') \xrightarrow{\sigma} C(S^*(G')) \to 0 \]

\((S^*(G')): \text{cosphere bundle of the Lie algebroid } A(G'))\)

**Theorem 1** *The analytic index*

\[ \text{Ind}_a: K^0(A^*(G)) \to K_0(C^*(G')) \]

*is induced by the tangent groupoid* \(G \times ]0, 1] \cup A(G) \times \{0\}.*

Think of \(A(G')\) as the tangent space.
\(G \times ]0, 1]\) is open and saturated \(\Rightarrow\)

\[
0 \rightarrow C^*(G \times ]0, 1]) \rightarrow C^*(G^T) \rightarrow C_0(A(G)) \rightarrow 0
\]

\(\Rightarrow K_*(C^*(G^T)) \xrightarrow{e_0} K_*(A^*(G))\)

and \(ind_a = e_1 \circ e_0^{-1} : K^*(A^*(G)) \rightarrow K_*(C^*(G))\).

**Remark** This is not a Fredholm index!
Define an embedding of manifolds with corners $i : M \to X$, and a commutative diagram:

$$
\begin{align*}
K_* (C^*(G(M))) & \xrightarrow{\sim} K_* (C^*(G(X))) \\
K^* (A(G(M))) & \xrightarrow{i!} K^* (A(G(X)))
\end{align*}
$$
The embedding $i : M \to X$ has to be such that:

- it is an embedding of manifolds with corners

$$K_\ast(C^\ast(G(M))) \simeq K_\ast(C^\ast(G(X))): G(X) \sim G(M)$$

sufficient

- $\text{ind}_{\alpha}^X$ is an isomorphism

Construction of $X$ such that:

- each open face of $X$ is contractible,

- each face of $M$ is the intersection of $M$ and of a face of $X$ transverse to $M$,

- there is a bijection between the open faces of $M$ and those of $X$. 
\[ K_\ast(C_\ast(G(M)))) \xrightarrow{\sim} K_\ast(C_\ast(G(X)))) \]

\[ ind^M_a \uparrow \sim \uparrow ind^X_a \]

\[ K_\ast(A(G(M)))) \xrightarrow{i_!} K_\ast(A(G(X)))) \]

\( ind_a \) can be defined through a deformation:

\[ G^T(M) = A(G(M)) \times \{0\} \cup G(M) \times (0, 1]. \]

Like in Atiyah-Singer: use \( U \) a tubular neighborhood of \( M \) in \( X \).

Two steps: diagram for the submersion \( U \rightarrow M \), then diagram for \( U \rightarrow X \).
To obtain the first commutative diagram: get a **double deformation**.

\[ A(G(U)) \to A(G(M)) \times_M U \times_M U \]

\[ A(G(M)) \times_M U \times_M U \times (0, 1) \]

**Idea:** \( U \to M \) fibration with smooth fibers, deform \( T_{vert} U \) in \( U \times_M U \) (\( t = 0 \)), then deform \( T_{hor} U = T_M \) in \( G(M) \).

\( G_{s=1} \) is equivalent to \( G(M) \).
Second step: $U$ is open in $X$, so that $G^T(U) = G^T(X)_U$ is open in $G^T(X)$.

\[
\begin{array}{c}
C^*(G(U)) \hookrightarrow C^*(G(X)) \\
\uparrow e_1 & \uparrow e_1 \\
C^*(G(U)^T) \hookrightarrow C^*(G(X)^T) \\
\downarrow e_0 & \downarrow e_0 \\
C_0(A(G(U))) \hookrightarrow C_0(A(G(X)))
\end{array}
\]
Everything put together:

\[
\begin{align*}
K^*_*(C^*_{\bullet}(G(M))) & \xleftarrow{\sim} K^*_*(C^*_{\bullet}(G(U))) \xrightarrow{} K^*_*(C^*_{\bullet}(G(X))) \\
K^*_{\bullet}(A(G(M))) & \xleftarrow{\sim} K^*_{\bullet}(A(G(U))) \xrightarrow{} K^*_{\bullet}(A(G(X)))
\end{align*}
\]