The Kähler Ricci flow on Fano manifolds
(joint work with Xiuxiong Chen)

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Ricci Flow

- On a closed manifold $M$ of dimension $m$, a smooth family of metrics $g(t)$ is called a Ricci flow solution if it satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

where $R_{ij}$ is the Ricci curvature.

- The Ricci flow was introduced by R. Hamilton in 1982. He proved the short time existence of the Ricci flow and showed that the flow will exist forever whenever Riemannian curvature is uniformly bounded. In the case $m = 3$ and the initial metric has positive Ricci curvature, the normalized flow will converge to a metric of constant positive sectional curvature.
Normalized Ricci flow

- The normalized Ricci flow solution.

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{r}{n} g_{ij}
\]

where \( r \) is the average of scalar curvature.

- The round sphere is fixed by the normalized Ricci flow solution. The round sphere evolves to a point under the unnormalized Ricci flow.
Ricci flow singularity

For the purpose to search Einstein manifold, it is better to consider the normalized Ricci flow. However, the flow may develop finite time singularity, i.e., maximal curvature norm blows up.

- Starting from every dumb bell metric on $S^2$, the normalized flow has global existence and converge to round sphere (Chow, 1991).
- Starting from some dumb bell metric on $S^n (n \geq 3)$, the normalized flow may develop neck pinch singularities.

There is a family of manifolds where the normalized flow do not have short time singularity. Actually, if regard $S^2$ as $\mathbb{C}P^1$, then we can find the high dimensional correspondence of $S^2$ as $\mathbb{C}P^n$. More generally, we can consider Fano manifolds.
Kähler manifold

- A Kähler manifold is a triple \((M, g, J)\), where \(g\) is a Riemannian metric, \(J\) is an almost-complex structure satisfying \(\nabla J = 0\), with respect to the Levi-Civita connection. In local charts, we have

\[
\nabla J = \left( \frac{\partial J^j_i}{\partial x^k} + J^j_p \Gamma^i_{pk} - J^j_i \Gamma^p_{ik} \right) \frac{\partial}{\partial x^j} \otimes dx^i \otimes dx^k = 0.
\]

- On each Kähler manifold, we have \(Ric \circ J = J \circ Ric\).
- Metric form: \(\omega(X, Y) = g(JX, Y)\) for vector fields \(X, Y\).
- Ricci form: \(\rho(X, Y) = Ric(JX, Y)\).
Fano manifolds

- A Fano manifold is a complex manifold \((M, J)\) such that \(K_M^{-1}\) is ample. In other words, for large enough integer \(\nu\), the map

\[
\iota : M \mapsto \mathbb{CP}^N, \\
x \mapsto [s_0(x) : s_1(x) : \cdots : s_N(x)].
\]

is an embedding map (Kodaira), where \(\{s_i\}_{i=0}^N\) is a basis of \(H^0(K_M^{-\nu})\).

- The metric form \(\omega = \frac{1}{\nu} \iota^* \omega_{FS}\) is Kähler and \([\omega] = 2\pi c_1(M, J)\).

- Note that \([\rho] = 2\pi c_1(M, J)\) always. Then \([\omega] - [\rho] = 0\).
Kähler Ricci flow on Fano manifolds

▶ Choose a metric form in the class $2\pi c_1(M, J)$, then run the parabolically normalized Ricci flow

$$\frac{\partial}{\partial t} g = -\text{Ric} + g. \quad (1)$$

▶ Kähler condition is preserved by the Ricci flow.

$$\left\{ \frac{\partial}{\partial t} (\nabla J) \right\}^j_{ik} = J^p_i \frac{\partial}{\partial t} \Gamma^j_{pk} - J^j_p \frac{\partial}{\partial t} \Gamma^p_{ik}$$

$$= (J^j_p R_{pi} - J^p_i R_{jp}),k + (J^j_p R_{pi} - J^p_i R_{jk}),p + (J^p_i R_{pk} - J^j_p R_{ik}),p = 0$$

since $\text{Ric} \circ J = J \circ \text{Ric}$ on Kähler manifold.

▶ Along the flow, the metric form stays in the class $2\pi c_1(M, J)$.

$$\frac{\partial}{\partial t} \omega = -\rho + \omega, \quad \frac{\partial}{\partial t} [\omega] = -[\rho] + [\omega] = 0.$$
Evolution of the Kähler potentials

- Since $[\omega_t]$ is always the same as the initial class $[\omega] = 2\pi c_1(M, J)$, we have

$$\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$$

where $\varphi_t$ is called the Kähler potential.

- Denote $\varphi = \varphi_t$, then the evolution of $\varphi$ along the Kähler Ricci flow is

$$\dot{\varphi} = \log \frac{\omega^n}{\varphi^n} + \varphi - h_\omega$$

where $h_\omega$ is the Ricci potential of $\omega$, i.e., $\rho(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega$.

For simplicity of notation, we call the Ricci flow solution in $2\pi c_1(M, J)$ of a Fano manifold $(M, J)$ an **anti-canonical Kähler Ricci flow**.
Flow singularities

- **No finite time singularity.** Based on the celebrated estimate of Yau, H.D. Cao proved (1985) that on each finite time interval $[0, T]$ and positive integer $k$, the potential function $\varphi$ has uniform $C^k$-bound. Therefore, the Kähler Ricci flow has no finite time singularity.

- **Classification of infinite time singularity.** Note the difference between the potential Kähler Ricci flow (equation (2)) and the metric Kähler Ricci flow (equation (1)). It is possible that $\varphi(t)$ does not converge smoothly but $g(t)$ has uniformly bounded Riemannian geometry. If this case happens, we see the “jumping” of complex structures.

We shall focus on the Riemannian geometry part of $g(t)$, or the behavior of the flow (1).
Cheeger-Gromov convergence, definitions

A sequence of pointed manifolds \((M^m_i, x_i, g_i)\) is said to converge to \((\bar{M}, \bar{x}, \bar{g})\) in \(\hat{C}^k\)-Cheeger-Gromov topology if

- For each fixed \(r > 0\), we have \(d_{GH}(B_{g_i}(x_i, r), B_{\bar{g}}(\bar{x}, r)) \to 0\).

\(\bar{M}\) has a regular-singular decomposition \(\mathcal{R} \cup S\), where \(\mathcal{R}\) is an open \(C^k\)-manifold of dimension \(m\) and \(S\) is measure (m-Hausdorff measure, or volume) zero. The regular part \(\mathcal{R}\) has an exhaustion \(\bigcup_{p=1}^{\infty} U_k\). For each \(p\) and large \(i\), there are diffeomorphisms \(\varphi_{p,i} : U_p \to \varphi_{p,i}(U_i) \subset B(x_i, i) \subset M_i\) such that

\[ \varphi_{p,i}^* g_i \xrightarrow{C^k} \bar{g}. \]
Cheeger-Gromov convergence, conventions

- In these notations, $\hat{C}^k$-Cheeger-Gromov is different from $C^k$-Cheeger-Gromov. The later case does not deal with singularities.
- If the diameters of $M_i$ are uniformly bounded, then $x_i$ will be dropped.
- More structures can be attached to $\hat{C}^k$-Cheeger-Gromov convergence. For example, if $(M_i, g_i, J_i)$ are Kähler manifolds, the notation

$$\left( M_i, x_i, g_i, J_i \right) \xrightarrow{\hat{C}^k} (\bar{M}, \bar{x}, \bar{g}, \bar{J})$$

has an extra meaning

$$\varphi_{p_i}^* J_i \xrightarrow{C^k} \bar{J}.$$
Cheeger-Gromov convergence, examples

- $(M_i, g_i)$ has uniformly bounded geometry, $S = \emptyset$.
- $(M_i, g_i)$ are non-collapsed Kähler Einstein surfaces with uniform $\int |Rm|^2$ bound, $(\bar{M}, \bar{g})$ is a Kähler Einstein orbifold surface (Anderson, Bando-Kasue-Nakajima, Tian, 1989-1990).
- $(M_i, g_i)$ are non-collapsed Kähler Einstein manifolds, then $S$ has Hausdorff co-dimension at least 4 (Cheeger-Colding-Tian theory, 2002). Furthermore, $\bar{M}$ is a projective normal variety if it is compact (Donaldson-Sun, 2012).
The Kähler Ricci flow on Fano manifolds

Singularity of the flow

Hamilton-Tian conjecture

**Conjecture**[Hamilton-Tian, 1997]:

Suppose \( \{(M, g(t), J), 0 \leq t < \infty\} \) is an anti-canonical Kähler Ricci flow. Then for each sequence \( t_i \to \infty \), by taking subsequence if necessary, we have

\[
(M, g(t_i), J) \xrightarrow{C^\infty} (\bar{M}, \bar{g}, \bar{J}).
\]

The limit space \( \bar{M} \) has a regular-singular decomposition \( \bar{M} = R \cup S \). The singular part \( S \) has Hausdorff codimension at least 4. The metric on regular part \( R \) satisfies Kähler Ricci soliton equation, i.e.,

\[
\text{Ric} + \nabla \tilde{\nabla} f - g = 0, \quad \nabla \nabla f = \tilde{\nabla} \tilde{\nabla} f = 0
\]

for some smooth function \( f \) on \( R \).
Perelman’s functional

Suppose $f$ is a smooth function on $(M, g)$, $\tau > 0$ a number, then

$$W(f, g, \tau) = (4\pi\tau)^{-\frac{m}{2}} \int_M \left\{ \tau (R + |\nabla f|^2) + f - m \right\} e^{-f} dv,$$

$$\mu(g, \tau) = \inf_f W(g, \tau),$$

where infimum is taken over $f$ satisfying $(4\pi\tau)^{-\frac{m}{2}} \int_M e^{-f} dv = 1$, $m$ is the real dimension of the manifold $M$. Along the anti-canonical Kähler Ricci flow, $m = 2n$, $\mu(g, \frac{1}{2})$ is monotonely increasing.

$$\frac{d}{dt} \mu \left( g, \frac{1}{2} \right)$$

$$= (2\pi)^{-n} \int_M \left\{ |Ric + \nabla \tilde{\nabla} f - g|^2 + |\nabla \nabla f|^2 + |\tilde{\nabla} \tilde{\nabla} f|^2 \right\} e^{-f} dv \geq 0,$$

where $f$ is the minimizer of the functional $\mu(g, \tau)$. 
Perelman’s estimates

Based on his functional, Perelman proved the following celebrated estimate along the Kähler Ricci flow.

- Diameter of \((M, g(t))\) is uniformly bounded.
- Scalar curvature of \((M, g(t))\) is uniformly bounded.
- Normalized Ricci potential is uniformly bounded in \(C^1\).

After some modifications, Perelman’s estimate can be written as

\[
Diam(g(t)) + |R|_{g(t)} + |\dot{\phi}| + |\nabla \dot{\phi}|_{g(t)} < C
\]

for all \(t \in [0, \infty)\).
Other important estimates

There are many important estimates in this field. Just name a few which are essentially related to our work.

- Perelman’s no-local-collapsing, $r^{-2n} \text{Vol}(B(x, r)) \geq \kappa$.
- Sobolev constant $C_S$ is uniformly bounded by Q.S. Zhang and R.G. Ye.
- Volume ratio upper bound by Q.S. Zhang and Chen-Wang. In other words, $r^{-2n} \text{Vol}(B(x, r)) \leq C$.

In short, we are considering volume doubling spaces with uniform Sobolev constant.
Progresses

Understanding of the convergence of the Kähler Ricci flow (1).

- Hamilton(1988), if $n = 1$ and the initial metric has positive $R$.
- Chow(1991), if $n = 1$. Hamilton and Chow’s results were proved much before the general conjecture was written down.
- Sesum(2004), in the case that $|Ric| < C$ along the flow.
- Chen-Wang(2009), if $n = 2$, based on $L^2$-bound of $|Rm|$, which holds by Perelman’s bound of scalar curvature.
- Tian-Zhang(2013), if $n = 3$, based on $L^4$-bound of $|Ric|$.
- Chen-Wang(2014), for every $n \geq 3$.

The above list only contains some progresses in the convergence of (1). For the convergence of (2), there are important works of Phong-Song-Sturm-Weinkove(2007), Tosatti(2008), Széklehydi(2009), etc.
A more general question

The key of Chen-Wang’s solution is that the weak compactness of Kähler Ricci flow solutions should be considered in a more general level. We need to consider the combination of the following structures.

- The metric space structure.
- The Ricci flow structure.
- The line bundle structure.

Combined the estimates from these three structures, we study the weak compactness of the evolving line bundles over a Fano manifold.
A more general question

Starting from the general weak-compactness theorem of evolving line bundles, by omitting extra structures and taking advantage of monotonicity of Perelman’s functional, we obtain the proof of Hamilton-Tian conjecture.

- Step 1. Restricting the weak compactness of the bundles to their base manifolds, we obtain weak compactness of Ricci flows.
- Step 2. Restricting the weak compactness of the Ricci flows to their time slices, we obtain weak compactness of manifolds.
- Step 3. Taking the metrics from a given Kähler Ricci flow with time tending to $\infty$, we obtain asymptotic weak compactness of a fixed flow. The singular set $S$ of the limit space has codimension at least 4.
- Step 4. Applying monotonicity of Perelman’s functional, we obtain the regular part of the limit space is a Kähler Ricci soliton.
Statement of the results

We call \( \mathcal{LM} = \{(M^n, g(t), J, L, h(t)), t \in (-T, T) \subset \mathbb{R}\} \) a polarized Kähler Ricci flow if

- \( \mathcal{M} = \{(M^n, g(t), J), t \in (-T, T)\} \) is a Kähler Ricci flow solution.
- \( L \) is a Hermitian line bundle over \( M \), \( h(t) \) is a family of smooth metrics on \( L \) whose curvature is \( \omega(t) \), the metric form compatible with \( g(t) \) and the complex structure \( J \).

The first Chern class of \( L \) is \([\omega(t)]\), which does not depend on time. So a polarized Kähler Ricci flow stays in a fixed integer Kähler class.
The evolution equation of $g(t)$ can be written as

$$
\frac{\partial}{\partial t} g_{ij} = -R_{ij} + \lambda g_{ij},
$$

(3)

where $\lambda = \frac{c_1(M)}{c_1(L)}$.

Since the flow stays in the fixed class, we can let $\omega_t = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi$. Then $\dot{\phi}$ is the Ricci potential, i.e.,

$$
\sqrt{-1} \partial \bar{\partial} \phi = -Ric + \lambda g.
$$

Note the choice of $\phi$ is unique up to adding a constant. So we can always modify the choice of $\phi$ such that $\sup_{M} \dot{\phi} = 0$. 

Moduli space

For simplicity, we denote $\mathcal{K}(n, A)$ as the collection of all the polarized Kähler Ricci flows $\mathcal{LM}$ satisfying the following estimate

$$
\begin{cases}
T \geq 2, \\
C_S(M) + \frac{1}{\text{Vol}(M)} + |\dot{\phi}|_{C^0(M)} + |R - n\lambda|_{C^0(M)} \leq A.
\end{cases}
$$

(4)

Here $C_S$ means the Sobolev constant, $A$ is a uniform constant. In this paper, we study the structure of polarized Kähler Ricci flows locating in the space $\mathcal{K}(n, A)$.

The motivation behind (4) arises from the fundamental estimate of diameter, scalar curvature, $C^1$-norm of Ricci potential, and Sobolev constant along the anti-canonical Kähler Ricci flows.
The Kähler Ricci flow on Fano manifolds

Main theorem

Weak compactness theorem

Theorem (Chen-Wang, 2014)

Suppose \( \mathcal{LM}_i \in \mathcal{K}(n, A) \), \( x_i \in M_i \). Then we have

\[
(\mathcal{LM}, x_i) \overset{\hat{C}_\infty}{\longrightarrow} (\overline{\mathcal{LM}}, \bar{x}),
\]

where \( \overline{\mathcal{LM}} \) is a polarized Kähler Ricci flow solution on an analytic normal variety \( \bar{M} \), whose singular set \( S \) has Minkowski codimension at least 4, with respect to each \( \bar{g}(t) \). Moreover, if \( \bar{M} \) is compact, then it is a projective normal variety with at most log-terminal singularities.
Asymptotic limit of a Kähler Ricci flow

Theorem (Chen-Wang, 2014)
Suppose \( \{(M^n, g(t)), 0 \leq t < \infty\} \) is an anti-canonical Kähler Ricci flow solution on a Fano manifold \((M, J)\). For every \( s > 1 \), define
\[
g_s(t) \triangleq g(t + s), \quad M_s \triangleq \{(M^n, g_s(t)), -s \leq t \leq s\}.
\]
Then for every sequence \( s_i \to \infty \), by taking subsequence if necessary, we have
\[
(M_{s_i}, g_{s_i}) \overset{\hat{C}^\infty}{\longrightarrow} (\bar{M}, \bar{g}),
\]
where the limit space-time \( \bar{M} \) is a Kähler Ricci soliton flow solution on a Q-Fano normal variety \((\bar{M}, \bar{J})\).

This theorem confirms the Hamilton-Tian conjecture(1997).
Model space: motivation

- Scalar-flat Ricci flow is Ricci-flat.
  \[
  \frac{d}{dt} R = \Delta R + 2|Ric|^2.
  \]
  
- “Almost” scalar-flat Ricci flow is “almost” Ricci-flat.

Therefore, the local structure of a Ricci flow with bounded scalar curvature should be approximated by Ricci-flat spaces. In our case, they are the non-collapsed, Calabi-Yau spaces with mild singularities.

- Attention: singularities cannot be avoided.
Model space: definition

Let $\mathcal{H}(n, \kappa)$ be the collection of length spaces $(X, g)$ with the following properties.

1. $X$ has a disjoint regular-singular decomposition $X = \mathcal{R} \cup \mathcal{S}$.
2. The regular part $\mathcal{R}$ is a nonempty, open Calabi-Yau manifold.
3. $\mathcal{R}$ is weakly convex.
4. $\dim_{\mathcal{M}} \mathcal{S} < 2n - 3$, where $\mathcal{M}$ means Minkowski dimension.
5. Let $v$ be the volume density function, i.e.,
   \[
   v(x) \triangleq \lim_{r \to 0} \frac{|B(x, r)|}{\omega_{2n} r^{2n}}
   \]
   for every $x \in X$. Then $v \equiv 1$ on $\mathcal{R}$ and $v \leq 1 - 2\delta_0(n)$ on $\mathcal{S}$.
6. The asymptotic volume ratio $\text{avr}(X) \geq \kappa$. 
The model space moduli has the following properties.

- $\mathcal{MS}(n, \kappa)$ is compact under the pointed $\hat{C}^\infty$-Cheeger-Gromov topology.
- One can further improve regularity of the space itself to obtain $\dim \mathcal{M} = 2n - 4$, $\mathcal{R}$ is strongly convex.

These can be proved following the classical route of the weak compactness theory developed by Cheeger, Colding, Tian and Naber, etc.

- Remark: Compactness, rather than weak-compactness, is important for a moduli space to be chosen as a model.
Model space: trivial extension

Fix \((M, g) \in \mathcal{K}(n, \kappa)\), we can trivially add more structures to \((M, g)\).

- Ricci flow structure, by letting \(g(t) \equiv g\).
- Line bundle structure, by attaching trivial line bundle to \(M\).

With these extra structures, the following fundamental work can be applied.

- Perelman’s work on the Ricci flow, reduced distance, reduced volume.
- Donaldson-Sun’s work on the Bergman functions, i.e., partial-\(C^0\)-estimate for non-collapsed Kähler Einstein manifolds.
Model space: Perelman’s tools for the Ricci flow

Following tools are important:

- Reduced distance $l = l((x, 0), (y, -\tau))$.
- Reduced volume $V = V((x, 0), \tau)$.
- $W$-functional $W = \mu(g, f, \tau)$.

On Ricci-flat space, we have

$$l = \frac{d^2}{4\tau}, \quad V((x, 0), \tau) = \int_X (4\pi \bar{\tau})^{-\frac{m}{2}} e^{-\frac{d^2}{4\tau}} dv,$$

$$\lim_{\tau \to \infty} V((x, 0), \tau) = \text{avr}(X) = \lim_{r \to \infty} \frac{|B(x, r)|}{\omega_m r^m}.$$
Model space: line bundle estimates

Bergman function is defined as

\[ b^{(k)}(x) = \log \sum_{i=0}^{N_k} \left\| S_i^{(k)} \right\|_h^2(x). \]

Donadson-Sun’s argument of estimating \( b^{(k)} \) can be applied, whenever the following conditions are satisfied.

- In the limit space, every tangent space is a “good” metric cone.
- The singular set of each tangent space has Minkowski codimension strictly greater than 2.
- Sobolev constants and Ricci potentials uniformly bounded.
Model space: compactness

In the non-collapsed Kähler Einstein case.

- By Colding’s result, volume converges. Every tangent space is a volume cone.
- By Cheeger-Colding’s theory, every volume cone is a metric cone.
- By Cheeger-Colding-Tian’s theory, singular set has Hausdorff codimension at least 4.
- By Colding-Naber’s work, regular set is convex.
- By Chen-Donaldson’s and Cheeger-Naber’s work, singular set has Minkowski codimension at least 4.
- By Donaldson-Sun’s work, compact Kähler Einstein limit has projective variety structure.
An intuitive parabolic generalization

For the Kähler Ricci flow, one expects to argue along a similar route.

- Reduced volume converges.
- Tangent flow is a flow over a metric cone.
- Regular set is convex.
- Singular set has Minkowski codimension at least 4.
- Compact limits are varieties.

Although the above route works for static Kähler Ricci flow (i.e. Kähler Einstein) solutions, it does not work directly for general moving Kähler Ricci flow. In order this to work, one must need the existence of a “limit Ricci flow”.
A workable parabolic generalization

We developed the parabolic version of weak compactness along another route.

- Define polarized canonical radius $pcr$.
- Show the convergence of polarized flows under the assumption $pcr \geq r_0$.
- Show that there is an a priori estimate of $pcr$ in the moduli $\mathcal{K}(n, A)$. 
Polarized canonical radius

For each $X \in \mathcal{K}\mathcal{I}(n, \kappa)$, there exist natural line bundle structure and flow structure attached to it. Denote the evolving line bundle by $\mathcal{L}\mathcal{X}$. Then the local structure of every polarized Kähler Ricci flow $\mathcal{L}\mathcal{M} \in \mathcal{KH}(n, A)$ can be locally “approximated” by some $\mathcal{L}\mathcal{X}$.

To make this approximation precise, we define the $\text{pcr}$ (polarized canonical radius) to be a positive quantity $r = \frac{1}{j}$ such that under the scale of $r$, the structure of $\mathcal{L}\mathcal{M}$ and $\mathcal{L}\mathcal{X}$ are close enough such that the weak compactness theory of $\mathcal{L}\mathcal{M}$ and $\mathcal{L}\mathcal{X}$ can be done following the same route. The important things are:

- $\text{pcr}(x, 0) \geq 1$ if $x \in X \in \mathcal{K}\mathcal{I}(n, \kappa)$ (x maybe singular point).
- $\text{prc}(y, 0) > 0$ if $y \in M$, and $\mathcal{L}\mathcal{M} \in \mathcal{KH}(n, A)$. 


Definition of PCR

Definition
Suppose \((M, g, J, L, h)\) is a polarized Kähler manifold satisfying

\[ \text{Osc}_M \phi + C_S(M) + |\lambda| \leq B, \; x \in M. \]

We say the polarized canonical radius of \(x\) is not less than 1 if

- \(\text{cr}(x) \geq 1.\)

- \(\sup_{1 \leq j \leq 2k_0} b^{(j)}(x) \geq -2k_0,\)

where \(k_0 = k_0(n, B)\). For every \(r = \frac{1}{j}, j \in \mathbb{Z}^+,\) we say the polarized canonical radius of \(x\) is not less than \(r\) if the rescaled polarized manifold \((M, j^2 g, J, L^j, h^j)\) has polarized canonical radius at least 1 at the point \(x\). Fix \(x,\) let \(\text{pcr}(x)\) be the supreme of all the \(r\) with the above property and call it as the polarized canonical radius of \(x.\)
Definition of Canonical radius

Definition
We say that the canonical radius (with respect to model space $\mathcal{K} S(n, \kappa)$) of a point $x_0 \in M$ is not less than $r_0$ if for every $r < r_0$, we have the following properties.

1. Volume ratio estimate: $\kappa \leq \omega^{-1} 2^{-n} r^{-2n} |B(x_0, r)| \leq \kappa^{-1}$.

2. Regularity estimate: $r^{2+k} |\nabla^k Rm| \leq 4 c_a^{-2}$ in the ball $B(x_0, \frac{1}{2} c_a r)$ for every $0 \leq k \leq 5$ whenever $\omega^{-1} 2^{-n} r^{-2n} |B(x_0, r)| \geq 1 - \delta_0$.

3. Density estimate: $r^{2p_0 - 2n} \int_{B(x_0, r)} v(r)(y)^{-2p_0} dy \leq 2E$.

4. Connectivity estimate: $B(x_0, r) \cap \mathcal{F}_{\frac{1}{50} c_b r} (M)$ is $\frac{1}{2} \epsilon_b r$-regular-connected on the scale $r$.

Then we define canonical radius of $x_0$ to be the supreme of all the $r_0$ with the properties mentioned above. We denote the canonical radius by $cr(x_0)$. 
Important technical difficulties

The strategy is simple. We define $p_{cr}$ for the purpose of weak compactness theory, then we use a quantitative maximum principle argument to obtain a lower bound of $p_{cr}$. Then difficulties are pushed to the estimates of the flows whenever $p_{cr}$ has a fixed lower bound.

- Rough metric equivalence.
- Long-time pseudo-locality.
- Local tangent flow structure.
Rough metric equivalence: motivation

Suppose $\mathcal{LM} \in \mathcal{K}(n, A)$. Let $\tilde{\omega}_t$ be the pull back of the Fubini-Study metric by orthonormal basis of $L$ with respect to $\omega_t$ and $h_t$. Then we have the evolution inequality of $\tilde{\omega}_t$:

$$-2A\tilde{\omega}_t \leq \frac{d}{dt} \tilde{\omega}_t \leq 2A\tilde{\omega}_t. \quad (7)$$

In particular, we have

$$e^{-2A|t|} \tilde{\omega}_0 \leq \tilde{\omega}_t \leq e^{2A|t|} \tilde{\omega}_0. \quad (8)$$
Rough metric equivalence: realization

Suppose \((M, L)\) is a polarized Kähler manifold satisfying the following conditions

1. \(|B(x, r)| \geq \kappa \omega_{2n} r^{2n}, \forall x \in M, 0 < r < 1.\)
2. \(|b| \leq 2c_0\) where \(b\) is the Bergman function.
3. \(\|\nabla S\| \leq C_1\) for every \(L^2\)-unit section \(S \in H^0(M, L)\).

For every positive number \(a\), define \(\Omega(x, a)\) to be the path connected component of

\[
\left\{ z \left| \|S\|^2(z) \geq e^{-2a-2c_0}, \|S\|^2(x) = e^{b(x)}, \int_M \|S\|^2 \, dv = 1 \right. \right\}.
\]

containing \(x\). Then we have

\[ B(x, r) \subset \Omega(x, a) \subset B(x, \rho) \]

for some \(r = r(n, \kappa, c_0, C_1, a)\) and \(\rho = \rho(n, \kappa, c_0, C_1, a)\).
Short-time pseudo-locality

**Theorem (Perelman)**

For every $0 < \alpha < \frac{1}{100m}$, there exists $\delta > 0$, $\epsilon > 0$ with the following property. Suppose we have a smooth solution to the Ricci flow on time $0 \leq t \leq \epsilon^2$, and assume that at $t = 0$ we have $R \geq -1$ and $I(B_{g_0}(x_0, 1)) \geq (1 - \delta)I(\mathbb{R}^m)$, where $R$ is scalar curvature, $I$ is isoperimetric constant. Then we have estimates

$$|Rm| \leq \alpha t^{-1} + \epsilon^{-2},$$

$$B_g(t)\left(x, \sqrt{t}\right)_{d\mu_g(t)} \geq \kappa' t^{\frac{m}{2}},$$

whenever $0 < t \leq \epsilon^2$, $d_g(t)(x, x_0) < \epsilon$. 
Theorem (Chen-Wang)

Suppose $\mathcal{L}M \in \mathcal{K}(n, A; 1)$. Suppose $x_0 \in M$, $\Omega = B_{g_0}(x_0, r)$, $\Omega' = B_{g_0}(x_0, \frac{r}{2})$. At time $t = 0$, suppose the isoperimetric constant estimate $I(\Omega) \geq (1 - \delta_0)I(\mathbb{C}^n)$ holds for $\delta_0 = \delta_0(n)$. Then we have

$$|\nabla^k Rm|(x, t) \leq C_k, \quad \forall \ k \in \mathbb{Z}^{\geq 0}, \ x \in \Omega', \ t \in [-1, 1],$$

where $C_k$ is a constant depending on $n, A, r$ and $k$. 

Long-time pseudo-locality
For simplicity, we focus on the forward direction only. Suppose $\omega_t$ is uniformly regular at $x$. Then the long-time-pseudolocality can be obtained as follows.

- Rough metric equivalence implies good heat kernel estimate.
- Chern-Lu’s inequality in the parabolic version, together with heat kernel estimate, to obtain bound of $\Lambda_{\omega_0}\omega_t$.
- Perelman’s short-time pseudo-locality.
- Blowup argument and Liouville’s theorem to show higher order estimate of $\omega_t$ around $x$. 
Local application of $W$-functional

Suppose $LM_i \in \mathcal{K}(n, A; 1)$ satisfies $\sup_{\mathcal{M}_i}(|R| + |\lambda|) \to 0$. Let $u_i$ be the fundamental solution of the backward heat equation $[-\frac{\partial}{\partial t} - \Delta + R] u_i = 0$ based at the space-time point $(x_i, 0)$. Then $u_i$ converges to a limit solution $\bar{u}$ on $\mathcal{R} \times (-1, 0]$, i.e.,

$$\left[-\frac{\partial}{\partial t} - \Delta + R\right] \bar{u} = 0.$$

Moreover, we have

$$\int\int_{\mathcal{R} \times (-1, 0]} 2|t| \left| Ric + \nabla \nabla \bar{f} + \frac{\bar{g}}{2t}\right|^2 \bar{u} d\bar{g} \leq C,$$

where $C = C(n, A)$, $\bar{u} = (4\pi|t|)^{-n} e^{-\bar{f}}$. 
Tangent flow is over a metric cone

Suppose $L M_i$ is a sequence of polarized Kähler Ricci flow solutions in $\mathcal{K}(n, A; r_0)$, $x_i \in M_i$. Let $(\tilde{M}, \tilde{x}, \tilde{g})$ be the limit space of $(M_i, x_i, g_i(0))$, $\tilde{y}$ be an arbitrary point of $\tilde{M}$. Then every tangent space of $\tilde{y}$ is an irreducible metric cone. Note that

$$\int \int_{\hat{R} \times (-\infty, 0]} 2|t| \left| Ric + \nabla \nabla \hat{f} + \frac{\hat{g}}{2t} \right|^2 \hat{u} \hat{v} = 0.$$

Since $Ric \equiv 0$ on $\hat{R}$, we have

$$\nabla \nabla \hat{f} + \frac{\hat{g}}{2t} = 0, \quad \text{on} \quad \hat{R}.$$

Then use the high-codimension of $\hat{S}$ and rigidity of the above equation, to obtain the metric cone structure.
Thanks!