Moduli Problems in Sasakian Geometry

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May 21, 2015,
Recent Advances in Kähler Geometry,
Vanderbilt University
Fundamental Problems

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1. Given a manifold determine how many contact structures $D$ of Sasaki type there are.

2. Given a contact structure or isotopy class of contact structures:
   - Determine the space of compatible Sasakian structures.
   - Determine the (pre)-moduli space of Sasaki classes.
   - Determine the (pre)-moduli space of extremal Sasakian structures.
   - Determine those of constant scalar curvature (cscS). How many?
   - Determine the (pre)-moduli space of Sasaki-Einstein and/or $\eta$-Einstein structures.
   - Determine the (pre)-moduli space of Sasakian structures with the same underlying CR structure.
   - Determine those having distinct underlying CR structures within the same isotopy class of contact structures.

We give partial answers to these problems for particular cases. My talk is based on joint work with various colleagues: Leonardo Macarini, Justin Pati, Christina Tønnesen-Friedman, and Otto van Koert.
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- A **contact 1-form** $\eta$ such that

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for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \text{Ker} \ \eta$ of $TM$ with a conformal symplectic structure. So $\{\text{oriented contact 1-forms in } \mathcal{D}\} \approx C^\infty(M)^+$
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- Unique vector field \( \xi \), called the **Reeb vector field**, satisfying
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Contact bundle $\mathcal{D} \rightarrow$ choose almost complex structure $J$ extend to an endomorphism $\Phi$ with $\Phi \xi = 0$ with a compatible metric $g = d\eta \circ (\Phi \otimes 1) + \eta \otimes \eta$. Quadruple $S = (\xi, \eta, \Phi, g)$ called contact metric structure.
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- The pair $(\mathcal{D}, J)$ is a **strictly pseudo-convex almost CR structure** ($s\psi$CR structure).

- If $(\mathcal{D}, J)$ is an **integrable** CR structure, and $\mathcal{L}_\xi g = 0$ then $S = (\xi, \eta, \Phi, g)$ is a **Sasakian** structure. Then contact manifold $(M, \mathcal{D})$ is of **Sasaki type**.
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- **Gray Stability Theorem:** On a closed contact manifold all deformations are trivial.
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Think of the cone $(M \times \mathbb{R}^+, \omega)$ and smoothing singularity at cone point.
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Distinguishing Contact Structures

- **Contact Invariants.**
  - **Gray Stability Theorem:** On a closed contact manifold all deformations are trivial.
  - A classical invariant: The **first Chern class**: \( c_1(D) \).
  - **contact homology:** has serious transversality problems, so we work with **fillings**.

**Definition**

A (strong) symplectic filling of \((M, D)\) is a compact symplectic manifold \((W, \omega)\) such that \(\partial W = M\), there is a local outward pointing vector field \(\Psi\) on \(W\) such that \(\mathcal{L}_\Psi \omega = \omega\) and \(D = \ker(\Psi \mid \omega)\mid_M\). If \(\Psi\) is globally defined \((W, \omega)\) is a **Liouville filling**. It is a **holomorphic filling** if \(W\) has a complex structure \(J\) such that \((M, J)\) is strictly pseudo-convex and \(D = TM \cap JTM\). It is a **Stein (Kähler) filling** if \((W, \omega)\) is biholomorphic to a Stein (Kähler) manifold.

- Think of the cone \((M \times \mathbb{R}^+, \omega)\) and smoothing singularity at cone point.
- Kähler fillability coincides with holomorphic fillability. **Stein fillable** implies **Liouville fillable**.
- For a **Liouville filling** \((W, \omega)\), the symplectic form \(\omega\) is **exact**.
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- Under the right assumptions \(SH^{+, S^1}(W)\) is a **contact invariant**.
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For a convenient Liouville filling $(W, \omega)$, the mean Euler characteristic is defined by

$$\chi_m(W) = \frac{1}{2} \left( \liminf_{N \to \infty} \frac{1}{N} \sum_{i=-N}^{N} (-1)^i sb_i(W) + \limsup_{N \to \infty} \frac{1}{N} \sum_{i=-N}^{N} (-1)^i sb_i(W) \right)$$

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- $\chi_m(W)$ and $SH^+_{i,S^1}(W)$ allows us to distinguish components of the Sasaki moduli space.
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On a highly connected manifold of dimension greater than five, any contact structure $D$ satisfies $c_1(D) = 0$.

On a simply connected rational homology sphere, $c_1(D) = 0$. 
Three Types of **Deformations** of Sasakian Structures

1. **Fix CR structure** $(D, J)$, deform characteristic foliation $F$. This gives rise to Sasaki cones. After this type of deformation the transverse holonomy becomes irreducible.

2. **Fix contact structure** $D$, deform transverse complex structure (CR) $J$. This gives rise to Sasaki bouquets. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\text{Con}(M, D)$.

3. **Fix characteristic foliation** $F$, deform contact structure $D$. This is used to obtain extremal Sasaki metrics. This type of deformation does not change the transverse holonomy nor the isotopy class of contact structure.

Denote by $F(M)$ the space of all Sasakian structures on $M$, and by $F(M, \xi, \bar{J})$ the subspace of $F(M)$ with Reeb vector field $\xi$ and transverse complex structure $\bar{J}$. The identification space $F(M)/F(M, \xi, \bar{J})$ is the pre-moduli space of Sasaki classes.

The moduli space $\mathcal{M}(M)$ of Sasaki classes is the quotient of $F(M)/F(M, \xi, \bar{J})$ by $\text{Diff}(M)$. $\mathcal{M}(M)$ can be non-Hausdorff.

We think of an element of $\mathcal{M}(M)$ as represented by a basic cohomology class $[d\eta]_{B} \in H^{1,1}(F_{\xi})$.

We are mainly interested in those classes with $c_{1}(F_{\xi})$ positive and with $c_{1}(D) = c$ which we denote by $\mathcal{M}(M, c)$. By the transverse Yau Theorem $\mathcal{M}(M, c)$ has a representative with positive Ricci curvature.
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1. Fix CR structure \((\mathcal{D}, J)\), deform characteristic foliation \(\mathcal{F}\). This gives rise to **Sasaki cones**. After this type of deformation the transverse holonomy becomes **irreducible**.

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3. Fix characteristic foliation \(F\), deform contact structure \(D\). This is used to obtain **extremal Sasaki metrics**. This type of deformation does not change the transverse holonomy nor the isotopy class of contact structure.
Three Types of **Deformations** of Sasakian Structures

1. Fix CR structure \((\mathcal{D}, J)\), deform characteristic foliation \(\mathcal{F}\). This gives rise to **Sasaki cones**. After this type of deformation the transverse holonomy becomes **irreducible**.

2. Fix contact structure \(\mathcal{D}\), deform transverse complex structure (CR) \(J\). This gives rise to **Sasaki bouquets**. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group \(\text{Con}(M, \mathcal{D})\).

3. Fix characteristic foliation \(\mathcal{F}\), deform contact structure \(\mathcal{D}\). This is used to obtain **extremal Sasaki metrics**. This type of deformation does not change the transverse holonomy nor the isotopy class of contact structure.

Denote by \(\mathcal{S}(M)\) the space of all Sasakian structures on \(M\), and by \(\mathcal{S}(M, \xi, \tilde{J})\) the subspace of \(\mathcal{S}(M)\) with Reeb vector field \(\xi\) and transverse complex structure \(\tilde{J}\). The identification space \(\mathcal{S}(M)/\mathcal{S}(M, \xi, \tilde{J})\) is the pre-moduli space of Sasaki classes.
Deformations of Sasakian Structures and Sasaki Classes

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The moduli space \(\mathcal{M}(M)\) of Sasaki classes is the quotient of \(\mathfrak{S}(M)/\mathfrak{S}(M, \xi, \bar{J})\) by \(\text{Diff}(M)\).
Deformations of Sasakian Structures and Sasaki Classes

Three Types of **Deformations** of Sasakian Structures

1. Fix **CR structure** \((\mathcal{D}, J)\), deform **characteristic foliation** \(\mathcal{F}\). This gives rise to **Sasaki cones**. After this type of deformation the **transverse holonomy** becomes irreducible.

2. Fix **contact structure** \(\mathcal{D}\), deform **transverse complex structure** (CR) \(J\). This gives rise to **Sasaki bouquets**. Here Sasaki cones in bouquets are related to **conjugacy classes of tori** in the contactomorphism group \(\mathcal{C}on(M, \mathcal{D})\).

3. Fix **characteristic foliation** \(\mathcal{F}\), deform **contact structure** \(\mathcal{D}\). This is used to obtain **extremal Sasaki metrics**. This type of deformation does not change the **transverse holonomy** nor the **isotopy class** of contact structure.

Denote by \(\mathcal{F}(M)\) the space of all Sasakian structures on \(M\), and by \(\mathcal{F}(M, \xi, \bar{J})\) the subspace of \(\mathcal{F}(M)\) with Reeb vector field \(\xi\) and **transverse complex structure** \(\bar{J}\). The identification space \(\mathcal{F}(M)/\mathcal{F}(M, \xi, \bar{J})\) is the **pre-moduli space** of Sasaki classes.

The moduli space \(\mathcal{M}(M)\) of Sasaki classes is the quotient of \(\mathcal{F}(M)/\mathcal{F}(M, \xi, \bar{J})\) by \(\text{Diff}(M)\).

\(\mathcal{M}(M)\) can be **non-Hausdorff**.
Three Types of **Deformations** of Sasakian Structures

1. **Fix CR structure** $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**. After this type of deformation the transverse holonomy becomes irreducible.

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The moduli space $\mathcal{M}(M)$ of Sasaki classes is the quotient of $\mathcal{S}(M)/\mathcal{S}(M, \xi, \bar{J})$ by $\text{Diff}(M)$.

$\mathcal{M}(M)$ can be non-Hausdorff.

We think of an element of $\mathcal{M}(M)$ as represented by a basic cohomology class $[d\eta]_B \in H^{1,1}(\mathcal{F}_\xi)$. 
Three Types of **Deformations** of Sasakian Structures

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We are mainly interested in those classes with \(c_1(\mathcal{F}_\xi)\) positive and with \(c_1(\mathcal{D}) = c\) which we denote by \(\mathcal{M}_{+,c}\).
Three Types of **Deformations** of Sasakian Structures

1. **Fix CR structure** $(D, J)$, deform characteristic foliation $F$. This gives rise to **Sasaki cones**. After this type of deformation the transverse holonomy becomes irreducible.

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Denote by $\mathcal{F}(M)$ the space of all Sasakian structures on $M$, and by $\mathcal{F}(M, \xi, \bar{J})$ the subspace of $\mathcal{F}(M)$ with Reeb vector field $\xi$ and transverse complex structure $\bar{J}$. The identification space $\mathcal{F}(M)/\mathcal{F}(M, \xi, \bar{J})$ is the pre-moduli space of Sasaki classes.

The **moduli space** $\mathcal{M}(M)$ of Sasaki classes is the quotient of $\mathcal{F}(M)/\mathcal{F}(M, \xi, \bar{J})$ by $Diff(M)$.

$\mathcal{M}(M)$ can be non-Hausdorff.

We think of an element of $\mathcal{M}(M)$ as represented by a basic cohomology class $[d\eta]_B \in H^{1,1}(F_\xi)$.

We are mainly interested in those classes with $c_1(F_\xi)$ positive and with $c_1(D) = c$ which we denote by $\mathcal{M}_{+,c}$.

By the transverse Yau Theorem $\mathcal{M}_{+,c}$ has a representative with **positive Ricci curvature**.
A Brieskorn manifold \( L(a) \) is a link of a Brieskorn-Pham polynomial \( f(z) = z^{a_0} + \cdots z^{a_n} \), namely \( L(a) = \{ f(z) = 0 \} \cap S^{2n+1} \) with \( a = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}_{\geq 2} \).
A Brieskorn manifold $L(\mathbf{a})$ is a link of a Brieskorn-Pham polynomial $f(\mathbf{z}) = z_0^{a_0} + \cdots z_n^{a_n}$, namely $L(\mathbf{a}) = \{f(\mathbf{z}) = 0\} \cap S^{2n+1}$ with $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 2}^{n+1}$.

$L(\mathbf{a})$ has a natural Sasakian structure.
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Simply connected Rational Homology Spheres in Dimension Five
A Brieskorn manifold $L(a)$ is a link of a Brieskorn-Pham polynomial $f (z) = z_0^{a_0} + \cdots z_n^{a_n}$, namely $L(a) = \{ f(z) = 0 \} \cap S^{2n+1}$ with $a = (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 2}^{n+1}$.

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**Simply connected Rational Homology Spheres in Dimension Five**

Smale manifolds $M_r$ with $H_2(M_r, \mathbb{Z}) = \mathbb{Z}_r + \mathbb{Z}_r$ and connected sums $kM_r$. 
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**Theorem (B-,Macarini,van Koert)**

On the rational homology spheres $M = S^5, M_2, M_3, M_5, 2M_3, 4M_2$ we have $|\pi_0(\mathcal{M}_{+0}(M))| = \aleph_0$.

Moreover, each component belongs to a distinct contact structure, so there are infinitely many inequivalent contact structures of positive Sasaki type on each of the above rational homology 5-spheres.
A Brieskorn manifold $L(a)$ is a link of a Brieskorn-Pham polynomial $f(z) = z_0^{a_0} + \cdots z_n^{a_n}$, namely $L(a) = \{f(z) = 0\} \cap S^{2n+1}$ with $a = (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 2}^{n+1}$.

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**Proof**: Represent $M$ by a sequence of Brieskorn links $L(a)$ and compute the mean Euler characteristic.
A Brieskorn manifold $L(a)$ is a link of a Brieskorn-Pham polynomial $f(z) = z_0^{a_0} + \cdots + z_n^{a_n}$, namely $L(a) = \{f(z) = 0\} \cap S^{2n+1}$ with $a = (a_0, \ldots, a_n) \in \mathbb{Z}_{>2}^{n+1}$.

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By smoothing singularity $L(a)$ is Stein hence Liouville fillable.

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**Simply connected Rational Homology Spheres in Dimension Five**

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**Proof:** Represent $M$ by a sequence of Brieskorn links $L(a)$ and compute the mean Euler characteristic.

**Example:** $M_2$ can be represented by the links $L(2, 3, 3, 3 + 6k)$ and $\chi_m(W) = \frac{3 + 10k}{6 + 4k}$.
A Brieskorn manifold \( L(\mathbf{a}) \) is a link of a Brieskorn-Pham polynomial \( f(z) = z_0^{a_0} + \cdots z_n^{a_n} \), namely \( L(\mathbf{a}) = \{ f(z) = 0 \} \cap S^{2n+1} \) with \( \mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}_{\geq 2} \).

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By smoothing singularity \( L(\mathbf{a}) \) is Stein hence Liouville fillable.

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Simply connected Rational Homology Spheres in Dimension Five

Smale manifolds \( M_r \) with \( H_2(M_r, \mathbb{Z}) = \mathbb{Z}_r + \mathbb{Z}_r \) and connected sums \( kM_r \).

**Theorem** (B-, Macarini, van Koert)

*On the rational homology spheres \( M = S^5, M_2, M_3, M_5, 2M_3, 4M_2 \) we have \( |\pi_0(\mathfrak{M}_{+0}(M))| = \aleph_0 \). Moreover, each component belongs to a distinct contact structure, so there are infinitely many inequivalent contact structures of positive Sasaki type on each of the above rational homology 5-spheres.*

**Proof**: Represent \( M \) by a sequence of Brieskorn links \( L(\mathbf{a}) \) and compute the mean Euler characteristic.

**Example**: \( M_2 \) can be represented by the links \( L(2, 3, 3, 3 + 6k) \) and \( \chi_m(W) = \frac{3+10k}{6+4k} \).

All except \( 4M_2 \) are known to admit **Sasaki-Einstein metrics**.
We denote the Sasaki-Einstein moduli space on $M$ by $\mathcal{M}^{SE}(M)$ (excludes standard round sphere).
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There is a natural map $c : \mathcal{M}^{SE}(M) \to \mathcal{M}_{+0}(M)$.
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There is a natural map \(c: \mathcal{M}^{SE}(M) \rightarrow \mathcal{M}_{+,0}(M)\).

82 families of SE metrics on \(S^5\) (B-, Galicki, Kollár; Ghigi, Kollár; B-, Macarini, van Koert; Sun, Li).

There are 6 pairs that cannot be distinguished by \(\chi_m(W)\) or \(\text{SH}_+, S_1(W)\).

55 components are single points.

There are other components of real dimension 2, 4, 6, 8, 10, 20.

SE metrics on higher homotopy spheres

On the 28 oriented homotopy spheres homeomorphic to \(S^7\), the lower bounds on \(|\pi_0(\mathcal{M}^{SE}(\Sigma^7))|\) vary between 424 and 229.

\(|\pi_0(\mathcal{M}^{SE}(S^9))| \geq 983\) and \(|\pi_0(\mathcal{M}^{SE}(\Sigma^9))| \geq 494\).

\(|\pi_0(\mathcal{M}^{SE}(S^{4n+1}))|\) grows double exponentially with dimension.

Other Results for \(\mathcal{M}_{+,0}\):

\(|\pi_0(\mathcal{M}^{+,0}(S^2 \times S^3))| = \aleph_0\) and \(|\pi_0(\mathcal{M}^{+,0}(S^2 \times S^2 \times S^2 \times S^2))| = |\pi_0(\mathcal{M}^{+,0}(S^2 \times S^2 \times \Sigma(4n+1)))| = \aleph_0\).
We denote the Sasaki-Einstein moduli space on $M$ by $\mathcal{M}^{SE}(M)$ (excludes standard round sphere).

There is a natural map $c : \mathcal{M}^{SE}(M) \longrightarrow \mathcal{M}_{+,0}(M)$.

82 families of SE metrics on $S^5$ (B-, Galicki, Kollár; Ghigi, Kollár; B-, Macarini, van Koert; Sun, Li).

Lower bound: $|\pi_0(\mathcal{M}^{SE}(S^5))| \geq 76$ (B-, Macarini, van Koert).
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SE metrics on higher homotopy spheres.
Sasaki-Einstein Moduli

We denote the Sasaki-Einstein moduli space on $M$ by $\mathcal{M}^{SE}(M)$ (excludes standard round sphere).

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Other Results for $\mathcal{M}_{+0}$
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Other Results for $\mathcal{M}_{+,0}$

$|\pi_0(\mathcal{M}_{+,0}(k(S^2 \times S^3))| = \aleph_0$ and

$|\pi_0(\mathcal{M}_{+,0}(S^{2n} \times S^{2n+1}))| = |\pi_0(\mathcal{M}_{+,0}(S^{2n} \times S^{2n+1} \# \Sigma^{4n+1}))| = \aleph_0$. 
Sasaki-Einstein Moduli

- We denote the Sasaki-Einstein moduli space on $\mathcal{M}$ by $\mathcal{M}^{SE}(\mathcal{M})$ (excludes standard round sphere).
- There is a natural map $c: \mathcal{M}^{SE}(\mathcal{M}) \to \mathcal{M}_{+},0(\mathcal{M})$.
- 82 families of SE metrics on $S^5$ (B-, Galicki, Kollár; Ghigi, Kollár; B-, Macarini, van Koert; Sun, Li).
- Lower bound: $|\pi_0(\mathcal{M}^{SE}(S^5))| \geq 76$ (B-, Macarini, van Koert).
- There are 6 pairs that cannot be distinguished by $\chi_m(W)$ or $SH^+;S^1(W)$.
- 55 components are single points.
- There are other components of real dimension 2, 4, 6, 8, 10, 20.

SE metrics on higher homotopy spheres

- On the 28 oriented homotopy spheres homeomorphic to $S^7$, the lower bounds on $|\pi_0(\mathcal{M}^{SE}(\Sigma^7))|$ vary between 424 and 229.
- $|\pi_0(\mathcal{M}^{SE}(S^9))| \geq 983$ and $|\pi_0(\mathcal{M}^{SE}(\Sigma^9))| \geq 494$.
- $|\pi_0(\mathcal{M}^{SE}(S^{4n+1}))|$ grows double exponentially with dimension.

Other Results for $\mathcal{M}_{+},0$

- $|\pi_0(\mathcal{M}_{+},0(k(S^2 \times S^3))| = \aleph_0$ and
- $|\pi_0(\mathcal{M}_{+},0(S^{2n} \times S^{2n+1}))| = |\pi_0(\mathcal{M}_{+},0(S^{2n} \times S^{2n+1} \# \Sigma^{4n+1}))| = \aleph_0$.
- $\mathcal{T} = \text{unit tangent sphere bundle over } S^{2n+1}$, then $|\pi_0(\mathcal{M}_{+},0(\mathcal{T}))| = \aleph_0$. 
Sasakian structure $S = (\xi, \eta, \Phi, g)$ with scalar curvature $s_g$. 
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Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g$.
Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $S = (\xi, \eta, \Phi, g)$ with scalar curvature $s_g$.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g$.
- Deform contact structure $\eta \mapsto \eta + td^c \varphi$ within its isotopy class where $\varphi$ is basic.
Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature $s_g$.

Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g$.

Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where $\varphi$ is basic.

This gives critical point of $E(g) \iff \partial_g^\# s_g$ is transversely holomorphic.
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Special case: constant scalar curvature Sasakian (CSC). If $c_1(D) = 0 \Rightarrow$ Sasaki-$$Einstein (S\eta E)$ with Ricci curvature $\text{Ric}_g = ag + b\eta \otimes \eta$, $a, b$ constants. If $b = 0$ get Sasaki-Einstein (SE).
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If $S = (\xi, \eta, \Phi, g)$ is extremal (or CSC) then so is $S_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0.$
Sasaki cones and bouquets

- Sasaki cones

1. The Lie algebra of $T_k$ of the Sasaki cone (unreduced):

   $\{\xi' \in t_k | \eta(\xi') > 0\}$

   where $S = (\xi, \eta, \Phi, g)$ belongs to $(D, J)$ is Sasaki.

2. $\kappa(D, J) = t_k + k(D, J)/W$ where $W$ is the Weyl group of $CR(D, J)$.

3. $\kappa(D, J)$ is finite dimensional moduli of Sasaki structures with underlying CR structure $(D, J)$.

4. $1 \leq \dim \kappa(D, J) \leq n + 1$ and if $\dim \kappa(D, J) = n + 1$, $M$ is toric Sasaki.

5. The set of extremal rays $e(D, J)$ is open in $\kappa(D, J)$.

Sasaki bouquets

- A contact structure $D$ of Sasaki type with a space of compatible CR structures $J(D)$

- A map $Q: J(D) \rightarrow \{conjugacy classes of tori in the contactomorphism group Con(M, D)\}$

- Get bouquet $[\alpha \kappa(D, J_\alpha)$ of Sasaki cones, $J_\alpha \in J(D)$, $\alpha$ ranges over distinct conjugacy classes.

- A bouquet consisting of $N$ Sasaki cones is called an $N$-bouquet, denoted by $B^N$. The Sasaki cones in an $N$-bouquet can have different dimension. The pre-moduli space is typically non-Hausdorff.
Sasaki cones and bouquets

- **Sasaki cones**
  - $t_k$ the Lie algebra of $T^k$
Sasaki cones and bouquets

**Sasaki cones**

1. $t_k$ the Lie algebra of $T^k$
2. **Sasaki cone** (unreduced): $t_k^+(\mathcal{D}, J) = \{ \xi' \in t_k \mid \eta(\xi') > 0 \}$ s.t. $S = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
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Charles Boyer (University of New Mexico)
Moduli Problems in Sasakian Geometry
May 21, 2015, Recent Advances in Kähler Geometry, UC Berkeley
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  4. $\kappa(D, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(D, J)$.

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Moduli Problems in Sasakian Geometry
May 21, 2015, Recent Advances in Kähler Geometry, 1/18
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  1. a contact structure $\mathcal{D}$ of Sasaki type with a space of **compatible CR structures** $\mathcal{J}(\mathcal{D})$
Sasaki cones and bouquets

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Charles Boyer (University of New Mexico)
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  2. a map $\mathcal{Q} : \mathcal{J}(\mathcal{D}) \to \{ \text{conjugacy classes of tori} \}$ in the contactomorphism group $\mathcal{Con}(M, \mathcal{D})$
  3. Get bouquet $\bigcup_{\alpha} \kappa(\mathcal{D}, J_{\alpha})$ of Sasaki cones, $J_{\alpha} \in \mathcal{J}(\mathcal{D})$, $\alpha$ ranges over distinct conjugacy classes.
Sasaki cones and bouquets

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3. **Sasaki cone (reduced):** \( \kappa(\mathcal{D}, J) = t_k^+ (\mathcal{D}, J) / W \) where \( W \) is the Weyl group of \( CR(\mathcal{D}, J) \).
4. \( \kappa(\mathcal{D}, J) \) is finite dim'l *moduli of Sasakian structures* with underlying CR structure \( (\mathcal{D}, J) \).
5. \( 1 \leq \dim \kappa(\mathcal{D}, J) \leq n + 1 \) and if \( \dim \kappa(\mathcal{D}, J) = n + 1 \), \( M \) is *toric Sasakian*.
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**Sasaki bouquets**

1. a contact structure \( \mathcal{D} \) of Sasaki type with a space of *compatible CR structures* \( \mathcal{J}(\mathcal{D}) \)
2. a map \( \mathcal{Q} : \mathcal{J}(\mathcal{D}) \to \{ \text{conjugacy classes of tori} \} \) in the contactomorphism group \( \text{Con}(M, \mathcal{D}) \).
3. Get *bouquet* \( \bigcup_{\alpha} \kappa(\mathcal{D}, J_{\alpha}) \) of Sasaki cones, \( J_{\alpha} \in \mathcal{J}(\mathcal{D}) \), \( \alpha \) ranges over distinct conjugacy classes.
4. A bouquet consisting of \( N \) Sasaki cones is called an *N-bouquet*, denoted by \( \mathcal{B}_N \). The Sasaki cones in an N-bouquet can have different dimension. The *pre-moduli space* is typically non-Hausdorff.
5. the Sasaki cones \( \kappa(\mathcal{D}, J_{\alpha}) \) can be distinguished by *equivariant Gromov-Witten invariants*.
**Join Construction**: Given quasi-regular Sasakian manifolds \( \pi_i : M_i \rightarrow \mathbb{Z}_i \) with 
\[ \text{Dim } M_i = 2n_i + 1 \] for \( i = 1, 2 \).

The dimension of 
\[ M_1 \ast_{l_1, l_2} M_2 \] is 
\[ 2(n_1 + n_2) + 1. \]
**Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \rightarrow Z_i$ with $\dim M_i = 2n_i + 1$ for $i = 1, 2$.

Form $(l_1, l_2)$-join $\pi : M_1 \star_{l_1,l_2} M_2 \rightarrow Z_1 \times Z_2$ as an $S^1$-orbibundle.
**Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \rightarrow \mathcal{Z}_i$ with $\dim M_i = 2n_i + 1$ for $i = 1, 2$.

- Form $(l_1, l_2)$-join $\pi : M_1 \star_{l_1,l_2} M_2 \rightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an $S^1$-orbibundle.
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $S_{l_1,l_2}$ for all relatively prime positive integers $l_1, l_2$. Fixing $l_1, l_2$ fixes the contact orbifold. It is a smooth manifold iff $\gcd(v_1 l_2, v_2 l_1) = 1$ where $v_i$ is the order of orbifold $\mathcal{Z}_i$. 

The dimension of $M_1 \star_{l_1,l_2} M_2$ is $2(n_1 + n_2) + 1$. The join $M_1 \star_{l_1,l_2} M_2$ has reducible transverse holonomy a subgroup of $U(n_1) \times U(n_2)$. 

Take $\pi_2 : M_2 \rightarrow \mathcal{Z}_2$ to be the $S^1$-orbibundle $\pi_2 : S^3_{w} \rightarrow \mathbb{CP}^1[w]$ determined by a weighted $S^1$ action on $S^3$ with weights $w = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_i) = 1$, and $M_1$ regular Sasakian manifold whose quotient is a compact Kähler manifold $N$. In this case the Join Construction and Admissible Construction of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman fit as hand and glove.

An $S^1$ orbibundle $M \star_{l_1,l_2} S^3_{w} \rightarrow N \times \mathbb{CP}^1[w]$, where $N$ is compact Kähler. The join $M \star_{l_1,l_2} S^3_{w}$ can be realized as a lens space bundle over $N$ with fiber the lens space $L(l_2; l_1 w_1, l_1, w_2)$. 

I present two fundamental theorems about $M \star_{l_1,l_2} S^3_{w}$ and then present brief outlines of their proofs. Finally, I discuss the special case of $S^3$-bundles over a Riemann surface $\Sigma_g$. 

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**Charles Boyer (University of New Mexico)**

**Moduli Problems in Sasakian Geometry**

May 21, 2015. Recent Advances in Kähler Geometry, Vanderbilt University
The Join Construction (B-, Galicki, Ornea)

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \to Z_i$ with $\text{Dim } M_i = 2n_i + 1$ for $i = 1, 2$.

- Form $(l_1, l_2)$-join $\pi : M_1 \star_{l_1, l_2} M_2 \to Z_1 \times Z_2$ as an $S^1$-orbibundle.

- $M_1 \star_{l_1, l_2} M_2$ has a natural quasi-regular Sasakian structure $S_{l_1, l_2}$ for all relatively prime positive integers $l_1, l_2$. Fixing $l_1, l_2$ fixes the contact orbifold. It is a smooth manifold iff $\gcd(v_1 l_2, v_2 l_1) = 1$ where $v_i$ is the order of orbifold $Z_i$.

- The dimension of $M_1 \star_{l_1, l_2} M_2$ is $2(n_1 + n_2) + 1$. 


**Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \rightarrow Z_i$ with $\text{Dim } M_i = 2n_i + 1$ for $i = 1, 2$.

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The dimension of $M_1 \star_{l_1, l_2} M_2$ is $2(n_1 + n_2) + 1$.

The join $M_1 \star_{l_1, l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.
The Join Construction (B-, Galicki, Ornea)

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- The join \( M_1 \star_{l_1, l_2} M_2 \) has **reducible transverse holonomy** a subgroup of \( U(n_1) \times U(n_2) \).

- Take \( \pi_2 : M_2 \rightarrow Z_2 \) to be the \( S^1 \) orbibundle \( \pi_2 : S^3_w \rightarrow CP^1[w] \) determined by a weighted \( S^1 \) action on \( S^3 \) with weights \( w = (w_1, w_2) \) satisfying \( \gcd(l_2, l_1 w_1) = 1 \), and \( M_1 = M \) regular Sasaki manifold whose quotient is a compact Kähler manifold \( N \).
**Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \to \mathbb{Z}_i$ with $\dim M_i = 2n_i + 1$ for $i = 1, 2$.

Form $(l_1, l_2)$-join $\pi : M_1 \ast_{l_1, l_2} M_2 \to \mathbb{Z}_1 \times \mathbb{Z}_2$ as an $S^1$-orbibundle.

$M_1 \ast_{l_1, l_2} M_2$ has a natural quasi-regular Sasakian structure $S_{l_1, l_2}$ for all relatively prime positive integers $l_1, l_2$. Fixing $l_1, l_2$ fixes the contact orbifold. It is a smooth manifold iff $\gcd(v_1 l_2, v_2 l_1) = 1$ where $v_i$ is the order of orbifold $\mathbb{Z}_i$.

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Take $\pi_2 : M_2 \to \mathbb{Z}_2$ to be the $S^1$ orbibundle $\pi_2 : S^3_w \to \mathbb{C}P^1[w]$ determined by a weighted $S^1$ action on $S^3$ with weights $w = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_1) = 1$, and $M_1 = M$ regular Sasaki manifold whose quotient is a compact Kähler manifold $N$.

In this case the **Join Construction** and **Admissible Contraction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman fit as hand and glove.
**Join Construction**: Given quasi-regular Sasakian manifolds \( \pi_i : M_i \longrightarrow Z_i \) with
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An \( S^1 \) orbibundle \( M \ast_{l_1, l_2} S^3_w \longrightarrow N \times \mathbb{CP}^1[w] \), where \( N \) is compact Kähler.
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An \( S^1 \) orbibundle \( M \star_{l_1, l_2} S^3_w \rightarrow N \times \mathbb{CP}^1[w] \), where \( N \) is compact Kähler.

The join \( M \star_{l_1, l_2} S^3_w \) can be realized as a lens space bundle over \( N \) with fiber the lens space \( L(l_2; l_1 w_1, l_1, w_2) \).
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I present two **fundamental theorems** about $M \star_{l_1, l_2} S^3_w$ and then present brief outlines of their proofs. Finally, I discuss the special case of $S^3$-bundles over a Riemann surface $\Sigma_g$. 
Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone.
Existence of extremal and CSC Sasaki metrics by deforming in the Sasaki cone

**Theorem (B-, Tønnesen-Friedman)**

Let $M_{l_1,l_2,w} = M \ast_{l_1,l_2} S^3_w$ be the $S^3_w$-join with a regular Sasaki manifold $M$ which is an $S^1$-bundle over a compact Kähler manifold $N$ with constant scalar curvature. Then for each vector $w = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_v$ in a 2-dimensional sub cone, the $w$-cone, of the Sasaki cone on $M_{l_1,l_2,w}$ such that the corresponding ray of Sasakian structures $S_a = (a^{-1} \xi_v, a \eta_v, \phi, g_a)$ has constant scalar curvature.

If the scalar curvature $s_N$ of $N$ is nonnegative, then the $w$-cone is exhausted by extremal Sasaki metrics. If the scalar curvature $s_N$ of $N$ is positive and $l_2$ is large enough there are infinitely many contact CR structures with at least 3 rays of CSC Sasaki structures in the $w$-cone. When $N$ is positive $KE$ get $SE$ metric on $M_{l_1,l_2,w}$ for appropriate choice of $(l_1, l_2)$. The $SE$ metrics of 3 were previously obtained by physicists (Gauntlett, Martelli, Sparks, Waldram) by another method. Most of the CSC Sasaki structures are irregular.

**Relation to CR Yamabe Problem** (Jerison and Lee): For a Sasaki structure the Webster pseudo-Hermitian metric coincides with the transverse Kähler metric. So a CSC Sasaki metric provides a solution to the CR Yamabe Problem. It is known that when the CR Yamabe invariant $\lambda(M)$ is nonpositive, the CSC metric is unique. However, when $\lambda(M) > 0$ there can be several CSC solutions. Our results provides many such examples.
Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

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1. If the scalar curvature $s_N$ of $N$ is nonnegative, then the $w$-cone is exhausted by **extremal** Sasaki metrics.

2. If the scalar curvature $s_N$ of $N$ is positive and $l_2$ is large enough there are infinitely many contact CR structures with at least 3 rays of CSC Sasakian structures in the $w$-cone.

3. When $N$ is positive constant Euler characteristic $SE$ metric on $M_{l_1/l_2,w}$ for appropriate choice of $(l_1, l_2)$. The $SE$ metrics of 3 were previously obtained by physicists (Gauntlett, Martelli, Sparks, Waldram) by another method.

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Fundamental Theorem (B-, Tønnesen-Friedman)

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3. When $N$ is positive KE get SE metric on $M_{l_1,l_2,w}$ for appropriate choice of $(l_1, l_2)$. 

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Most of the **CSC** Sasaki structures are **irregular**.
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1. If the scalar curvature $s_N$ of $N$ is **nonnegative**, then the $w$-cone is exhausted by **extremal** Sasaki metrics.
2. If the scalar curvature $s_N$ of $N$ is **positive** and $l_2$ is large enough there are infinitely many contact CR structures with at least 3 rays of **CSC** Sasakian structures in the $w$-cone.
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- Relation to **CR Yamabe Problem** (Jerison and Lee): For a Sasaki structure the Webster pseudo-Hermitian metric coincides with the transverse Kähler metric. So a **CSC** Sasaki metric provides a solution to the CR Yamabe Problem. It is know that when the **CR Yamabe invariant** $\lambda(M)$ is **nonpositive**, the CSC metric is unique. However, when $\lambda(M) > 0$ there can be several CSC solutions. Our results provides many such examples.
Outline of proof of Fundamental Theorem:

- The existence of an extra Hamiltonian Killing vector field from $S^3_w$ gives the 2-dimensional Sasaki $w$-cone $t^+_w$. 

Lifing to $M_{l_1, l_2, w}$ gives extremal (CSC) Sasaki metrics in the quasi-regular case. The irregular case uses a continuity argument together with the fact that quasi-regular Sasaki structures are dense in the Sasaki cone.

The existence of multiple rays of CSC Sasaki metrics comes from a sign changing count.
Outline of proof of Fundamental Theorem:

- The existence of an extra Hamiltonian Killing vector field from $S^3_w$ gives the 2-dimensional Sasaki $w$-cone $t^+_w$.
- The quotient space of the $S^1$-action generated by any quasi-regular Reeb vector field $\xi_v \in t^+_w$ is a ruled orbifold $(S_n, \Delta_{mv_1, mv_2})$ with a branch divisor

$$\Delta_{mv_1, mv_2} = \left(1 - \frac{1}{mv_1}\right)D_1 + \left(1 - \frac{1}{mv_2}\right)D_2$$

consisting of the zero $D_1$ and infinity $D_2$ sections of the projective bundle $S_n = \mathbb{P}(1 \oplus L_n)$ over $N$ with ramification indices $mv_1, mv_2$, respectively and $n$ an integer determined by $l_1, l_2, w, v$.
Outline of proof of Fundamental Theorem:

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- For $n \neq 0$, apply the admissible construction of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1, mv_2})$. 

When $\Theta(z)(1 + rz)dz$ is a $(d + 3)$ order $(d + 2)$ order polynomial we get extremal (CSC) Kähler metrics. Here $d$ is the complex dimension of $N$. Lifiting to $M_{l_1, l_2, w, v}$ gives extremal (CSC) Sasaki metrics in the quasi-regular case. The irregular case uses a continuity argument together with the fact that quasi-regular Sasaki structures are dense in the Sasaki cone. The existence of multiple rays of CSC Sasaki metrics comes from a sign changing count.
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- For $n \neq 0$, apply the admissible construction of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S^n, \Delta_{mv_1, mv_2})$.
- This gives the Kähler orbifold metric $g(S^n, \Delta) = \frac{1+r_3}{r} g_{\Sigma_g} + \frac{d_3^2}{\Theta(3)} + \Theta(3)\theta^2$ where $\theta$ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(3) > 0$ and $-1 < 3 < 1$, $\Theta(\pm1) = 0$, $\Theta'(-1) = \frac{2}{mv_2}$, $\Theta'(1) = -\frac{2}{mv_1}$.
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- The existence of an extra Hamiltonian Killing vector field from $S^3_w$ gives the 2-dimensional Sasaki $w$-cone $t^+_w$.
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- For $n \neq 0$, apply the admissible construction of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1, mv_2})$

This gives the Kähler orbifold metric $g(S_n, \Delta) = \frac{1+r^3}{r} g_{\Sigma g} + \frac{d^2}{\Theta(3)} + \Theta(3) \theta^2$ where $\theta$ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(3) > 0$ and $-1 < 3 < 1$, $\Theta(\pm 1) = 0$, $\Theta'(1) = -\frac{2}{mv_2}$, $\Theta'(1) = -\frac{2}{mv_1}$.

- When $\Theta(3)(1 + r^3)^d$ is a $(d + 3)$ order $(d + 2)$ order polynomial we get extremal (CSC) Kähler metrics. Here $d$ is the complex dimension of $N$. 
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- This gives the Kähler orbifold metric $g_{(S_n, \Delta)} = \frac{1+r_3}{r} g_{\Sigma} + \frac{d_3^2}{\Theta(3)} + \Theta(3) \theta^2$ where $\theta$ is a connection 1-form, $d\theta = n \omega_N$, $0 < r < 1$, $\Theta(3) > 0$ and $-1 < 3 < 1$, $\Theta(\pm 1) = 0$, $\Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(3)(1 + r_3)^d$ is a $(d + 3)$ order ($(d + 2)$ order) polynomial we get extremal (CSC) Kähler metrics. Here $d$ is the complex dimension of $N$.
- Lifing to $M_{l_1, l_2, w}$ gives extremal (CSC) Sasaki metrics in the quasi-regular case.
Outline of proof of Fundamental Theorem:

- The existence of an extra Hamiltonian Killing vector field from $S^3_w$ gives the 2-dimensional Sasaki $w$-cone $t^+_w$.

- The quotient space of the $S^1$-action generated by any quasi-regular Reeb vector field $\xi_v \in t^+_w$ is a ruled orbifold $(S_n, \Delta_{mv_1, mv_2})$ with a branch divisor

$$\Delta_{mv_1, mv_2} = \left(1 - \frac{1}{mv_1}\right)D_1 + \left(1 - \frac{1}{mv_2}\right)D_2$$

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- The irregular case uses a continuity argument together with the fact that quasi-regular Sasaki structures are dense in the Sasaki cone.
- The existence of multiple rays of CSC Sasaki metrics comes from a sign changing count.
When $g = 0$ we get Sasakian structures on the two $S^3$-bundles over the $S^2$ for all relatively prime positive integers $l_1, l_2$. (B-,B-Pati) (Also E. Legendre).
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So \( Y^{p,q} \) and \( Y^{p',q'} \) map to the same component of \( \mathcal{M}_{+,0} \) under \( c \).
$S^3$-bundles over Riemann surface $\Sigma_g$ of genus $g$: Case 1: genus $g = 0$

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Example: A regular 4-bouquet $\mathcal{B}_4(D_{-6})$ on $S^2 \times S^3$ with $l_2 = 1$ and $c_1(D) = -6\gamma$. The base spaces are Hirzebruch surfaces $S_0, S_2, S_4, S_6$, respectively.
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If we take \( l_2 > 1 \) we get \( c_1(D) = (2l_2 - 8)\gamma \) and we loose the product base \( S_0 = \mathbb{CP}^1 \times \mathbb{CP}^1 \) and regularity giving a 3-bouquet on \( S^2 \times S^3 \) with orbifold Hirzebruch surfaces \( (S_2, \Delta_{l_2}), (S_4, \Delta_{l_2}), (S_6, \Delta_{l_2}) \) as base spaces. In each case the fiber is \( \mathbb{CP}^1 / \mathbb{Z}/l_2 \).
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If $l_2 > 53$ all three Sasaki cones have 3 CSC rays of Sasaki metrics.
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Similar results hold for the non-trivial $S^3$-bundle over $S^2$, but no SE metrics.
When \( g > 0 \) we need \( l_2 = 1 \) to get \( S^3 \)-bundles over a Riemann surface \( \Sigma_g \). There are two diffeomorphism types, the trivial bundle \( \Sigma_g \times S^3 \), the non-trivial bundle \( \Sigma_g \tilde{\times} S^3 \).
When $g > 0$ we need $l_2 = 1$ to get $S^3$-bundles over a Riemann surface $\Sigma_g$. There are two diffeomorphism types, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.

On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a unique ray of CSC Sasakian structures.
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When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(D_k, J)$ on $S^3$-bundles over $\Sigma_g$ are exhausted by extremal Sasaki metrics.
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When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(D_k, J)$ on $S^3$-bundles over $\Sigma_g$ are exhausted by extremal Sasaki metrics

For $g \geq 20$ there are rays in $\kappa(D_k, J)$ which admit no extremal Sasaki metrics.
When $g > 0$ we need $l_2 = 1$ to get $S^3$-bundles over a Riemann surface $\Sigma_g$. There are two
\textbf{diffeomorphism types}, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.

On both manifolds there is a countably infinite number of inequivalent \textbf{contact structures} $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a unique ray of \textbf{CSC} Sasakian structures.

When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on $S^3$-bundles over $\Sigma_g$ are
exhausted by \textbf{extremal Sasaki metrics}.

For $g \geq 20$ there are rays in $\kappa(\mathcal{D}_k, J)$ which admit \textbf{no} extremal Sasaki metrics.

For any genus $g \geq 1$ and for each positive integer $k$, the contact manifold $(\Sigma_g \times S^3, \mathcal{D}_k)$ has
a \textbf{$k$-bouquet} $\mathcal{B}_k$ of 2-dimensional Sasaki cones.
When \( g > 0 \) we need \( l_2 = 1 \) to get \( S^3 \)-bundles over a **Riemann surface** \( \Sigma_g \). There are two **diffeomorphism types**, the trivial bundle \( \Sigma_g \times S^3 \), the non-trivial bundle \( \Sigma_g \wedge S^3 \).

On both manifolds there is a countably infinite number of inequivalent **contact structures** \( \mathcal{D}_k \) admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a unique ray of **CSC** Sasakian structures.

When \( 0 < g \leq 4 \) all 2-dimensional Sasaki cones \( \kappa(\mathcal{D}_k, J) \) on \( S^3 \)-bundles over \( \Sigma_g \) are exhausted by **extremal Sasaki metrics**.

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Example: The 4-bouquet in the \( g = 0 \) case persists on \( \Sigma_g \times S^3 \) for all genera \( g \), but the base spaces are **pseudo-Hirzebruch surfaces** in this case.
When $g > 0$ we need $l_2 = 1$ to get $S^3$-bundles over a Riemann surface $\Sigma_g$. There are two \textbf{diffeomorphism types}, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \widetilde{\times} S^3$.

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Example: The 4-bouquet in the $g = 0$ case persists on $\Sigma_g \times S^3$ for all genera $g$, but the base spaces are \textbf{pseudo-Hirzebruch surfaces} in this case.

The distinct Sasaki cones in the bouquet $\mathcal{B}_k$ correspond to distinct conjugacy classes of maximal tori in $\mathcal{Con}(D_{l_1}, l_2, w)$. Uses the work of Buşe on \textbf{equivariant Gromov-Witten invariants}.
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The construction can be ‘twisted’ by reducible representations of the fundamental group $\pi_1(\Sigma_g)$. The irreducible representations of $\pi_1(\Sigma_g)$ give 1-dimensional Sasaki cones. They arise from **stable** rank two vector bundles and have CSC Sasaki metrics.
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When $l_2 > 1$ some of the same type of results have been obtained on 5-manifolds whose fundamental group is a non-Abelian extension of $\pi_1(\Sigma_g)$ in Castañoeda’s thesis.
THANK YOU!
References
