1. Introduction

Let H be a separable complex Hilbert space, $\mathcal{B}(H)$ be the Banach space consisting of all bounded linear operators from H to H. A von Neumann algebra $\mathcal{M}$ is defined to be a selfadjoint subalgebra of $\mathcal{B}(H)$ which is closed in the strong operator topology. Factors are the von Neumann algebras whose centers are scalar multiples of the identity. The factors are classified by means of a relative dimension function into type I, II, III factors.

In this talk, we focus mainly on the type II_1 von Neumann algebras. Some interesting examples of type II_1 factors come from actions of the group on a measure space and from complex group algebras. (ref. [10])

1.1 Factors from complex group algebras

The Hilbert space H is $l^2(G)$. We assume that G is countable so that H is separable. For each g in G, let L_g denote the left translation of functions in $l^2(G)$ by g^{-1}. Then $g \to L_g$ is a faithful unitary representation of G on H. Let $L(G)$ be the von Neumann algebra generated by $\{L_g : g \in G\}$.

Proposition 1 *([12]) If G is an* infinite conjugacy class *(i.c.c) group, then $L(G)$ is a II_1 factor.*

Examples:

1. $L(F(n))$, where $F(n)$ is the free group with $n(> 1)$ generators.

2. $L(\Pi)$, where Π is the permutation group of $\mathbb{Z}$ (consisting of those permutations that leave fixed all but a finite set of $\mathbb{Z}$.

3. $L(SL(3, \mathbb{Z}))$, where $SL(3, \mathbb{Z})$ is the 3×3 special linear group with integer entries.

1.2 Factors from actions of group on a measure space

The group-measure construction of II_1 factors involves a (countable separated) measure space $(S, \mathcal{S}, m)$ on which a countable group G acts freely. The algebra $\mathcal{A}_0$ of multiplications by essentially bounded measurable functions on $L^2(S, m) = H$ is maximal abelian in $B(H)$. Each g gives rise to a unitary operator U_g on H that "normalizes" $\mathcal{A}_0$ (that is, $U_g \mathcal{A}_0 U_g^* = \mathcal{A}_0$ and $A \to U_g A U_g^*$ is a * automorphism of $\mathcal{A}_0$). Let H_g be H and $\mathcal{K}$ be $\oplus_{g \in G} H_g$.

If we describe the elements of $B(\mathcal{K})$ as matrices with columns and rows indexed by the elements of G and entries from $B(H)$, those diagonal matrices, with the same element of $\mathcal{A}_0$ at the each diagonal entry, form an abelian von Neumann subalgebra $\mathcal{A}$ of $B(\mathcal{K})$.

Let V_g be the matrix with $\delta_{p,gq}U_g$ at the p, q entry, where $\delta_{p,q}$ is 0 if $p \neq q$ and I if $p = q$, and $\mathcal{R}(\mathcal{A}_0, G)$ be the von Neumann algebra generated by $\mathcal{A}$ and $\{V_g : \ g \in G\}$.

Proposition 2 *([12]) If $m(S) < \infty$, G acts ergodically on S, and m has no atoms, then $\mathcal{R}(\mathcal{A}_0, G)$ is a factor of type II$_1$ and $\mathcal{A}$ is a "Cartan subalgebra" of $\mathcal{R}(\mathcal{A}_0, G)$.*

A Cartan subalgebra $\mathcal{A}$ of a II$_1$ von Neumann algebra $\mathcal{M}$ is a maximal abelian von Neumann subalgebra of $\mathcal{M}$, such that the normalizer of $\mathcal{A}$, denoted by $\mathcal{N}(\mathcal{A})$, generates $\mathcal{M}$, where

$$\mathcal{N}(\mathcal{A}) = \{u \in \mathcal{M} | u \text{ unitary}, \ u\mathcal{A}u^* = \mathcal{A}\}.$$

1.3 Questions

A very natural question was asked:

Does every type II_1 von Neumann algebra have a Cartan subalgebra? What about $L(\mathbf{F}(n))$?

By his theory of free entropy, Voiculescu ([17]) showed that $L(\mathbf{F}(n))$ has no Cartan subalgebra, which answered the question negatively.

2. A brief introduction to free probability and free entropy theory

2.1 Free probability theory

In the early 1980s, D. Voiculescu began the development of the theory of free probability and free entropy. This new and powerful tool was crucial in solving some old open problems in the field of von Neumann algebras.

A noncommutative W$^*-$*probability space* is a pair $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a von Neumann algebra and τ is a normal state. We assume that $\mathcal{M}$ has a separable predual and τ is a faithful normal tracial state. (So that $\mathcal{M}$ is a finite von Neumann algebra). Elements of $\mathcal{M}$ are called non-commuting *random variables*.

Definition 1 *(From Voiculescu [19]) The* distribution *of a random variable A in $(\mathcal{M}, \tau)$ is a linear functional μ on $\mathbb{C}[x]$, the polynomial ring with variable x and coefficients in $\mathbb{C}$, such that $\mu(\psi(x)) = \tau(\psi(A))$ for all $\psi(x)$ in $\mathbb{C}[x]$.*

Example: A semicircular element A in $\mathcal{M}$ is one whose distribution $\gamma_{a,r}$ satisfies the *"semicircle law"* centered at $a \in \mathbb{R}$ and of radius $r > 0$ where that law $\gamma_{a,r} : \mathbb{C}[x] \to \mathbb{C}$ is defined by

$$\gamma_{a,r}(\psi) = \frac{2}{\pi r^2} \int_{a-r}^{a+r} \psi(t)\sqrt{r^2 - (t-a)^2}dt.$$

Definition 2 *(From Voiculescu [19]) The* joint distribution *of a family of random variables A_i, $i \in \mathcal{I}$, in $(\mathcal{M}, \tau)$ is a linear functional μ on $\mathbb{C} < x_i, i \in \mathcal{I} >$, the noncommutative polynomial ring with noncommuting variables x_i, such that $\mu(\psi(x_{i_1}, \ldots, x_{i_n})) = \tau(\psi(A_{i_1}, \ldots, A_{i_n}))$ for every ψ in $\mathbb{C} < x_i, i \in \mathcal{I} >$.*

Definition 3 *(From Voiculescu [19]) The von Neumann subalgebras $\mathcal{M}_i, i \in \mathcal{I}$ of $\mathcal{M}$ are* free *with respect to the trace τ if $\tau(A_1 \ldots A_n) = 0$ whenever $A_j \in \mathcal{M}_{i_j}, i_1 \neq \ldots \neq i_n$ and $\tau(A_j) = 0$ for $1 \leq j \leq n$ and every n in $\mathbb{N}$.*

Theorem 1 Central Limit Law *(From Voiculescu [19]): Let* $(\mathcal{M}, \tau)$ *be a noncommutative probability space, and let* $(a_j)_{j=1}^{\infty}$ *be a free family of random variables in* $\mathcal{M}$ *such that:*

1. $\psi(a_j) = 0 \ \forall j \geq 1$;
2. $\sup_{j \geq 1} |\psi(a_j^k)| < \infty \ \forall k \geq 2$;
3. $\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \psi(a_j^2) = \frac{r^2}{4}$.

Then letting s_n *be* $\frac{1}{\sqrt{n}}(a_1 + \cdots + a_n)$*, the sequence* $(s_n)_{n=1}^{\infty}$ *converges in distribution to the semicircle law* $\gamma_{0,r}$.

Using the model of random matrices, D. Voiculescu proved the following important theorem:

Theorem 2 *(Voiculescu [19]) Let* $n \geq 2$ *and let* S *be a set with at least two elements. Then*

$$L(\mathbf{F}(|S|)) \simeq M_n(\mathbb{C}) \otimes L(\mathbf{F}(n^2(|S|-1)+1)).$$

In particular,

$$\begin{array}{l} L(\mathbf{F}(2)) \simeq M_n(\mathbb{C}) \otimes L(\mathbf{F}(n^2+1)) \\ L(\mathbf{F}(2)) \simeq M_2(\mathbb{C}) \otimes L(\mathbf{F}(5)) \end{array}.$$

Also by using the model of random matrices, Radulescu was able to show the following,

Theorem 3 *(Radulescu [14]) The fundamental group of* $L(\mathbf{F}(\infty))$ *is* $\mathbb{R}^+$*. That is, in essence,*

$$L(\mathbf{F}(\infty)) \simeq M_n(\mathbb{C}) \otimes L(\mathbf{F}(\infty)),$$

for any $n \geq 1$.

2.2 Free entropy

Let $M_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$ and τ_k be the normalized trace on $M_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}$Tr, where Tr is the usual trace on $M_k(\mathbb{C})$. Let M_k^{sa} denote the self-adjoint complex matrices. The euclidean norm $\| \, \|_e$ on $(M_k^{sa})^n$ is given by

$$\|(A_1, \ldots, A_n)\|_e^2 = \text{Tr}\,(A_1^2 + \cdots + A_n^2),$$

for each $(A_1, \ldots, A_n)$ in $(M_k^{sa})^n$. Let Λ denote the Lebesgue measure on $(M_k^{sa})^n$ induced by the euclidean norm $\| \, \|_e$.

Let $(\mathcal{M}, \tau)$ be a II_1 von Neumann algebra, $X_1, \ldots, X_n$ be self-adjoint elements in $\mathcal{M}$. For $\epsilon, R > 0$, $m, k \in \mathbb{N}$, let

$$\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon)$$

be a subset of $(M_k^{sa})^n$ consisting of all $(A_1, \ldots, A_n)$ in $(M_k^{sa})^n$ such that

$$|\tau(X_{i_1} \ldots X_{i_n}) - \tau_k(A_{i_1} \ldots A_{i_n})| \leq \epsilon,$$

for all $1 \leq p \leq m$, $(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$and $\|A_j\| < R, 1 \leq j \leq m.$

Then, we define successively,

$$\chi_R(X_1,\ldots,X_n;m,k,\epsilon)$$
$$= \log\Lambda(\Gamma_R(X_1,\ldots,X_n;m,k,\epsilon)),$$
$$\chi_R(X_1,\ldots,X_n;m,\epsilon)$$
$$= \limsup_{k\to\infty}\left(k^{-2}\chi_R(X_1,\ldots,X_n;m,k,\epsilon) + \frac{n}{2}\log k\right),$$
$$\chi_R(X_1,\ldots,X_n)$$
$$= \inf\left\{\chi_R(X_1,\ldots,X_n;m,\epsilon) : m\in\mathbb{N}, \epsilon>0\right\},$$
$$\chi(X_1,\ldots,X_n)$$
$$= \sup_{R>0}\chi_R(X_1,\ldots,X_n).$$

In [16], D. Voiculescu proved the following

Theorem 4 Basic Properties of $\chi(X_1, \ldots, X_n)$,

1. One Variable Case

   $$\chi(X) = \iint \log|s-t| d\mu(s)d\mu(t) + \frac{3}{4} + \frac{1}{2}\log 2\pi,$$

   where μ is the distribution of X.

2. Assume $\chi(X_j) > -\infty, 1 \leq j \leq n$. Then,

   $$\chi(X_1, \ldots, X_n) = \chi(X_1) + \cdots + \chi(X_n)$$

   iff $X_1, \ldots, X_n$ are freely independent.

Therefore, for free group factor $L(\mathbf{F}(n))$, there are self-adjoint elements $X_1, \ldots, X_n$, such that

$$\chi(X_1, \ldots, X_n) > -\infty.$$

For technical reasons, D. Voiculescu [21] defined modified free entropy $\chi(X_1,\ldots,X_n : Y_1,\ldots,Y_m)$, as with $\chi(X_1,\ldots,X_n)$, using

$$\Gamma_R(X_1,\ldots,X_n : Y_1,\ldots,Y_m; m,k,\epsilon)$$
$$= pr_{1,\ldots,n}\Gamma_R(X_1,\ldots,X_n,Y_1,\ldots,Y_m; m,k,\epsilon),$$

and successively.

2.3 Free entropy dimension The results applying free entropy can be nicely stated in the language of free entropy dimension (also developed by the D. Voiculescu). (See [16]).

Definition 4 *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra and the trace on it. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ be selfadjoint random variables in $(\mathcal{M}, \tau)$. The* modified free entropy dimension *is defined by*

$$\delta_0(X_1, \ldots, X_n : Y_1, \ldots, Y_m) = n + \limsup_{\epsilon \to 0} \frac{\chi(X_1 + \epsilon S_1, ..., X_n + \epsilon S_n : S_1, ..., S_n, Y_1, ..., Y_m)}{|\log \epsilon|}$$

where $\{S_1, \ldots, S_n\}$ is a semicircular family and $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ and $\{S_1, \ldots, S_n\}$ are free. If $m = 0$, we write $\delta_0(X_1, \ldots, X_n)$.

Definition 5 *Free entropy dimension of a von Neumann algebra M, fdim(M) is defined as*

$$\text{fdim}(M) = \sup\{\delta_0(X_1, \ldots, X_n), \; X_1 \ldots X_n \text{ is a family of generators of } M\}$$

2.4 Some earlier results

1. Voiculescu ([16]) showed that for any n in $\mathbb{N}$

$$\text{fdim}(L(\mathbf{F}(n))) \geq n.$$

2. Voiculesu [17] showed that if von Neumann algebra M has a Cartan subalgebra, then

$$\text{fdim}(M) \leq 1.$$

3. The fifty-year-old question of Ambrose and Singer was answered by Ge [4] (also using Voiculescu's free entropy) when he showed that if the von Neumann algebra M has simple maximal abelian von neumann subalgebras, then

$$\text{fdim}(M) \leq 2.$$

4. Later, Ge [5] showed that if von Neumann algebra M is not prime, i.e., is a tensor product of two infinite-dimensional von Neumann algebras, then

$$\text{fdim}(M) \leq 1.$$

   In particular, $L(\mathbf{F}(n))(n > 1)$ is prime.

   This also solves a very old question.

5. K. Dykema [3] computed the free entropy dimension for the von Neumann algebra with property C.

6. M. Stefan [15] showed that the free group factors $L(\mathbf{F}(n))$ don't have nonprime subfactors with finite index.

2.5 Our main result (Joint work with Liming Ge)

Theorem 5 *Let $\mathcal{M}$ be a type II_1 von Neumann algebra. If there is a sequence of "Haar" unitaries $\{u_j\}_{j=1}^{\infty}$ in $\mathcal{M}$ such that*

1. *$\{u_j\}_{j=1}^{\infty}$ generate $\mathcal{M}$, and*

2. *$u_{j+1}u_ju_{j+1}^*$ is in the von Neumann subalgebra generated by $\{u_1, \ldots, u_j\}$ for all $j \geq 1$,*

then

$$fdim(\mathcal{M}) \leq 1,$$

where the unitary u is called a Haar unitary if $\tau(u^n) = 0$ for $n \neq 0$.

Collorary 1 $L(\mathbf{F}(n))$ *has no Cartan subgebras for any* $n \geq 2$*.*

This is the result by Voiculescu in [17].

Collorary 2 $L(\mathbf{F}(n))$ *is not prime for any* $n \geq 2$*.*

This is the result by Liming Ge in [5]

Collorary 3 *fdim*$(L(SL(2n+1,\mathbb{Z}))) \leq 1.$

This extends an earlier result by Voiculescu in [18].

3. A sketch of the proof of our main theorem

The idea of the proof of the our main result is motivated by the papers from Liming Ge [5] and D. Voiculescu [17, 18].

Let k be a large integer and denote

$$W_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i(k-1)/k} \end{pmatrix}$$

For any integers g, p and any positive numbers ω, D, D_j's , inductively we define the following constants:

$$
\begin{aligned}
{}_{g-1} &= \frac{\omega}{13D\sqrt{g}}, && \delta_{g-1} = \epsilon_{g-1}^{p}; \\
\epsilon_{g-2} &= \frac{\delta_{g-1}}{45D_{g-1}}, && \delta_{g-2} = \epsilon_{g-2}^{p}; \\
&\cdots && \cdots \\
\epsilon_1 &= \frac{\delta_2}{45D_2}, && \delta_1 = \epsilon_1^{p}; \\
\epsilon_0 &= \frac{\delta_1}{9D_1}.
\end{aligned}
$$

Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra. We assume that there is a sequence of "Haar" unitaries $\{u_j\}_{j=1}^{\infty}$ in $\mathcal{M}$ such that

1. $\{u_j\}_{j=1}^{\infty}$ generate $\mathcal{M}$, and

2. $u_{j+1}u_j u_{j+1}^*$ is in the von Neumann subalgebra generated by $\{u_1, \ldots, u_j\}$ for all $j \geq 1$.

Let X_1, X_2, …, X_n be self-adjoint elements in $\mathcal{M}$ with $\|X_j\| \leq 1$ for all j. For any $\omega > 0$, assume that there exist an integer $g > 0$ and non-commutative polynomial $\Psi_i(x_1, \ldots, x_g)$, $i = 1, \ldots, n$, such that

$$\|X_i - \Psi_i(u_1, \ldots, u_g)\|_2 < \omega.$$

In the von Neumann algebra $\mathcal{M}$, we have

$$
\begin{aligned}
& X_1, X_2, \ldots, X_n \\
& u_1, u_2, \ldots, u_g \\
& v_2 = u_2 u_1 u_2^* \\
& \qquad \vdots \\
& v_j = u_{j+1} u_j u_{j+1}^*, \\
& \qquad \vdots \\
& v_{g-1} = u_g u_{g-1} u_g^*
\end{aligned}
$$

and the relations:

$$
\begin{aligned}
& \|X_i - \Psi_i(u_1, \ldots, u_g)\|_2 < \omega \\
& \|u_{j+1} u_j - v_j u_{j+1}\|_2 = 0 \\
& \|v_j - \psi_j(u_1, \ldots, u_j)\|_2 < \delta_j / 45
\end{aligned}
$$

We have the correspondence between the von Neumann algebra $\mathcal{M}$ and the matrix algebra $M_k(\mathbb{C})$:

$$\text{von Neumann algebra} \to \text{ matrix algebra}$$
$$X_i \to A_i$$
$$u_j \to K_j$$
$$v_j \to S_j$$

So, we have the following lemma:

Lemma 1 *For any*

$$(A_1, \ldots, A_n, K_1, \ldots, K_g, S_1, \ldots, S_{g-1})$$

in

$\Gamma_{10/9}(X_1, \ldots, X_n, u_1, \ldots, u_g, v_1, \ldots, v_{g-1}; k, m, \epsilon)$

and any given

$$\omega, \epsilon_0, \ldots, \epsilon_{g-1},$$

we can choose m *large and epsilon small enough so that*

1. $||A_i - \Psi_i(K_1, \ldots, K_g)\|_2 < \omega$
2. *for* $1 \leq j \leq g-1$,
   $\|K_{j+1}K_j - S_jK_{j+1}\|_2 < \epsilon_0$,
   $\|S_j - \psi_j(K_1, \ldots, K_j)\|_2 < \frac{\delta_j}{45}$
3. *there is a family of unitary matrices* $\tilde{U}_j, \tilde{V}_j$
   $\in M_k(\mathbb{C})$*, such that*
   $\|K_j - \tilde{U}_jW_1\tilde{U}_j^*\|_2 < \epsilon_0$ *and*
   $\|S_j - \tilde{V}_jW_1\tilde{V}_j^*\|_2 < \epsilon_0$

For any positive number δ,

Lemma 2 *Assume that A, B, C are given elements in $M_k(\mathbb{C})$ satisfying:*

1. *A is conjugate to the unitary matrix W_1;*
2. *$||B|| \leq 2, ||C|| \leq 2$ and B is normal (i.e. $B^*B = BB^*$);*
3. *$||AC - CB||_2 < \delta$.*

Define the following two sets:

$$\Sigma(A, B, \delta) = \{G \in M_k(\mathbb{C}) : \\ \|G\| \leq 2, \|AG - GB\|_2 < \delta\};$$

$$\Omega(A, B, \delta) = \{H \in M_k(\mathbb{C}) : \\ \|H - G\|_2 < \epsilon = \delta^{1/p} \text{ for some } G \text{ in } \Sigma(A, B, \delta)\}$$

Then $C \in \Sigma(A, B, \delta)$ and the covering number $\mu(\Omega(A, B, \delta), 6\epsilon)$ of $\Omega(A, B, \delta)$ with balls of radius 6ϵ (with respect to the normalized trace norm $\| \ \|_2$) satisfies

$$\mu(\Omega(A, B, \delta), 6\epsilon) \leq (\frac{2}{\epsilon})^{2\pi k^2 \delta^{1-\frac{1}{p}}}.$$

We will use $\mathcal{T}(A, B, \delta)$ to denote a set of covering balls of radius 6ϵ for $\Omega(A, B, \epsilon)$ that has minimal cardinality. So the cardinality of $\mathcal{T}(A, B, \delta)$ is bounded by $(2/\epsilon)^{2\pi k^2 \delta^{1-\frac{1}{p}}}$.

In our applications, we will be interested in a subset, denoted by $\mathcal{T}_1(A, B, \delta)$, of $\mathcal{T}(A, B, \delta)$ consisting of all balls that contain an element of the form $U^* W_1 U$ for some unitary U in $\mathcal{U}(k)$. in the following, we will choose and fix such an element, denoted by W_j, in each ball in $\mathcal{T}_1(A, B, \delta)$.

Our procedure for handling K_j's

We proceed as follows:

1. For K_1, we already know that there is a unitary matrix $\tilde{U}_1$, such that

$$\|K_1 - \tilde{U}_1 W_1 \tilde{U}_1^*\|_2 < \epsilon_0.$$

2. From the relations:

$$||K_2 K_1 - S_1 K_2\|_2 < \epsilon_0,$$

$$\|S_1 - \psi_1(K_1)\|_2 < \frac{\delta_1}{45}$$

we know that

$$\|\tilde{U}_1^* K_2 \tilde{U}_1 W_1 - \psi_1(W_1)\tilde{U}_1^* K_2 \tilde{U}_1\|_2 < \epsilon_1.$$

3. Apply our Lemma 2, we know that there is some W_2 from $\mathcal{T}_1(W_1, \psi(W_1), \epsilon_1), \delta_2)$ such that

$$\|W_2 - \tilde{U}_1^* K_2 \tilde{U}_1\|_2 < 13\epsilon_1,$$

and

$$\text{Card}(\mathcal{T}_1(W_1, \psi(W_1), \epsilon_1), \delta_2)) \leq (\frac{2}{\epsilon_1})^{2\pi k^2 \delta_1^{1-\frac{1}{p}}}$$

4. Repeating step 2 again, we can show that

Lemma 3 *There are some unitary matriices $\tilde{U}_1$ and W_j in $\mathcal{U}(k)$, such that*

$$\|W_j - \tilde{U}_1^* K_j \tilde{U}_1\|_2 \leq 13\epsilon_{j-1}.$$

Therefore

$$\|\tilde{U}_1^* A_j \tilde{U}_1 - \Psi_j(W_1, W_2, \ldots, W_g)\|_2 < 3\omega$$

for $j = 1, \ldots, g$, where W_j and ϵ_j are all described as above, and

$$\text{\# of choices of } W_j \leq (\frac{2}{\epsilon_{j-1}})^{2\pi k^2 \delta_{j-1}^{1-\frac{1}{p}}} \quad (j \geq 1).$$

Theorem 6

$$\chi(X_1, \ldots, X_n : U_1, \ldots, U_g) \\ \leq \log(8C) + \frac{n}{2}\log(6e\pi) + (n-1)\log\omega,$$

where $C \leq 3\pi e^{3\pi}$ *is a constant.*

With the von Neumann algebra $\mathcal{M}$ as above, we have

Theorem 7 *Suppose* $X_1, \ldots, X_n$ *are self-adjoint elements in* $\mathcal{M}$ *that generate* $\mathcal{M}$ *as a von Neumann algebra. Then the free entropy dimension*

$$\delta(X_1, X_2, \ldots, X_n) \leq 1.$$

Therefore,

$$\text{fdim}(\mathcal{M}) \leq 1.$$

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