Finite propagation operators which are Fredholm

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September 20, 2003

The translation algebra

Let X be a metric space. We will assume for simplicity that X is uniformly discrete and proper (in this context, the latter condition means that balls are finite sets).

Consider infinite matrices parameterized by $X \times X$. Such a matrix $\{a_{xy}\}$ has finite propagation if there is a constant $r > 0$ such that $a_{xy} = 0$ whenever $d(x, y) > r$. The uniformly bounded finite propagation matrices form an algebra under matrix multiplication

$$(a \cdot b)_{xz} = \sum_{y \in X} a_{xy} b_{yz}. \tag{1}$$

This is the translation algebra.

There is a natural 'regular' representation of the translation algebra as bounded operators on $\ell^2(X)$ (by matrix multiplication), and this representation is faithful.

(2) DEFINITION: *The (uniform) translation C^*-algebra $C_u^*(X)$ is the completion of the translation algebra in the norm of operators on $\ell^2(X)$.*

Higson and Yu observed that in case X is the underlying metric space of a discrete group Γ, we have

$$C_u^*(X) = \ell^\infty(\Gamma) \rtimes \Gamma \tag{3}$$

Notice that in any event $C_u^*(X)$ is a large scale (coarse) invariant of X.

Coarse index theory

If X is (a uniform lattice in a) complete Riemannian manifold of bounded geometry, then the K-theory groups $K_*(C_u^*(X))$ are the natural receptacles for the indices of bounded geometry operators. (The related groups $K_*(C^*(X))$ are the natural receptacles for indices without the bounded geometry condition.)

These indices are almost never ordinary Fredholm indices. In fact one has

(4) PROPOSITION: *Suppose that X is path connected and non-compact, and that D is Fredholm in the ordinary sense. Then the coarse index of D (in $K_*(C^*(X))$ or $K_*(C_u^*(X))$) is zero.*

PROOF: Use the functoriality of these K-theory groups and the computation for the case that X is a half line. $\square$

RRR ask — Nevertheless, what is the index theory for finite propagation operators that happen to be Fredholm?

Limit operators

Take $X = \mathbb{Z}$. Let T be a bounded operator on $\ell^2(X)$. Let U denote the bilateral shift.

(5) DEFINITION: *The operator T is of limit class if for every ultrafilter ω on $\mathbb{Z}$, the limit $T_\omega = \lim_{n\to\omega} U^{-n}TU^n$ exists in the strong operator topology.*

(Slight adaptation of a definition of Rabinovich, Roch and Silbermann.) The operators of limit class form a C^*-algebra.

(6) PROPOSITION: *Every $T \in C^*_u(X)$ is of limit class.*

PROOF: A net of operators of fixed finite propagation converges strongly iff it converges weakly. But the unit ball of $\mathfrak{B}(H)$ is weakly compact. $\square$

For an operator of limit class, the operator spectrum is the collection of limit operators T_ω. The assignment $T \mapsto T_\omega$ is a $*$-homomorphism.

Limit operators and Fredholmness

The operator U is clearly of limit class. A multiplication operator (by an ℓ^∞ function) is of limit class. These two facts provide an alternative proof that $C_u^*(X)$is made up of limit class operators, using the result of Higson and Yu mentioned above.

Every compact operator is of limit class, and its operator spectrum is zero. Hence every Fredholm operator in $C_u^*(X)$ has operator spectrum made up of invertibles.

(7) THEOREM: *(RRS) An operator $T \in C_u^*(X)$ is Fredholm if and only if every T_ω is invertible (and the norms $\|T_\omega^{-1}\|$ are bounded away from zero).*

Question: How can you express $\mathrm{Index}(T)$ in terms of the limit operators?

Plus and minus indices

Let P denote the orthogonal projection from $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{Z}^+)$, and let $Q = 1 - P$ denote the complementary projection.

For every $T \in C^*_u(X)$ let $T_+ = PTP + Q$ and $T_- = P + QTQ$. Then

$$T_+T_- = T_-T_+ = PTP + QTQ = T + \text{compact} \qquad (8)$$

Thus if T is Fredholm so are T_+ and T_- and the index of T is the sum of the indices of T_+ and T_-. Call these the plus-index and minus-index of T.

The index theorem

Let T belong to the translation C^*-algebra of $X = \mathbb{Z}$. The operator spectrum $\sigma_{op}(T)$ of T can be written as the union of two pieces: $\sigma_+(T)$ corresponding to ultrafilters which tend to $+\infty$ and $\sigma_-(T)$ corresponding to ultrafilters which tend to $-\infty$.

(9) THEOREM: *Let $T \in C^*_u(X)$ be Fredholm. Then*

(a) *For every $T_+ \in \sigma_+(T)$ the operator $PT_+P + Q$ is Fredholm and $\mathrm{Index}(PT_+P + Q) = \mathrm{Index}(PTP + Q)$;*

(b) *For every $T_- \in \sigma_-(T)$ the operator $QT_-Q + P$ is Fredholm and $\mathrm{Index}(QT_-Q + P) = \mathrm{Index}(QTQ + P)$;*

(c) *Consequently,*

$$\mathrm{Index}(T) = \mathrm{Index}(PT_+P + Q) + \mathrm{Index}(QT_-Q + P) \tag{10}$$

Thus we have expressed $\mathrm{Index}(T)$ as the sum of two 'local Toeplitz indices at infinity'.

Proof of the theorem

Let A denote the C^*-subalgebra of $C_u^*(X)$ which is the unitalization of the ideal generated by P. Thus A consists of those operators of the form $S = PTP + \lambda Q + \text{compact}$.

(11) LEMMA: *We have $K_1(A/\mathfrak{K}) \cong \mathbb{Z}$, with the isomorphism being implemented by the Fredholm index.*

Grant this. Let ω be an ultrafilter tending to $+\infty$. If S is as above then $S_\omega = T_\omega$. The map $S \mapsto S_\omega$ is a $*$-homomorphism from $A/\mathfrak{K}$ to $C_u^*(X)$. It follows that $\varphi\colon S \mapsto PS_\omega P + Q$ is a $*$-homomorphism from $A/\mathfrak{K}$ to the Calkin algebra.

(12) LEMMA: *φ induces the identity on K_1.*

PROOF: Given the previous lemma, we need only check this on the unilateral shift $V = PUP + Q$. But $\varphi(V) = V$. $\square$

Proof continued

It follows from this that $\mathrm{Index}(S) = \mathrm{Index}(PS_\omega P + Q)$ for a Fredholm $S \in A$, and applying this to $S = PTP + Q$ and recalling that $S_\omega = T_\omega$, we get Theorem 9(a).

It remains to prove Lemma 11. This is done in three stages:

(i) Compute $K_*(C^*_u(X))$ using the Pimsner-Voiculescu sequence for the crossed product $\ell^\infty(\mathbb{Z}) \rtimes \mathbb{Z}$;

(ii) Apply the coarse Mayer-Vietoris principle 'in reverse' to compute the K-theory of the ideals associated to the positive and negative half-lines. We conclude that $K_1(A) = 0$;

(iii) Use the fact that A contains an operator of index one (namely V) to deduce that the Fredholm index gives an isomorphism from $K_1(A/\mathfrak{K})$ to $\mathbb{Z}$.

Note that the same phenomenon (K_1 vanishes for a half line) which showed that the coarse index of a Fredholm operator is zero, is the one that allows us to prove this more refined index theorem.

An example: Atiyah-Singer for the circle

This is an example of the dual control perspective. See the forthcoming PhD thesis of V.–T. Luu.

Consider a scalar pseudodifferential operator D on the circle, of order zero. Its symbol is given by a continuous function on the cosphere bundle of S^1, i.e. a pair of functions f, g on the circle. Up to compact perturbation

$$D = PM_fP + QM_gQ \tag{13}$$

where P is the Hardy projection and Q is its complement.

Regard D as controlled over $X = \mathbb{Z} = \widehat{S^1}$ via Fourier series.

Then $D \in C^*_u(X)$. Moreover, each D_ω for $\omega \to +\infty$ is convolution by the Fourier series of f, and each D_ω for $\omega \to -\infty$ is convolution by the Fourier series of g. Our index theorem therefore gives

$$\mathrm{Index}(D) = -wn(f) + wn(g) \tag{14}$$

in agreement with the result of the Atiyah-Singer theorem in this case.

Questions and prospects

1. How is the Fredholmness result of RRS, Theorem 7, related to the coarse groupoid exact sequence: if G is the Skandalis-Tu-Yu groupoid of X we have an exact sequence
$$0 \to \mathfrak{K} \to C^*_r(G) = C^*_u(X) \to C^*_r(G_\infty) \to 0 \qquad (15)$$
provided that X has property A?

2. Relate this to the 'total ellipticity theorem' for index theory on singular manifolds (Monthubert-Nistor).

3. Thinking about pseudodifferential operators from the dual control perspective, as above, leads us to consider the algebra $C(h\Gamma) \rtimes \Gamma$ where $h\Gamma$ is the Higson compactification. What is the significance of this algebra? (Note that it is not a coarse invariant.)

4. What about a higher-dimensional version of the main theorem? Should we look at homotopy classes of appropriate 'symbol' maps from S^{n-1} to a suitable restricted general linear group?