RANKIN-COHEN DEFORMATIONS ALONG THE TRANSVERSE FUNDAMENTAL CLASS

The symmetry in transverse geometry comes organized in the form of certain Hopf algebras $\mathcal{H}_n$, which play a role similar to that of GL_n as structure group for the frame bundle. Much like a classical Lie algebra deforms to a QUE algebra, a simplified version of $\mathcal{H}_1$ also admits a nontrivial deformation, given by a natural extension of the Rankin-Cohen brackets for modular forms. I will explain in this light the results of [1], which in turn are based on the framework developed in [2].

A. Connes, H. Moscovici:

[1] Rankin-Cohen brackets and the Hopf Algebra of Transverse Geometry, *arXiv:math.QA/0304316*;

[2] Modular Hecke Algebras and their Hopf Symmetry, *arXiv:math.QA/0301089*.

Prototype = the geometry of the *space of leaves* of a foliation $(V, \mathcal{F})$, in terms of (M, Γ), with M = *complete transversal* and $\Gamma =$ *holonomy pseudogroup*. The 'coordinates' of this transverse space are embodied by the algebra

$$\mathcal{A}_M^\Gamma := C_c^\infty(FM) \rtimes \Gamma\,, \qquad \text{where } FM = J^1(M)\,,$$

consisting of finite sums of monomials of the form

$$\sum f\,U_\phi^*\,, \quad f \in C_c^\infty(FM)\,, \phi \in \Gamma\,,$$

and with the product given by $\quad f\,U_\phi^* \cdot g\,U_\psi^* = (f \cdot g|\phi)\,U_{\psi\phi}^*\,.$

W.l.o.g. assume M equipped with a *flat affine connection* (ω_j^i) and let $\{X_k\}$, $\{Y_i^j\}$ be the standard horizontal and vertical vector fields, implementing the action of the affine extension $\mathfrak{a}(n, \mathbb{R})$ of $\mathfrak{gl}(n, \mathbb{R})$,

$$[Y_i^j, Y_k^\ell] = \delta_k^j Y_i^\ell - \delta_i^\ell Y_k^j\,, \quad [Y_i^j, X_k] = \delta_k^j X_i\,, \quad [X_k, X_\ell] = 0\,.$$

Each $Z \in \mathfrak{a}(n, \mathbb{R})$ acts on $\mathcal{A}_M^\Gamma$ as a linear transformation: $\quad Z(fU_\psi^*) = Z(f)U_\psi^*\,.$

Let $\mathcal{H}_n$ be the algebra of linear transformations of $\mathcal{A}_M^\Gamma$ generated by the operators $\{ X_k,\, Y_i^j,\, \delta^i_{jk,\ell_1\dots,\ell_r} \}$, where

$$\delta^i_{jk,\ell_1\dots,\ell_r}(f\,U^*_\varphi) \;=\; \gamma^i_{jk,\ell_1\dots\ell_r}\cdot f\,U^*_\varphi\,, \quad \text{with}$$
$$\gamma^i_{jk} = \langle \widetilde{\varphi}^*\omega^i_j,\, X_k\rangle \quad , \quad \gamma^i_{jk,\ell_1\dots\ell_r} = X_{\ell_r}\cdots X_{\ell_1}(\gamma^i_{jk})\,.$$

The generators satisfy obvious product rules when acting on $\mathcal{A}_M^\Gamma$:

$$\begin{aligned} Y_i^j(a^1a^2) &= Y_i^j(a^1)a^2 + a^1Y_i^j(a^2)\,, \\ X_k(a^1a^2) &= X_k(a^1)a^2 + a^1X_k(a^2) + \delta^i_{jk}(a^1)Y_i^j(a^2), \\ \delta^i_{jk}(a^1a^2) &= \delta^i_{jk}(a^1)\,a^2 + a^1\,\delta^i_{jk}(a^2)\,. \end{aligned}$$

By multiplicativity $\quad h(a^1a^2) \;=\; \sum\, h_{(1)}(a^1)\,h_{(2)}(a^2)\,, \quad h\,\in\,\mathcal{H}_n\,, \quad a_1, a_2\,\in\,\mathcal{A}_M^\Gamma\,,$

giving a *coproduct* $\Delta : \mathcal{H}_n \,\to\, \mathcal{H}_n\otimes\mathcal{H}_,$,

$$\Delta\,h = \sum\, h_{(1)}\otimes h_{(2)}\,, \qquad \forall\, h\in\mathcal{H}\,,$$

compatible with the algebra structure, and satisfying all the Hopf algebra axioms. The resulting Hopf algebra $\mathcal{H}_n$ is actually independent of the choices made, and can alternatively be described as a bicrossproduct of G. Kac type.

The Hopf algebra $\mathcal{H}_1$

As *algebra* $\mathcal{H}_1 :=$ the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n \,; n \geq 1\}$ and brackets

$$[Y,X] = X, \quad [Y,\delta_n] = n\delta_n, \quad [X,\delta_n] = \delta_{n+1}, \quad [\delta_k,\delta_\ell] = 0.$$

As *coalgebra*,

$$\Delta\, Y = Y \otimes 1 + 1 \otimes Y, \quad \Delta\, X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y\,, \quad \Delta\, \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1\,,$$

together with the multiplicativity property

$$\Delta(h^1\, h^2) = \Delta h^1 \cdot \Delta h^2\,, \quad \forall\, h^1\,, h^2 \in \mathcal{H}_1\,.$$

The *antipode* is determined by

$$S(Y) = -Y, \quad S(X) = -X + \delta_1 Y, \quad S(\delta_1) = -\delta_1\,,$$

and the property

$$S(h^1\, h^2) = S(h^2)\, S(h^1)\,, \quad \forall\, h^1\,, h^2 \in \mathcal{H}_1\,.$$

The *counit* is $\varepsilon(h) =$ constant term of $h \in \mathcal{H}_1$.

STANDARD ACTIONS OF $\mathcal{H}_1$

Given a 1-dimensional oriented manifold M^1 and a subgroup $\Gamma \subset \mathrm{Diff}^+(M^1)$ acting on $J^1_+(M^1) \simeq M \times \mathbb{R}^+$ (with coordinates (x, x_1)) by natural prolongation

$$\varphi(x, x_1) = (\varphi(x),\, \varphi'(x) \cdot x_1)\,,$$

$\mathcal{H}_1$ acts on the crossed product algebra $\mathcal{A} = C^\infty_c(J^1_+(M^1)) \rtimes \Gamma$ by

$$Y(fU^*_\varphi) \;=\; x_1 \,\frac{\partial f}{\partial x_1}\, U^*_\varphi\,, \qquad X(fU^*_\varphi) = x_1\,\frac{\partial f}{\partial x}\, U^*_\varphi\,,$$

$$\delta_n(fU^*_\varphi) \;=\; x_1^n\,\frac{d^n}{dx^n}\left(\log \frac{d\varphi}{dx}\right)\, fU^*_\varphi\,.$$

In addition

$$\tau(fU^*_\varphi) \;=\; \begin{cases} \int_{J^1_+(M^1)} f(x, x_1)\, \frac{dx \wedge dx_1}{x_1^2} & \text{if } \varphi = 1\,, \\ 0 & \text{if } \varphi \neq 1\,, \end{cases}$$

gives a trace satisfying the invariance property

$$\tau(h(a)) = \nu(h)\,\tau(a)\,, \quad \forall\, h \in \mathcal{H}_1\,,$$

where $\nu(Y) = 1, \quad \nu(X) = 0, \quad \nu(\delta_n) = 0$, $\nu \in \mathcal{H}_1^*$.

$S^2 \neq \mathrm{Id}$, but the *twisted antipode* $\widetilde{S} := \nu * S\,,$ does satisfy $\widetilde{S}^2 = \mathrm{Id}$.

PROJECTIVE REDUCTION

In particular take $M^1 = P^1(\mathbb{R}) \simeq S^1$ and $\Gamma = \mathrm{PSL}(2,\mathbb{R})$ acting by projective (i.e. linear fractional) transformations. One has $\varphi \in \Gamma \Leftrightarrow$ its *Schwarzian derivative*

$$\{\varphi; x\} := \frac{d^2}{dx^2}\left(\log \frac{d\varphi}{dx}\right) - \frac{1}{2}\left(\frac{d}{dx}\left(\log \frac{d\varphi}{dx}\right)\right)^2 \equiv 0$$

$\Leftrightarrow \delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$ acts as 0 on $\mathcal{A} = C_c^\infty(J_+^1(P^1(\mathbb{R}))) \rtimes \Gamma$.

Thus, the effective action on $\mathcal{A}$ is that of the quotient Hopf algebra

$$\mathcal{H}_{\mathrm{pr}} := \mathcal{H}_1/\left(\delta_2'\right)$$

by the ideal – and coideal – generated by the *primitive* element $\delta_2' \in \mathcal{H}_1$.

In $\mathcal{H}_{\mathrm{pr}}$ one has

$$\delta_n = \frac{(n-1)!}{2^{n-1}}\delta_1^n .$$

Thus, $\mathcal{H}_{\mathrm{pr}}$ is finitely generated as an algebra, but still *infinite-dimensional* as vector space.

- $\delta_1 \in \mathcal{H}_1$ gives a class $[\delta_1] \in HC^1_{\mathrm{Hopf}}(\mathcal{H}_1)$,

$$b(\delta_1) = 1 \otimes \delta_1 - \Delta\delta_1 + \delta_1 \otimes 1 = 0, \quad \tau_1(\delta_1) = \widetilde{S}(\delta_1) = S(\delta_1) = -\delta_1,$$

which is mapped by the characteristic homomorphism

$$\chi_\tau : HC^*_{\mathrm{Hopf}}(\mathcal{H}_1) \to HC^*(C_c^\infty(J^1_+(M^1)) \rtimes \Gamma),$$
$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0\, h^1(a^1) \ldots h^n(a^n))$$

into the Godbillon-Vey class:

$$\chi_\tau(\delta_1)\big(f^0 U^*_\varphi, f^1 U^*_{\varphi^{-1}}\big) = \tau\big(f^0 U^*_\varphi \cdot \delta_1(f^1 U^*_{\varphi^{-1}})\big) = \int_{J^1_+(M^1)} f^0(x, x_1) \cdot f^1(\varphi(x, x_1)) \cdot x_1 \frac{d}{dx}(\log \varphi'(x)) \cdot \frac{dx \wedge dx_1}{x_1^2}.$$

- $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$ is also primitive, hence a cyclic cocycle. One has $\delta_2' = B(\delta_1 \otimes X + \frac{1}{2}\,\delta_1^2 \otimes Y)$, thus $[\delta_2'] \in HC^1_{\mathrm{Hopf}}(\mathcal{H}_1)$ vanishes in $PHC^{\mathrm{odd}}_{\mathrm{Hopf}}(\mathcal{H}_1)$.

- Last but not least $T := X \otimes Y - Y \otimes X - \delta_1\, Y \otimes Y$ is a Hopf cyclic 2-cocycle, mapped by χ_τ into the *transverse fundamental class* of $PHC^{\mathrm{ev}}(C_c^\infty(J^1_+(M^1)) \rtimes \Gamma)$.

Definition. A *deformation* of a Hopf algebra $\mathcal{H}$ is a Hopf algebra structure on the topological module $\mathcal{H}[[t]]$ over the ring $\mathbb{C}[[t]]$, such that

$$m_t = m + tm_1 + t^2 m_2 + \dots \quad \text{and} \quad \Delta_t = \Delta + t\Delta_1 + t^2\Delta_2 + \dots\ .$$

Twisting. Let $F \in \mathcal{H}[[t]] \otimes_{\mathbb{C}[[t]]} \mathcal{H}[[t]]$ be an invertible element such that $(\Delta \otimes \mathrm{Id})(F) \cdot F \otimes 1 \;=\; (\mathrm{Id} \otimes \Delta)(F) \cdot 1 \otimes F$ and $(\varepsilon \otimes \mathrm{Id})(F) \;=\; 1 \otimes 1 \;=\; (\mathrm{Id} \otimes \varepsilon)(F)$. Then $\Delta_t(h) = F^{-1}\Delta(h)F$ (while $m_t = m$) twists $\mathcal{H}[[t]]$ into a Hopf algebra $\mathcal{H}[[t]]^F$.

Example: QUE algebras [Drinfeld, 1983]. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra, and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be skew-symmetric. There exists a deformation $\mathfrak{A}(\mathfrak{g})[[t]]^F$ of $\mathfrak{A}(\mathfrak{g})$ given by a twisting of the form

$$F = 1 \otimes 1 + tF_1 + t^2 F_2 + \dots, \qquad \text{with} \qquad F_1 - F_1^{21} = r$$

if and only if r satisfies the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \;=\; 0\,.$$

Moreover, $R = (F^{21})^{-1}F$ satisfies QYBE $\quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

THE RC-QUANTIZED HOPF ALGEBRA $\mathcal{H}_{\rm pr}[[t]]^{RC}$

Given a Hopf algebra $\mathcal{H}$, a necessary condition for $r \in \mathcal{H} \otimes \mathcal{H}$ to define the *direction* $F_1 = r$ of a twisting element $F = 1 \otimes 1 + tF_1 + t^2 F_2 + \ldots$ is that

$$(\Delta \otimes \mathrm{Id})(r) + r \otimes 1 = (\mathrm{Id} \otimes \Delta)(r) + 1 \otimes r \, ,$$

which is precisely the same as $b(r) = 0$.

Theorem A. *The series* $F^{RC} := \sum_{n \geq 0} t^n F_n^{RC} \in \mathcal{H}_{\rm pr}[[t]] \otimes_{\mathbb{C}[[t]]} \mathcal{H}_{\rm pr}[[t]]$, *where*

$$F_n^{RC} \; : = \; \sum_{k=0}^{n} \frac{S(X)^k}{k!} (2Y + k)_{n-k} \otimes \frac{X^{n-k}}{(n-k)!} (2Y + n - k)_k \, ,$$

$$Z_k \; = \; Z \, (Z+1) \ldots . (Z + k - 1) \qquad \textit{and}$$

$$S(X)^k \; = \; \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{\delta_1^j}{2^j} X^{k-j} (2Y + k - j)_j \, ,$$

defines a twisting of $\mathcal{H}_{\rm pr}$ *along the direction* $F_1^{RC} = -2T = S(X) \otimes 2Y + 2Y \otimes X$.

To illustrate its degree of complexity, here is the 3rd component:

$$
\begin{aligned}
F_3^{RC} &= -2\,X\otimes X^2 - 2\,X\otimes X^2.Y + 2\,X^2\otimes X + 6\,X^2\otimes X.Y + 4\,X^2\otimes X.Y^2 \\
&-\frac{2\,X^3\otimes Y}{3} - 2\,X^3\otimes Y^2 - \frac{4\,X^3\otimes Y^3}{3} + \frac{2\,Y\otimes X^3}{3} + 2\,Y^2\otimes X^3 + \frac{4\,Y^3\otimes X^3}{3} \\
&-6\,X.Y\otimes X^2 - 6\,X.Y\otimes X^2.Y - 4\,X.Y^2\otimes X^2 - 4\,X.Y^2\otimes X^2.Y \\
&+2\,X^2.Y\otimes X + 6\,X^2.Y\otimes X.Y + 4\,X^2.Y\otimes X.Y^2 - 2\,\delta_1.X\otimes X - 6\,\delta_1.X\otimes X.Y \\
&-4\,\delta_1.X\otimes X.Y^2 + 2\,\delta_1.X^2\otimes Y + 6\,\delta_1.X^2\otimes Y^2 + 4\,\delta_1.X^2\otimes Y^3 + 2\,\delta_1.Y\otimes X^2 \\
&+2\,\delta_1.Y\otimes X^2.Y + 6\,\delta_1.Y^2\otimes X^2 + 6\,\delta_1.Y^2\otimes X^2.Y + 4\,\delta_1.Y^3\otimes X^2 + 4\,\delta_1.Y^3\otimes X^2.Y \\
&-\delta_1^2.X\otimes Y - 3\,\delta_1^2.X\otimes Y^2 - 2\,\delta_1^2.X\otimes Y^3 + \delta_1^2.Y\otimes X + 3\,\delta_1^2.Y\otimes X.Y \\
&+2\,\delta_1^2.Y\otimes X.Y^2 + 3\,\delta_1^2.Y^2\otimes X + 9\,\delta_1^2.Y^2\otimes X.Y + 6\,\delta_1^2.Y^2\otimes X.Y^2 + 2\,\delta_1^2.Y^3\otimes X \\
&+6\,\delta_1^2.Y^3\otimes X.Y + 4\,\delta_1^2.Y^3\otimes X.Y^2 + \frac{1}{3}\,\delta_1^3.Y\otimes Y + \delta_1^3.Y\otimes Y^2 + \frac{2}{3}\,\delta_1^3.Y\otimes Y^3 \\
&+\delta_1^3.Y^2\otimes Y + 3\,\delta_1^3.Y^2\otimes Y^2 + 2\,\delta_1^3.Y^2\otimes Y^3 + \frac{2}{3}\,\delta_1^3.Y^3\otimes Y + 2\,\delta_1^3.Y^3\otimes Y^2 \\
&+\frac{4}{3}\,\delta_1^3.Y^3\otimes Y^3 - 6\,\delta_1.X.Y\otimes X - 18\,\delta_1.X.Y\otimes X.Y - 12\,\delta_1.X.Y\otimes X.Y^2 \\
&-4\,\delta_1.X.Y^2\otimes X - 12\,\delta_1.X.Y^2\otimes X.Y - 8\,\delta_1.X.Y^2\otimes X.Y^2 + 2\,\delta_1.X^2.Y\otimes Y \\
&+6\,\delta_1.X^2.Y\otimes Y^2 + 4\,\delta_1.X^2.Y\otimes Y^3 - 3\,\delta_1^2.X.Y\otimes Y - 9\,\delta_1^2.X.Y\otimes Y^2 \\
&-6\,\delta_1^2.X.Y\otimes Y^3 - 2\,\delta_1^2.X.Y^2\otimes Y - 6\,\delta_1^2.X.Y^2\otimes Y^2 - 4\,\delta_1^2.X.Y^2\otimes Y^3\,.
\end{aligned}
$$

Some explanatory comments:

1^0. Let $\mathcal{H}$ be a Hopf algebra acting on an algebra $\mathcal{A}$. Then any twisting element $F \in \mathcal{H}[[t]] \otimes_{\mathbb{C}[[t]]} \mathcal{H}[[t]]$ induces an *associative* deformation $\mathcal{A}[[t]]^F$ of $(\mathcal{A}, \mu)$,

$$\mu_t^F(a \otimes b) \, = \, \mu\left(F(a \otimes b)\right), \qquad a, b \in \mathcal{A}.$$

2^0. Assume $\mathcal{H}_{\mathrm{pr}}$ acts on a graded algebra $\mathcal{A} = \sum_{n \geq 0} \mathcal{A}_n$ such that $Y|\mathcal{A}_n \, = \, \frac{n}{2}$ and $\delta_1 \equiv 0$. Then $\forall\, a \in \mathcal{A}_k$, $b \in \mathcal{A}_\ell$, one has

$$\mu\left(F_n^{RC}(a \otimes b)\right) = [a,\, b]_{X,n}^{(k,\ell)} \, := \, \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} X^r(a) X^s(b),$$

i.e. $=$ [Zagier]'s *standard Rankin-Cohen brackets*.

3^0. The original nth RC-bracket of two modular forms $f \in \mathcal{M}_k$, $g \in \mathcal{M}_\ell$ is

$$[f,\, g]_n^{(k,\ell)} \, := \, \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f^{(r)} g^{(s)} \quad \in \, \mathcal{M}_{k+\ell+2n}\,.$$

The proof of Theorem A involves the framework of modular forms.

Let $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$, acting on the upper half-plane $\mathfrak{H} = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ by linear fractional transformations. A *modular form of weight* k is a holomorphic function f satisfying $\quad f|_k\,\gamma = f\,,\ \forall\,\gamma \in \Gamma(1) = \mathrm{SL}(2,\mathbb{Z})\,, \quad$ where

$$f|_k\,\alpha\,(z) = \det(\alpha)^{k/2}\,(cz+d)^{-k} f(\frac{az+b}{cz+d})\,, \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(\mathbb{R}) := \mathrm{GL}^+(2,\mathbb{R})\,,$$

which is also holomorphic at ∞, ie at $q = 0\,, \quad q = e^{2\pi i z}$. Modular forms make up a graded algebra $\mathcal{M}(\Gamma(1)) := \Sigma^{\oplus}\,\mathcal{M}_{2k}(\Gamma(1))\,, \quad$ resp. $\quad \mathcal{M}^0(\Gamma(1))$.

A richer algebra emerges when besides the modular group, one considers its principal congruence subgroups and the projective limit of Riemann surfaces,

$$\mathfrak{H}_{\mathbb{A}} := \varprojlim_{N} \Gamma(N)\backslash\mathfrak{H}\,, \quad \Gamma(N) := \left\{ \gamma \in \mathrm{SL}(2,\mathbb{Z})\,; \quad \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

Equivalently, $\mathfrak{H}_{\mathbb{A}} \simeq \mathrm{GL}(2,\mathbb{Q})\backslash\,\mathrm{GL}(2,\mathbb{A})^0/K_\infty\,Z_\infty$, where $K_\infty \simeq S^1$, $Z_\infty \simeq \mathrm{GL}(1,\mathbb{R})$ viewed as diagonal matrices in $\mathrm{GL}(2,\mathbb{R})$, and the subgroup $\mathrm{GL}(2,\mathbb{A})^0$ is defined by requiring that the determinant belongs to $\mathbb{Q}^* \times \mathbb{R}^* \subset \mathrm{GL}(1,\mathbb{A})$.

For each $N \geq 1$ one has a graded algebra of *forms of level* N. Taken together they define the algebra of modular forms of all levels: $\mathcal{M} := \varinjlim_{N\to\infty} \mathcal{M}(\Gamma(N))$.

The group $G^+(\mathbb{Q}) := \mathrm{GL}^+(2,\mathbb{Q})$ acts 'sideways' on the tower defining the projective limit $\mathfrak{H}_{\mathbb{A}}$. One can then form the crossed product algebra

$$\mathcal{A} \equiv \mathcal{A}_{G^+(\mathbb{Q})} := \mathcal{M} \rtimes G^+(\mathbb{Q}),$$

consisting of finite sums of symbols of the form $\sum f\, U_\gamma^*$, $\quad f \in \mathcal{M}, \gamma \in G^+(\mathbb{Q})$, with the product given by: $\quad f\, U_\alpha^* \cdot g\, U_\beta^* = (f \cdot g|\alpha)\, U_{\beta\,\alpha}^*$.

There are 'natural' actions of the Hopf algebra $\mathcal{H}_1$ on $\mathcal{A}_{G^+(\mathbb{Q})}$ analogous to the action of $\mathcal{H}_1$ on the crossed products of the polynomial functions on the frame bundle of S^1 by discrete subgroups of $\mathrm{Diff}(S^1)$.

'Most natural' action. First, let X act on $\mathcal{M}$ as the 'Ramanujan operator'

$$X := \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{12\pi i}\,\frac{d}{dz}(\log\Delta)\cdot Y \;= \frac{1}{2\pi i}\,\frac{d}{dz} - \frac{1}{2\pi i}\,\frac{d}{dz}(\log\eta^4)\cdot Y\,;$$

$\Delta(z) = (2\pi)^{12}\,\eta^{24}(z) = (2\pi)^{12}\, q\,\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi i z}$ is the discriminant cusp form of weight 12, and $\eta(z)$ is the Dedekind eta-function.

Secondly, let Y act on $\mathcal{M}$ as the grading operator $Y(f) = \frac{k}{2} \cdot f\,, \quad \forall\, f \in \mathcal{M}_k\,.$

For any $\gamma \in G^+(\mathbb{Q})$ and $f \in \mathcal{M}_k$, one has

$\big(X(f|_k\,\gamma^{-1})\big)\,|_{k+2}\,\gamma \;=\; X(f) \,-\, \mu_\gamma \cdot Y(f)\,,$ where $\mu_\gamma\,(z) \;=\; \dfrac{1}{12\pi i}\,\dfrac{d}{dz}\log\dfrac{\Delta|\gamma}{\Delta}\,.$

Equivalently, $\mu_\gamma\,(z) = \frac{1}{2\,\pi^2}\left(G_2^*|\gamma\,(z) - G_2^*(z) + \frac{2\pi i\, c}{cz+d}\right)\,,$ where

$$G_2^*(z) \;=\; 2\zeta(2) \,+\, 2\sum_{m\geq 1}\,\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2}$$

is the holomorphic (but not modular) Eisenstein series of weight 2. One can show that $\mu_\gamma \,\in\, \mathcal{E}_2(\mathbb{Q}) \,=$ space of Eisenstein series with rational constant term at any cusp.

Proposition. *There is a unique Hopf action of* $\mathcal{H}_1$ *on* $\mathcal{A}_{G^+(\mathbb{Q})}$ *, determined by*

$$X(f\,U_\gamma^*) = X(f)\,U_\gamma^*\,, \quad Y(f\,U_\gamma^*) = Y(f)\,U_\gamma^*\,, \quad \text{and} \quad \delta_n(f\,U_\gamma^*) = X^{n-1}(\mu_\gamma)\cdot f\,U_\gamma^*\,.$$

Remark. The Schwarzian cocycle $\delta_2' = \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$, acts as *inner* derivation, $\delta_2'(a) = [\omega\,,\, a]\,,$ implemented by the weight 4 modular form

$$\omega = (2\pi i)^{-2}\{Z;z\}\,, \qquad 72\,\omega = E_4 := 1 + 240\,\sum_1^\infty n^3 \frac{q^n}{1-q^n}.$$

Perturbations by 1-cocycles. A 1-*cocycle* $\gamma \in Z^1(\mathcal{H}_1, \mathcal{A})$ is an invertible element of the convolution algebra of *linear* maps $\mathrm{Hom}(\mathcal{H}, \mathcal{A})$, such that $\gamma(h\,h') = \sum \gamma(h_{(1)})\,h_{(2)}(\gamma(h'))\,,\ h, h' \in \mathcal{H}$. The γ-perturbed action of $\mathcal{H}_1$ on $\mathcal{A}$ is

$$h \cdot_\gamma a := \sum \gamma(h_{(1)})\,h_{(2)}(a)\,\gamma^{-1}(h_{(3)})\,.$$

Proposition. *$\forall\,\nu \in \mathcal{M}_2\,,\ \exists!\ 1$-cocycle $\gamma_\nu \in Z^1(\mathcal{H}_1, \mathcal{A}_{G^+(\mathbb{Q})})$ such that $\gamma(X) = 0 = \gamma(Y)$ and $\gamma(\delta_1) = \nu$. Then $X \mapsto X_\nu = X + \nu Y$ and the Schwarzian cocycle acts as $(\delta_2')_\nu \cdot a = [X(\nu) + \frac{\nu^2}{2} + \omega,\ a]$. There is no choice of ν for which $(\delta_2')_\nu \equiv 0$.*

The ‘normal order’ form of $S(X^n)$ in Theorem A, suggests that the higher Rankin-Cohen brackets for an action of $\mathcal{H}_{\rm pr}$ on a graded algebra $\mathcal{A} = \sum_{n \geq 0} \mathcal{A}_n$, such that $Y|\mathcal{A}_n = \frac{n}{2}$, should be of the form:

$$RC_n(a,\, b) := \sum_{k=0}^{n} (\frac{S(X)^k}{k!}\,(2Y+k)_{n-k})(a)(\frac{X^{n-k}}{(n-k)!}\,(2Y+n-k)_k)(b)\,.$$

However, these are not stable under 1-cocycle perturbations. E.g., by direct computation,

$$\begin{aligned} RC_2(a,\, b) := \quad & S(X)^2(a)\,Y(2Y+1)(b) \,+\, S(X)\,(2\,Y+1)(a)\,X(2Y+1)(b) \\ & + Y(2\,Y+1)(a)\,X^2(b) \,-\, Y(a)\,\Omega\,Y(2Y+1)(b) - \,Y(2Y+1)(a)\,\Omega\,Y(b) \end{aligned}$$

with $\Omega = X(\nu) + \frac{\nu^2}{2}$. The ‘stable’ formula turns out to be of the form

$$RC_n(a,b) := \sum_{k=0}^{n} \frac{A_k}{k!}\,(2Y+k)_{n-k}(a)\;\frac{B_{n-k}}{(n-k)!}\,(2Y+n-k)_k(b)\,,$$

with A_n, B_n obtained from recurrence relations.

The following ‘naturality’ property of the resulting RC-brackets plays a crucial role.

Lemma. *Let $\mathcal{H}_1$ act on an algebra $\mathcal{A}$ with δ_2' inner, and let $u \in \mathcal{A}$ be invertible such that, with $\nu := u^{-1}\delta_1(u)\,,\ X(u) = 0,\ Y(u) = 0,\ \delta_n(\nu) = 0,\ \forall\, n \in \mathbb{N}$. Then*

1^0. $RC_n(x\,u,\ y) = RC_n(x,\ u\,y)\,, \quad \forall\, x,\ y \in \mathcal{A}\,.$

2^0. $RC_n(u\,x,\ y) = u\,RC_n(x,\ y)\,, \quad RC_n(x,\ y\,u) = RC_n(x,\ y)\,u\,.$

When applied to $\mathcal{A}_{G^+(\mathbb{Q})} = \mathcal{M} \rtimes G^+(\mathbb{Q})$, this yields canonical extensions of the original RC-brackets brackets $[\,,\,]_n$ and shows that RC_n are entirely determined by their restriction to to $\mathcal{M}$. By [Zagier] and [Cohen-Manin-Zagier], this gives:

Theorem B. *The following $*$-product defines an associative deformation of $\mathcal{A}_{G^+(\mathbb{Q})}$*

$$a *_t b := \sum_{n \geq 0} t^n RC_n(a, b)\,, \quad a, b \in \mathcal{A}_{G^+(\mathbb{Q})}\,.$$

More generally, for any $\kappa \in \mathbb{C} \cup \{\infty\}\,,$ *the $*^\kappa$–product*

$$a *_t^\kappa b := \sum_{n \geq 0} t^n\, RC_n\left(\mathbf{t}_n^\kappa(Y \otimes 1, 1 \otimes Y)(a \otimes b)\right)\,, \quad a, b \in \mathcal{A}_{G^+(\mathbb{Q})}\,, \qquad \textit{where}$$

$$\mathbf{t}_n^\kappa(\alpha, \beta) := (-\frac{1}{4})^n \sum_{j \geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa - \frac{3}{2}}{j}\binom{\frac{1}{2} - \kappa}{j}}{\binom{-\alpha - \frac{1}{2}}{j}\binom{-\beta - \frac{1}{2}}{j}\binom{n + \alpha + \beta - \frac{3}{2}}{j}}\,, \qquad \kappa \in \mathbb{C}\,,$$

gives an associative deformation.

Extension of scalars

Denote $\mathcal{P} := \mathbb{C}[Z_0, Z_1, Z_2, \dots, Z_n, \dots]$, on which we let $\mathcal{H}_1$ act as derivations,

$$Y(Z_j) := (j+2)\, Z_j\,,\; X(Z_j) := Z_{j+1}\,, \quad \forall\, j \geq 0\,, \quad \text{while} \quad \delta_k(P) := 0\,, \forall\, P \in \mathcal{P}\,.$$

Next form the algebra

$$\widetilde{\mathcal{H}}_1 = \mathcal{P} \rtimes \mathcal{H}_1 \ltimes \mathcal{P} \qquad \big(= \mathcal{P} \otimes \mathcal{P} \otimes \mathcal{H}_1 \quad \text{as vector space}\big)$$

with product $P \rtimes h \ltimes Q \,\cdot\, P' \rtimes h' \ltimes Q' \;:=\; \sum P\, h_{(1)}(P') \ltimes h_{(2)} h' \rtimes h_{(3)}(Q')\, Q\,.$

$\widetilde{\mathcal{H}}_1$ is a $\mathcal{P}$-bimodule, with ‘left’ and ‘right’ action

$$\alpha(P) := P \rtimes 1 \ltimes 1\,, \quad \text{and} \quad \beta(Q) := 1 \rtimes 1 \ltimes Q\,, \quad \forall\, P, Q \in \mathcal{P}\,.$$

$\exists$ ‘coproduct’ $\Delta : \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}_1 \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_1\,, \quad \Delta(P \rtimes h \ltimes Q) := \sum P \rtimes h_{(1)} \ltimes 1 \otimes 1 \rtimes h_{(2)} \ltimes Q\,.$

The element $\quad \widetilde{\delta}_2' \;:=\; \delta_2 - \frac{1}{2}\delta_1^2 - \alpha(Z_0) + \beta(Z_0) \;\in \widetilde{\mathcal{H}}_1 \quad$ satisfies

$$\Delta(\widetilde{\delta}_2') \;=\; \widetilde{\delta}_2' \otimes 1 \;+\; 1 \otimes \widetilde{\delta}_2'$$

and we let $\widetilde{\mathcal{H}}_{\rm pr}$ denote the quotient of $\widetilde{\mathcal{H}}_1$ by the ideal generated by $\widetilde{\delta}_2'$.

Remark. *Any algebra $\mathcal{A}$ on which $\mathcal{H}_1$ acts such that δ_2' is inner and implemented by an $\Omega \in \mathcal{A}$ satisfying $\delta_k(\Omega) = 0\,, \forall k \in \mathbb{N}$, can be turned into an $\widetilde{\mathcal{H}}_{\rm pr}$-module in a fairly obvious way. The converse is also true.*

By Remark and the ‘perturbation’ Proposition, $\forall\, \nu \in \mathcal{M}_2$ one gets such an $\widetilde{\mathcal{H}}_{\rm pr}$–action on $\mathcal{A}_{G^+(\mathbb{Q})}$. Let

$$\sigma_\nu^{(n)} : \underbrace{\widetilde{\mathcal{H}}_{\rm pr} \otimes_{\mathcal{P}} \ldots \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_{\rm pr}}_{n\text{–times}} \quad \longrightarrow \quad \mathrm{Hom}\big(\underbrace{\mathcal{A}_{G^+(\mathbb{Q})} \otimes \ldots \otimes \mathcal{A}_{G^+(\mathbb{Q})}}_{n\text{–times}} , \mathcal{A}_{G^+(\mathbb{Q})}\big)$$

be the map of $(\mathcal{P}, \mathcal{P})$–bimodules defined by

$$\sigma_\nu^{(n)}(h^1 \otimes_{\mathcal{P}} \ldots \otimes_{\mathcal{P}} h^n)\,(a_1, \ldots, a_n) \;=\; (h^1 \cdot_\nu a_1) \cdots (h^n \cdot_\nu a_n)\,.$$

‘Ellipticity’ Lemma. *For each* $n \in \mathbb{N}$, *one has* $\bigcap_{\nu \in \mathcal{M}_2} \mathrm{Ker}\, \sigma_\nu^{(n)} = 0\,.$

In particular, each RC-bracket $RC_n : \mathcal{A}_{G^+(\mathbb{Q})} \otimes \mathcal{A}_{G^+(\mathbb{Q})} \to \mathcal{A}_{G^+(\mathbb{Q})}$ uniquely determines an element $\widetilde{RC}_n \in \widetilde{\mathcal{H}}_{\rm pr} \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_{\rm pr}$. Moreover, by Theorem B one gets:

Theorem C. $\widetilde{F}^{RC} := \sum_{n \geq 0} t^n\, \widetilde{RC}_n \in \widetilde{\mathcal{H}}_{\rm pr}[[t]] \otimes_{\mathcal{P}[[t]]} \widetilde{\mathcal{H}}_{\rm pr}[[t]]$ *is a twisting element*.

The **proof** of Theorem A follows by specializing $\widetilde{F}^{RC}$ at $Z_i = 0\,, i \geq 0$, i.e. sending $\widetilde{\mathcal{H}}_{\rm pr}$ onto $\mathcal{H}_{\rm pr}$ via the Hopf-algebraic homomorphism $P \rtimes h \ltimes Q \mapsto P(0)\, Q(0)\, h$.

We end by recording the most significant consequence of Theorem C.

Theorem D. 1^0. *Let* $\mathcal{A}$ *be an algebra on which* $\mathcal{H}_1$ *acts such that* δ_2' *is inner and implemented by an element* $\Omega \in \mathcal{A}$ *satisfying* $\delta_k(\Omega) = 0\,,\ \forall k \in \mathbb{N}$. *Then*

$$a *_t b := \sum_{n\geq 0} t^n RC_n(a, b)\,,$$

defines an associative deformation of $\mathcal{A}$.

2^0. *If in addition* $\mathcal{A}$, *is graded by the action of* Y, $Y|\mathcal{A}_n = \frac{n}{2}$, *one obtains a* 1*-parameter family of associative deformations, whose products are given for any* $\kappa \in \mathbb{C} \cup \{\infty\}$ *by the formula*

$$a *_t^\kappa b := \sum_{n\geq 0} t^n\, RC_n\left(\mathbf{t}_n^\kappa(Y\otimes 1, 1\otimes Y)(a\otimes b)\right)\,, \quad a, b \in \mathcal{A}\,,$$

where

$$\mathbf{t}_n^\kappa(\alpha,\beta) := (-\frac{1}{4})^n \sum_{j\geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-\alpha-\frac{1}{2}}{j}\binom{-\beta-\frac{1}{2}}{j}\binom{n+\alpha+\beta-\frac{3}{2}}{j}}\,, \qquad \kappa \in \mathbb{C}\,.$$