On C*-algebras and K-theory for infinite-dimensional Fredholm manifolds

Dorin Dumitraşcu
Dartmouth College

joint work with Jody Trout
preliminary report

September 21, 2003

ECOAS 2003, Vanderbilt University

Main Results

Goal 1. Associate to each M of a non-trivial class of infinite dimensional manifolds a non-commutative C^*-algebra $\mathcal{A}(M)$ that plays the role of $C_0(M)$.

Let M be a Hilbert manifold, with metric g, and Fredholm filtration $\mathcal{F} = \{M_n\}_n$ (*i.e.*, dim $M_n = n$, $M_n \subset M_{n+1} \subset M$, $M_\infty = \bigcup_{n \geq k} M_n$ is dense in M, and $M_\infty \hookrightarrow M$ is a homotopy equivalence). Then:

- $\mathcal{A}(M_n) = C_0(\mathbb{R}) \hat{\otimes} C_0(M_n, \mathrm{Cliff}(TM_n))$ (g used)
- $\mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$ (g, $\mathcal{F}$ used)
- $\mathcal{A}(M) = \mathcal{A}(M, g, \mathcal{F}) = \varinjlim_{n \to \infty} \mathcal{A}(M_n)$.

Example. The C^*-algebra $\mathcal{A}(\mathcal{E})$, where $\mathcal{E}$ is a separable infinite-dimensional real Hilbert space, used by Higson-Kasparov-Trout, is an example of our construction.

Goal 2. Use $\mathcal{A}(M)$ in relation to the K-theory of M.

- $K_{j+1}(\mathcal{A}(M)) = K^{\infty - j}(M)$ (M spin, def. Mukherjea)
- $K_{j+1}(\mathcal{A}(M)) = K_j^c(M)$ (M spin, Poincaré duality)

Fredholm Manifolds

We use the following notation:

$\mathcal{E}$ = a real separable Hilbert space

$\mathcal{L}(\mathcal{E})$ = the real C^*-algebra of bdd. lin. ops. on $\mathcal{E}$

$\mathcal{K} = \mathcal{K}(\mathcal{E})$ = the closed ideal of compact operators

$GL(\mathcal{E})$ = the Banach-Lie group of units of $\mathcal{L}(\mathcal{E})$

$GL_{\mathcal{K}}(\mathcal{E}) = \{T = I + K \mid T \in GL(\mathcal{E}), K \in \mathcal{K}(\mathcal{E})\}$

Definition. Let M be a manifold modeled on $\mathcal{E}$. A *Fredholm structure on* M is an integrable reduction of the principal $GL(\mathcal{E})$-bundle of M to $GL_{\mathcal{K}}(\mathcal{E})$. A *Fredholm manifold* is a Hilbert manifold with a specified Fredholm structure.

Fredholm filtrations

The following decomposition theorem is crucial in the study of Fredholm manifolds:

Theorem 1. [Mukherjea,1970] *Let* M *be a Fredholm manifold. There exists a sequence* $\{M_n\}_{n=k}^{\infty}$ *of finite dimensional closed submanifolds such that:*

(i) *dim* $M_n = n$*;* $M_n \subset M_{n+1}$*;*

(ii) *the inclusions* $M_n \hookrightarrow M_{n+1}$ *and* $M_n \hookrightarrow M$ *have trivial normal bundles;*

(iii) $M_\infty = \bigcup_{n \geq k} M_n$ *is dense in* M*; and*

(iv) *if* M_∞ *is given the direct limit topology, the natural inclusion* $M_\infty \hookrightarrow M$ *is a homotopy equivalence.*

A sequence $\{M_n\}_{n=k}^{\infty}$ as in the theorem above is called a *Fredholm filtration of* M.

Examples

Examples. (i) The Euclidean space $M = \mathcal{E}$ has an obvious Fredholm structure, determined by a single chart $I : \mathcal{E} \to \mathcal{E}$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{E}$, and E_n be the linear span of $\{e_1, e_2, \ldots, e_n\}$. The *flag* $\{E_n\}_n$ forms a Fredholm filtration of $\mathcal{E}$.

(ii) The unit sphere of $\mathcal{E}$, $S = \{x \in \mathcal{E} \mid \|x\| = 1\}$, gets by restriction from $\mathcal{E}$ a Fredholm structure, with Fredholm filtration $S^1 \subset S^2 \subset \cdots \subset S^n \subset \cdots \subset S$.

(iii) ([Eells-Elworthy,ICM 1970]) Let X be a complete finite-dimensional Riemannian manifold, and $a \in X$. Let $M = P_a(X)$ be the space of paths $\gamma : [0, 1] \to X$, with $\gamma(0) = a$ and γ absolutely continuous with square integrable derivative. Then M is a contractible Riemannian Fredholm manifold.

Non-example. The sequence

$$\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \cdots \subset \mathbb{R}P^n \subset \cdots \subset \mathbb{R}P^\infty$$

is *not* a Fredholm filtration of the infinite dimensional real projective space.

Some more geometry

Theorem 2. *Let M be a Fredholm manifold. There exists a Fredholm filtration $\{M_n\}_{n=k}^{\infty}$ of M for which geodesically defined exponential neighborhoods Z_n of M_n in M can be constructed satisfying:*

$$Z_n \subset Z_{n+1} \text{ and } \bigcup_{n\geq k} Z_n = M.$$

Moreover $U_n = Z_n \bigcap M_{n+1}$ is a tubular neighborhood of M_n in M_{n+1}, for each $n \geq k$.

(The above statement is a combination of remarks of Mukherjea, Eells and Elworthy.)

Definition. A Fredholm filtration $\{M_n\}_{n=k}^{\infty}$ as in the statement of theorem, together with the tubular neighborhoods $\{Z_n\}_n$ and $\{U_n\}_n$, is called an *augmented Fredholm filtration* and is denoted by $\mathcal{F} = (\{M_n\}_n, \{U_n\}_n)$.

Some more geometry

From now on (M, g) is a Riemannian Fredholm manifold. Denote the inclusions of an augmented Fredholm filtration by

$$i_n : M_n \to M_{n+1} \text{ and } j_n : M_n \to M.$$

• Using the induced metric $g_n = j_n^*(g)$, M_n is a (compact) Riemannian manifold, for each $n \geq k$.

• The inclusion $i_n : M_n \to M_{n+1}$ induces a split-exact sequence:

$$0 \to TM_n \to TM_{n+1}|_{M_n} \to \nu M_n \to 0,$$

where νM_n is the *normal bundle*. The metric gives the splitting, and $TM_{n+1}|_{M_n}$ has a natural (Levi-Civita) covariant derivative ∇^n, obtained from the one on M by pull-back(s). This endows νM_n with a connection $\nabla^{\nu M_n}$.

• νM_n is diffeomorphic to U_n

Where are we heading?

The geometric considerations led to:

$$\begin{array}{ccccc} \nu M_n & \xrightarrow{\text{diffeo}} & U_n & \xhookrightarrow{\text{open}} & M_{n+1} \\ \downarrow & & & & \\ M_n & & & & \end{array}$$

We shall translate this into the following commutative diagram of C^*-algebras:

$$\begin{array}{ccccc} \mathcal{A}(\nu M_n) & \xrightarrow{\simeq} & \mathcal{A}(U_n) & \xhookrightarrow{\text{incl}} & \mathcal{A}(M_{n+1}) \\ \text{Thom} \uparrow & & & \nearrow & \\ \mathcal{A}(M_n) & & \cdots & & \end{array}$$

The dotted arrow will give the connecting map $\mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$.

The C*-algebra of a finite dimensional manifold

Let V be a finite-dimensional Euclidean vector space. The Clifford algebra of V, $\mathrm{Cliff}(V)$, is the universal unital complex C^*-algebra that contains V as a real linear subspace such that $v^2 = \|v\|^2 1$, and $v^* = v$, for all $v \in V$.

$$\mathrm{Cliff}(V \oplus W) \cong \mathrm{Cliff}(V) \widehat{\otimes} \mathrm{Cliff}(W).$$

Definition. Let X be a *finite-dimensional* smooth Riemannian manifold. Let $\mathrm{Cliff}(TX) \to X$ denote the Clifford bundle of the tangent bundle TX. We define the *C^*-algebra of* X by

$$\mathcal{A}(X) = C_0(\mathbb{R}) \widehat{\otimes} C_0(X, \mathrm{Cliff}(TX)).$$

Lemma 1. *The* C^**-algebra* $C_0(X, \mathrm{Cliff}(TX))$ *has a canonical* $\mathbb{Z}_2$*-graded* $C_0(X)$*-algebra structure, and up to* $\mathbb{Z}_2$*-graded isomorphism, is independent of the Riemannian metric on* X*.*

We mention some useful functoriality properties:

Corollary. *Let* $\phi : X \to Y$ *be a diffeomorphism of Riemannian manifolds. There is an induced* C^**-algebra isomorphism*

$$\phi^* : \mathcal{A}(Y) \to \mathcal{A}(X).$$

Lemma 2. *Let* U *be an open subset of the Riemannian manifold* X*. The inclusion* $j : U \hookrightarrow X$ *induces a short exact sequence of* C^**-algebras*

$$0 \longrightarrow \mathcal{A}(U) \xrightarrow{1 \hat{\otimes} j_*} \mathcal{A}(X) \longrightarrow \mathcal{A}(X \setminus U) \longrightarrow 0.$$

Thus $\mathcal{A}(U) \subset \mathcal{A}(X)$ *as a* C^**-ideal.*

Thom *-homomorphism

Consider a smooth vector bundle $p : E \to B$, where:

• B is a finite-dimensional Riemannian manifold

• E is a finite rank *Euclidean* bundle, with a *connection* ∇ compatible with the metric

The above data endows E with a Riemannian metric, because $TE = p^*E \oplus p^*TB$. Consequently we can construct $\mathcal{A}(E)$.

Theorem 3. [Trout] *Let* (E, ∇, B) *be as above. There is a graded* $*$*-homomorphism*

$$\Psi_p : \mathcal{A}(B) \to \mathcal{A}(E).$$

See Jody Trout, *'A Thom isomorphism for infinite rank Euclidean bundles,'* Homology, Homotopy and Applications, vol 5, 2003.

The C*-algebra of a Frehdolm manifold

Putting together all the above, we obtain the following commutative diagram of C^*-algebras:

$$\begin{array}{ccccc} \mathcal{A}(\nu M_n) & \xrightarrow{\simeq} & \mathcal{A}(U_n) & \hookrightarrow & \mathcal{A}(M_{n+1}) \\ \text{Thom} \uparrow & & & \nearrow & \\ \mathcal{A}(M_n) & & & & \end{array}$$

The dotted arrow gives the connecting map $\alpha_n : \mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$.

Definition. Let $(M, g, \mathcal{F})$ be a Riemannian Fredholm manifold with an augmented Fredholm filtration. The C^**-algebra of the triple* $(M, g, \mathcal{F})$ is the direct limit C^*-algebra of the directed system $\{\mathcal{A}(M_n), \alpha_n\}$:

$$\mathcal{A}(M, g, \mathcal{F}) = \varinjlim_{n \to \infty} \mathcal{A}(M_n).$$

Question. What is the dependence of $\mathcal{A}(M, g, \mathcal{F})$ on g and $\mathcal{F}$?

An example

Example. Consider $M = \mathcal{E}$, with metric given by the inner product of $\mathcal{E}$, and Fredholm filtration given by a flag $\{E_n\}_n$. An approximation argument can be used to deal with the dense algebra of compactly supported functions, and our construction recovers the C^*-algebra of Higson-Kasparov-Trout:

$$\mathcal{A}(\mathcal{E}) = \varinjlim_{a} \mathcal{A}(E_a),$$

where the direct limit is this time over the directed set of *all* finite dimensional subspaces $E_a \subset \mathcal{E}$.

We can at least show that the K-theory of $\mathcal{A}(M, g, \mathcal{F})$ does not depend on g and $\mathcal{F}$ when M is a spin manifold.

Note. For all the computations in K-theory we ignore the grading on the C^*-algebras constructed above.

Spin structures on Fredholm manifolds

$\mathcal{O}(\mathcal{E})$ = orthogonal operators on $\mathcal{E}$
$\mathrm{GL}_P(\mathcal{E}) = \mathrm{GL}(\mathcal{E}) \bigcap (I + P)$, P perturbation class
$\mathcal{O}(\mathcal{E})_P = \mathcal{O}(\mathcal{E}) \bigcap \mathrm{GL}_P(\mathcal{E})$
$\mathcal{SO}(\mathcal{E})_P$ = the connected component of I in $\mathcal{O}(\mathcal{E})_P$
$\mathcal{F}(\mathcal{E})$ = the finite rank operators

P_q = the closure of $\mathcal{F}(\mathcal{E})$ under the norm

$$\| T \|_q = \left(\text{Trace}(T^* T)^{q/2} \right)^{1/q}.$$

The universal covering group $\mathrm{Spin}(\mathcal{E})_{P_q}$ of $\mathcal{SO}(\mathcal{E})_{P_q}$ was constructed for $q = 1$ (P. de la Harpe) and $q = 2$ (R. Plymen & R. Streater).

We use $\mathcal{SO}$ and $\boldsymbol{\rho}$: **Spin** $\to \mathcal{SO}$, for $q = 1, 2$.

Definition. ([Anastasiei]) Let $\xi : E \to M$ be a vector bundle over M, with Hilbertable fibers, endowed to a reduction of the structural group to $\mathcal{SO}$. A *spin structure* on ξ is an extension associated to $\boldsymbol{\rho}$ of the principal $\mathcal{SO}$ bundle of linear frames of ξ.

A *spin structure* on M is a spin structure on TM.

There is a theory of characteristic classes for vector bundles over Hilbert manifolds, and the above definition is equivalent to the vanishing of the second Stiefel-Whitney class $w_2(\xi) \in H^2(M;\mathbb{Z}_2)$.

Proposition 1. *Given spin structures on two out of the three vector bundles ξ_1, ξ_2, and $\xi_1 \oplus \xi_2$ on M, there is a uniquely determined spin structure on the third.*

Proposition 2. *If $\xi : E \to M$ admits a spin structure and $f : N \to M$ is smooth, then $f^*\xi : f^*E \to N$ admits a spin structure.*

Corollary. *Let M be a spin Fredholm manifold with Fredholm filtration $\{M_n\}_n$. Then each of the M_n's is a spin manifold.*

Indeed, associated to $j_n : M_n \to M$ we have a split short exact sequence

$$0 \to TM_n \to TM|_{M_n} \to \mu M_n \to 0.$$

The normal bundle μM_n has a spin structure being trivial, $TM|_{M_n}$ has one because of Proposition 2, and finally Proposition 1 gives the result.

The K-theory of M

Proposition 3. *Let* M_{2n} *be an oriented Riemannian* $2n$*-manifold. If* M_{2n} *is spin, we have the Morita equivalence* $\mathcal{A}(M_{2n}) \sim_{ME} C_0(\mathbb{R} \times M_{2n})$ *and*

$$K_j(\mathcal{A}(M_{2n})) \simeq K^{j+1}(M_{2n}). \tag{1}$$

The next sequence of isomorphisms shows that, the K-theory of $\mathcal{A}(M) = \mathcal{A}(M, g, \mathcal{F})$, M spin, coincides with the K-theory of M as defined by Mukherjea:

$$\begin{aligned}
K^{\infty-j}(M) &= \lim_{n\to\infty} K^{n-j}(M_n) && \text{(def., Mukherjea)}\\
&\simeq \lim_{n\to\infty} K^{2n-j}(M_{2n}) \\
&\simeq \lim_{n\to\infty} K^{j+2}(M_{2n}) && \text{(Bott periodicity)}\\
&\simeq \lim_{n\to\infty} K_{j+1}(\mathcal{A}(M_{2n})) && (1)\\
&= K_{j+1}(\mathcal{A}(M)).
\end{aligned}$$

Poincaré duality

Theorem 4. *For M spin Fredholm manifold*

$$K_j^c(M) = K_{j+1}(\mathcal{A}(M)) = K^{\infty-j}(M),$$

where $K_j^c(M) = \varinjlim_{X \subset M}^{cpct} K_j(X)$ *is the compactly supported K-homology of M.*

Indeed:

$$\begin{aligned}
K_j^c(M) &= K_j^c(M_\infty) && \text{(htpy. invariance of } K_*^c\text{)} \\
&= \lim_{n\to\infty} K_j(M_n) && \text{(direct limit top.)} \\
&= \lim_{n\to\infty} K_j(M_{2n}) && \\
&= \lim_{n\to\infty} K^{2n-j}(M_{2n}) && \text{(Poincaré duality)} \\
&= \lim_{n\to\infty} K^{j+2}(M_{2n}) && \text{(Bott periodicity)} \\
&= K_{j+1}(\mathcal{A}(M, g, \mathcal{F})). && \square
\end{aligned}$$

Corollary. *The K-theory of $\mathcal{A}(M)$ does not depend on g and $\mathcal{F}$.*