Orthogonal Polynomials in the Complex Plane and Applications to Moment Problems

Nikos Stylianopoulos
University of Cyprus

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Dedicated to Ed Saff
a Friend, a Colleague, a Teacher
Recovery from moments

The general moment problem

Given an infinite sequence of complex moments

$$\mu_{m,k} := \int z^m \overline{z}^k d\mu(z), \quad m, k = 0, 1, 2, \ldots,$$

where $\mu$ is a (non-trivial) finite positive Borel measure with compact support on $\mathbb{C}$, recover the support $S := \text{supp}(\mu)$ of $\mu$.

We are interested on a truncated version of the above:

Given a finite section of the infinite sequence of complex moments $\{\mu_{m,n}\}_{m,k=0}^n$, compute an approximation to $S$.

The related problem of when a double sequence of complex numbers $\{a_{m,n}\}_{m,n=0}^\infty$ defines a moment sequence has been solved by Atzmon in the 1970’s.
The main tools

Let $L^2(\mu)$ denote the associated Hilbert space and consider the series of the (unique) orthonormal polynomials (ONM)

$$p_n(\mu, z) := \kappa_n(\mu) z^n + \cdots, \quad \kappa_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

generated by the inner product

$$\langle f, g \rangle_\mu = \int f(z) \overline{g(z)} d\mu(z), \quad \|f\|_{L^2(\mu)} := \langle f, f \rangle_\mu^{1/2}.$$

Our main recovery tool will be the sequence of the ONM polynomials and the Christoffel functions $\{\lambda_n(\mu, z)\}_{n=0}^{\infty}$, where

$$\lambda_n(\mu, z) := 1/\sum_{k=0}^{n} |p_n(\mu, z)|^2, \quad z \in \mathbb{C},$$

and in particular its square root $\Lambda_n(\mu, z) := \sqrt{\lambda_n(\mu, z)}$. 
Construction of the ONM polynomials

**Algorithm: Monomial Gram-Schmidt (GS)**

Apply the Gram-Schmidt process to the monomials

\[ 1, z, z^2, z^3, \ldots \]

Main ingredient: the moments

\[ \mu_{m,k} := \langle z^m, z^k \rangle = \int z^m \overline{z}^k d\mu(z), \quad m, k = 0, 1, \ldots \]

The above algorithm has been used in practice for the area and the arclength measure by pioneers of Numerical Conformal Mapping, like P. Davis and D. Gaier, in the 1960’s. It was still treated as the standard procedure for constructing ONM polynomials by leaders of the subject, like P. Henrici (*Computational Complex Analysis*, Vol. I, II and III), in the late 1980’s.
Instability Indicator

The GS method is notorious for its instability. For measuring it, when orthonormalizing a system \( S_n := \{u_0, u_1, \ldots, u_n\} \) of functions, in a Hilbert space with norm \( \| \cdot \| \), the following instability indicator has been proposed by J.M. Taylor, (Proc. R.S. Edin., 1978):

\[
I_n := \frac{\|u_n\|^2}{\min_{u \in \text{span}(S_{n-1})} \|u_n - u\|^2}, \quad n \in \mathbb{N}.
\]

Note that, when \( S_n \) is an orthonormal system, then \( I_n = 1 \). When \( S_n \) is linearly dependent then \( I_n = \infty \). Also, if \( G_n := [\langle u_m, u_k \rangle]^n_{m,k=0} \) denotes the Gram matrix associated with \( S_n \) then,

\[
\kappa(G_n) \geq I_n,
\]

where \( \kappa(G_n) := \|G_n\| \|G_n^{-1}\| \) is the spectral condition number of \( G_n \).
Instability of the Monomial GS process

In the case where $d_\mu$ is the area measure on a finite union $G$ of bounded Jordan domains, then the application of the monomial GS process (i.e., $S_n = \{1, z, z^2, \ldots, z^n\}$) leads to the following estimate in the case when the boundary $\Gamma$ of $G$ satisfies an interior cone condition at the point $z_0$, where $|z_0| = \max\{|z| : z \in \Gamma\}$:

$$c_1(\Gamma) L^{2n} \leq l_n \leq c_2(\Gamma) L^{2n},$$

with,

$$L := \frac{\max\{|z| : z \in \Gamma\}}{\text{cap}(\Gamma)} \quad (\geq 1).$$

Note that $L = 1$, iff $G \equiv \mathbb{D}_r$ and that $l_n$ is sensitive to the relative position of $G$ w.r.t. the origin. When $G$ is the $8 \times 2$ rectangle centered at the origin, then $L = 3/\sqrt{2} \approx 2.12$. In this case, $l_{25} \approx 10^{16}$ and the method breaks down in MATLAB or FORTRAN, for $n = 25$. 
The Arnoldi algorithm for ONM polynomials

The Arnoldi method was used in Numerical Linear Algebra, as a part of the Krylov subspace methods since the 1960’s. However, it took almost 30 years to find its place in Computation Complex Analysis.

Arnoldi GS

At the $n$-th step of the GS, where the ONM polynomials $p_0, p_1, \ldots, p_{n-1}$, have already been constructed, orthonormalize the polynomial $zp_{n-1}$, instead of $z^n$, as in the monomial GS.

Stability of the Arnoldi GS

Theorem (St, Constr. Approx (2013))

In the Arnoldi GS, the instability indicator $I_n$ satisfies

$$1 \leq I_n \leq \|Z\|_{L^\infty(\text{supp}(\mu))} \frac{\kappa_n^2(\mu)}{\kappa_{n-1}^2(\mu)}, \quad n \in \mathbb{N}.$$

- When $d\mu \equiv |dz|$ on a system of smooth Jordan curves (Szegö polynomials,), we have

$$c_1(\mu) \leq \frac{\kappa_n(\mu)}{\kappa_{n-1}(\mu)} \leq c_2(\mu), \quad n \in \mathbb{N}.$$

- When $d\mu \equiv w(x)dx$ on $[a, b] \subset \mathbb{R}$, then this ratio tends to a constant.
\( d\mu \equiv dA|_G \) on a Jordan domain \( G \)

\[
\Omega := \overline{\mathbb{C}} \setminus \overline{G} \quad \Gamma := \partial G
\]

\[
\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots \quad \text{cap}(\Gamma) = 1/\gamma
\]

\[
\psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \quad \text{cap}(\Gamma) = b
\]

We use the simplification \( p_n(z) := p_n(A|_G, z) \) and \( \kappa_n := \kappa_n(A|_G) \), so that

\[
p_n(z) = \kappa_n z^n + \cdots, \quad \kappa_n > 0.
\]

The \( p_n(z) \)'s are called the **Bergman** polynomials of \( G \).
Strong asymptotics when $\Gamma$ is piecewise analytic


Assume that $\Gamma$ is piecewise analytic without cusps. Then, for $n \in \mathbb{N}$,

\[
\frac{n+1}{\pi} \frac{\gamma^{2(n+1)}}{\kappa_n^2} = 1 - \alpha_n, \quad \text{where} \quad 0 \leq \alpha_n \leq c(\Gamma) \frac{1}{n},
\]

and for any $z \in \Omega$,

\[
p_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi^n(z) \Phi'(z) \{1 + A_n(z)\},
\]

where

\[
|A_n(z)| \leq \frac{c_1(\Gamma)}{\text{dist}(z, \Gamma) |\Phi'(z)|} \frac{1}{\sqrt{n}} + c_2(\Gamma) \frac{1}{n}.
\]
Strong asymptotics when $\Gamma$ is analytic

T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which $\Phi$ is conformal in $\text{ext}(L_\rho)$, then

$$\frac{n + 1}{\pi} \gamma^{2(n+1)} \frac{\gamma^2(n+1)}{\kappa_n^2} = 1 - \alpha_n,$$

where $0 \leq \alpha_n \leq c_1(\Gamma) \rho^{2n}$,

$$\rho_n(z) = \sqrt{\frac{n + 1}{\pi}} \Phi^n(z) \Phi'(z) \{1 + A_n(z)\}, \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \leq c_2(\Gamma) \sqrt{n} \rho^n, \quad z \in \overline{\Omega}.$$
Strong asymptotics when $\Gamma$ is smooth

We say that $\Gamma \in C(p, \alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if $\Gamma$ is given by $z = g(s)$, where $s$ is the arclength, with $g^{(p)} \in \text{Lip}_\alpha$. Then both $\Phi$ and $\Psi := \Phi^{-1}$ are $p$ times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\Delta \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \text{Lip}_\alpha$.


Assume that $\Gamma \in C(p + 1, \alpha)$, with $p + \alpha > 1/2$. Then, for $n \in \mathbb{N}$,

$$\frac{n + 1}{\pi} \frac{\gamma^{2(n+1)}}{\kappa_n^2} = 1 - \alpha_n,$$

where $0 \leq \alpha_n \leq c_1(\Gamma) \frac{1}{n^{2(p + \alpha)}}$,

$$p_n(z) = \sqrt{\frac{n + 1}{\pi}} \Phi^n(z) \Phi'(z) \{1 + A_n(z)\},$$

where

$$|A_n(z)| \leq c_2(\Gamma) \frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$
Ratio asymptotics for $\kappa_n$


$$\sqrt{\frac{n + 1}{n}} \frac{\kappa_{n-1}}{\kappa_n} = \text{cap}(\Gamma) + \xi_n,$$

where $|\xi_n| \leq c(\Gamma) \frac{1}{n}$, $n \in \mathbb{N}$.

The above relation provides the means for computing approximations to the capacity of $\Gamma$, by using only the leading coefficients of the Bergman polynomials. In addition, it implies for the Instability Indicator $I_n$ of the Arnoldi GS:

**Corollary**

$$c_1(\Gamma) \leq I_n \leq c_2(\Gamma), \quad n \in \mathbb{N}.$$

*Hence, the Arnoldi GS for Bergman polynomials in the Jordan case is stable.*

For any \( z \in \Omega \), and sufficiently large \( n \in \mathbb{N} \),

\[
\sqrt{\frac{n}{n+1}} \frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1 + B_n(z)\},
\]

where

\[
|B_n(z)| \leq \frac{c_1(\Gamma)}{\sqrt{\text{dist}(z, \Gamma) |\Phi'(z)|}} \frac{1}{\sqrt{n}} + c_2(\Gamma) \frac{1}{n}.
\]

The above relation, combined with the Arnoldi GS, provides an efficient method for computing approximations to \( \Phi: \Omega \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \).

Note: The kernel polynomials \( K_n(z, z_0) := \sum_{j=0}^{n} p_j(z_0) p_j(z) \) are used in the **Bergman kernel method** for computing approximations to the interior conformal map \( \varphi: G \to \mathbb{D} \).
Recovery of \( G \) from complex area moments

Truncated Moment Problem

Given the finite \( n + 1 \times n + 1 \) section \( [\mu_{m,k}]_{m,k=0}^{n} \),

\[
\mu_{m,k} := \int_{G} z^{m} \overline{z}^{k} \, dA(z),
\]

of the infinite complex area moment matrix \( [\mu_{m,k}]_{m,k=0}^{\infty} \), associated with \( G \), compute an approximation to its boundary \( \Gamma \).

This leads to applications in 2D geometric tomography, through the Radon transform. Regarding well-definiteness we have the following result.

Theorem (Davis & Pollak, Trans. AMS, 1956)

The infinite matrix \( [\mu_{m,k}]_{m,k=0}^{\infty} \) defines uniquely \( \Gamma \).
Recovery of a Jordan domain

Recovery Algorithm (St, Constr. Approx., 2013)

(I) Use the Arnoldi GS to compute \( p_0, p_1, \ldots, p_n \).

(II) Compute the coefficients of the Laurent series of the ratio

\[
\sqrt{\frac{n}{n+1}} \frac{p_n(z)}{p_{n-1}(z)} = \gamma^{(n)}(z) + \gamma_0^{(n)} + \frac{\gamma_1^{(n)}}{z} + \frac{\gamma_2^{(n)}}{z^2} + \frac{\gamma_3^{(n)}}{z^3} + \cdots \quad (1)
\]

(III) Invert (1) and truncate to obtain

\[
\psi_n(w) := b^{(n)}w + b_0^{(n)} + \frac{b_1^{(n)}}{w} + \frac{b_2^{(n)}}{w^2} + \frac{b_3^{(n)}}{w^3} + \cdots + \frac{b_n^{(n)}}{w^n}.
\]

(IV) Approximate \( \Gamma \) by \( \tilde{\Gamma} := \{ z : z = \psi_n(e^{it}), \ t \in [0, 2\pi] \} \).
Numerical Examples

Recovery of the square, with $n = 16$. 

![Graph showing recovery of the square with $n = 16$.]
Recovery of the 3-cusped hypocycloid, with $n = 10$ and $n = 20$. 
Other methods

Recovery of the square, with $n = 16$.

Other methods

Recovery of the square, with \( n = 16 \).

The exponential transform algorithm, see Gustafsson, He, Milanfar & Putinar, Inverse Problems (2000).
Bergman polynomials on an archipelago

\[ G_j, j = 1, \ldots, N, \text{ a system of disjoint and mutually exterior Jordan curves in } \mathbb{C}, \quad G_j := \text{int}(\Gamma_j), \quad \Gamma := \bigcup_{j=1}^{N} \Gamma_j, \quad G := \bigcup_{j=1}^{N} G_j. \]

\[ \langle f, g \rangle_G := \int_{G} f(z) \overline{g(z)} \, dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle_G^{1/2} \]

The Bergman polynomials \( \{p_n\}_{n=0}^{\infty} \) of \( G \) are the unique orthonormal polynomials w.r.t. the area measure on \( G \):

\[ \langle p_m, p_n \rangle_G = \int_{G} p_m(z) \overline{p_n(z)} \, dA(z) = \delta_{m,n}, \]

with

\[ p_n(z) = \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \ldots. \]
Discovery of an archipelago

Research in Pairs, Oberwolfach, January 2008

Bjorn Gustafsson, Ed Saff, Mihai Putinar
Asymptotics in an archipelago

Let \( g_{\Omega}(z, \infty) \) denote the Green function of \( \Omega := \overline{C} \setminus \overline{G} \) with pole at \( \infty \).

**Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)**

Assume that every \( \Gamma_j \) is analytic. Then, for \( n \in \mathbb{N} \):

(i) There exists a positive constant \( C \), so that

\[
|p_n(z)| \leq \frac{C}{\text{dist}(z, \Gamma)} \sqrt{n} \exp\{n g_{\Omega}(z, \infty)\}, \quad z \notin \overline{G}.
\]

(ii) For every \( \epsilon > 0 \) there exist a constant \( C_\epsilon > 0 \), such that

\[
|p_n(z)| \geq C_\epsilon \sqrt{n} \exp\{n g_{\Omega}(z, \infty)\}, \quad \text{dist}(z, \text{Co}(\overline{G})) \geq \epsilon.
\]
Leading coefficients for an archipelago

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every $\Gamma_j$ is analytic, $j = 1, 2, \ldots, N$. Then, for $n \in \mathbb{N}$,

\[ c_1(\Gamma) \leq \kappa_n \text{cap}(\Gamma)^{n+1} \sqrt{\frac{\pi}{n + 1}} \leq c_2(\Gamma) \]

In view of the strong asymptotics in the single island case, where $c_1(\Gamma) \equiv 1$ and $c_2(\Gamma) \equiv 1 + O(1/n)$, this looks a bit awkward...
Leading coefficients for a lemniscate

However, the following result shows that we can not expect to do any better, in general, than the previous double inequality.

**Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)**

Consider the lemniscate \( G := \{ z : |z^m - 1| < r^m \}, \ m \geq 2, \ 0 < r < 1, \) and note that \( \text{cap}(\Gamma) = r. \) Then, the sequence

\[
\kappa_n \text{cap}(\Gamma)^{n+1} \sqrt{\frac{\pi}{n + 1}}, \quad n \in \mathbb{N},
\]

has exactly \( m \) limit points:

\[
\frac{1}{r^{m-1}}, \frac{1}{r^{m-2}}, \ldots, \frac{1}{r}, 1.
\]
A task regarding ratio asymptotics

The important class \textbf{Reg} of measures of orthogonality was introduced by Stahl and Totik in \textit{General Orthogonal Polynomials}, CUP (1992). Recall that \( \mu \in \text{Reg} \) if

\[
\lim_{n \to \infty} \kappa_{n}^{1/n}(\mu) = \frac{1}{\text{cap}(\Gamma)}.
\]

Motivated by the crucial properties of the ratio asymptotics outlined above, we have proposed the following:

\textbf{Problem}

\textit{Characterise measures \( \mu \), for which it holds that:}

\[
\lim_{n \to \infty} \frac{\kappa_{n+1}(\mu)}{\kappa_{n}(\mu)} = \frac{1}{\text{cap}(\Gamma)}.
\]
Discovery of an archipelago

Archipelago Recovery Algorithm
(Gustafsson, Putinar, Saff & St, Adv. Math., 2009.)

(I) Use the Arnoldi GS to compute \( p_0, p_1, \ldots, p_n \), from \( [\mu_{m,k}]_{m,k=0}^n \).

(II) Form the recovery functional

\[
\Lambda_n(z) := \left[ \sum_{k=0}^{n} |p_k(z)|^2 \right]^{-1/2}.
\]

(III) Plot the zeros of \( p_j, j = 1, 2, \ldots, n \). (Fejer’s Theorem!)

(IV) Plot the level curves of the function \( \Lambda_n(x + iy) \), on a suitable rectangular frame for \((x, y)\) that surrounds the plotted zero set.
Three-disks

Zeros of the Bergman polynomials $p_{140}$, $p_{150}$ and $p_{160}$.

Recovery of three disks

Level lines of $\Lambda_n(x + iy)$ on $\{(x, y) : -1 \leq x \leq 4, -2 \leq y \leq 2\}$, for $n = 25, 50, 75, 100$. 
Pentagon and disk

Recovery of pentagon and disk

Level lines of $\Lambda_n(x + iy)$ on $\{(x, y) : -2 \leq x \leq 5, -2 \leq y \leq 2\}$, for $n = 25, 50$. 
Why the recovery algorithm works?

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every $\Gamma_j$ is analytic and let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$. Then, as $n \to \infty$:

\[
\Lambda_n(z) \asymp \begin{cases} \text{dist}(z, \Gamma), & z \in G. \\ \frac{1}{n}, & z \in \Gamma. \\ \frac{1}{\sqrt{n}} \exp\{-ng_{\Omega}(z, \infty)\}, & z \in \Omega. \end{cases}
\]

Note: The values of the Green function $g_{\Omega}(z, \infty)$ increase from 0 on $\Gamma$ to $+\infty$ at the point of infinity.
Archipelagoes with Lakes

Research in Pairs, Oberwolfach, January 2011

Ed Saff, Vilmos Totik, Herbert Stahl
Bergman polynomials on archipelago with lakes

With $K$ is a compact subset of $G$, set $G^* := G \setminus K$ and consider

$$\langle f, g \rangle_{G^*} := \int_{G^*} f(z) \overline{g(z)} \, dA(z), \quad \|f\|_{L^2(G^*)} := \langle f, f \rangle_{G^*}^{1/2}.$$

The Bergman polynomials $\{p_n^*\}_{n=0}^\infty$ of $G^*$ are the unique orthonormal polynomials w.r.t. the area measure on $G^*$:

$$\langle p_m^*, p_n^* \rangle_{G^*} = \int_{G^*} p_m^*(z) \overline{p_n^*(z)} \, dA(z) = \delta_{m,n},$$

with

$$p_n^*(z) = \kappa_n^* z^n + \cdots, \quad \kappa_n^* > 0, \quad n = 0, 1, 2, \ldots.$$
Comparable asymptotics for $\Gamma_j$ Jordan

$$\Lambda_n^*(z) := \left[ \sum_{k=0}^{\infty} |p_n^*(z)|^2 \right]^{-1/2}$$

**Theorem (Saff, Stahl, St & Totik, arXiv 2014)**

Assume that every $\Gamma_j$ is a Jordan curve. Then:

(i) $\lim_{n \to \infty} \kappa_n^*/\kappa_n = 1$.

(ii) $\lim_{n \to \infty} \|p_n^* - p_n\|_{L^2(G)} = 0$.

(iii) $\lim_{n \to \infty} p_n^*(z)/p_n(z) = 1$, locally uniformly in $\overline{C \setminus Co(G)}$.

(iv) $\lim_{n \to \infty} \Lambda_n^*(z)/\Lambda_n(z) = 1$, locally uniformly in $\overline{C \setminus \overline{G}}$. 
Comparable asymptotics for $\Gamma_j$ smooth

**Theorem (Saff, Stahl, St & Totik, arXiv 2014)**

Assume that every boundary curve $\Gamma_j$ is $C^{p+\alpha}$-smooth, with some $p \in \mathbb{N}$ and $0 < \alpha < 1$. Then,

(i) $\frac{\kappa_n^*}{\kappa_n} = 1 + O \left( \frac{1}{n^{2(p+\alpha-1)}} \right)$.

(ii) $\|p_n^* - p_n\|_{L^\infty(G)} = O \left( \frac{1}{n^{p+\alpha-2}} \right)$.

(iii) $p_n^*(z)/p_n(z) = 1 + O \left( \frac{1}{n^{p+\alpha-1}} \right)$, locally uniformly in $\mathbb{C} \setminus \overline{G}$.

(iv) $\Lambda_n^*(z)/\Lambda_n(z) = 1 + O \left( \frac{1}{n^{2(p+2\alpha)-3}} \right)$, locally uniformly in $\mathbb{C} \setminus \overline{G}$.

Furthermore, if every $\Gamma_j$ is analytic, then the above hold with $O(\varrho^n)$, where $0 < \varrho < 1$. 

Context:

- University of Cyprus
- Construction
- Jordan
- Archipelago
- Lakes
- General
- Theory
- Recovery
Conclusion

The Bergman polynomials on an archipelago are "determined" by a strip near the outer boundaries.

This leads to an reconstruction algorithm for an archipelago having lakes.
Recovery of $G^*$ from complex area moments

Truncated Moment Problem

Given the finite $n + 1 \times n + 1$ section $[\mu_{m,k}^*]_{m,k=0}^{n}$,

$$\mu_{m,k}^* := \int_{G^*} z^m \overline{z}^k dA(z),$$

of the infinite complex area moment matrix $[\mu_{m,k}^*]_{m,k=0}^{\infty}$, associated with an archipelago $G^*$ with lakes $K$, compute a good approximation to both outer boundary $\Gamma$ and the lakes $K$. 
Discovery of an archipelago having lakes

**Recovery Algorithm - Phase A: Recovery of** $G$
(Saff, Stahl, St & Totik, arXiv 2014)

(I) Use the Arnoldi GS to compute $p_0^*, p_1^*, \ldots, p_n^*$, from the given set of moments $\mu_{m,k}^*$ of $G^*$.

(II) Form the recovery functional

$$\Lambda_n^*(z) := \left[ \sum_{k=0}^{n} |p_k^*(z)|^2 \right]^{-1/2}.$$ 

(III) Plot the zeros of $p_j^*$, $j = 1, 2, \ldots, n$. (Fejer again!)

(IV) Plot the level curves of the function $\Lambda_n^*(x + iy)$, on a suitable rectangular frame for $(x, y)$ that surrounds the plotted zero set.

The outer-most level curves will provide an approximation to the boundary of $G$. Denote by $\hat{G}$ the region(s) bounded by this approximation.
Recovery Algorithm – Phase B: Recovery of $K$

(I) Use the approximation $\hat{G}$ of $G$ to calculate the moments

$$\hat{\mu}_{m,k} := \int_{\hat{G}} z^i \overline{z}^j \, dA(z), \quad m, k = 0, 1, \ldots, n.$$ 

(II) Compute the approximate moments $\mu'_m,k$ for the lakes $K$ by taking the difference $\hat{\mu}_{m,k} - \mu^*_{m,k}$.

(III) Repeat steps I-IV of Phase A with data $\mu'_m,k$ in the place of $\mu^*_{m,k}$, to produce an approximation $\hat{K}$ to $K$.

No rivers will be recovered though...
Why this recovery algorithm works?

Theorem (Saff, Stahl, St & Totik, arXiv 2014)

Under the assumption that $\Gamma$ consists of a finite union of Jordan curves:

(i) There exists a positive constant $C$ such that

$$\Lambda_n(G^*, z) \geq C \text{dist}(z, \Gamma), \quad z \in G.$$ 

(ii) For every compact subset $B$ of $\Omega$, there exists a positive constant $C(B)$ such that

$$\Lambda_n(G^*, z) \leq C(B) \exp\{-n g_{\Omega}(z, \infty)\}, \quad z \in B.$$
Recovery of pentagon $G_1$ (with a disk lake $K$) and disk $G_2$

Level lines of $\Lambda_n(x + iy)$ and $\hat{\Lambda}_n(x + iy)$ on
\[\{(x, y) : -2 \leq x \leq 5, -2 \leq y \leq 2\},\text{ for } n = 80.\]
Why the position of zeros is important?

Recovery of pentagon $G_1$ (with a disk lake $K$) and disk $G_2$

Level curves of $\Lambda_{80}(G^*, x + iy)$, for the inappropriately frame $\{(x, y) : 3 \leq x \leq 6, -2 \leq y \leq 2\}$. 
Why the square root?

Recall: \( \Lambda_n(z) := \sqrt{\lambda_n(z)} \).

Level curves of \( \lambda_{80}(G^*, x + iy) \), on \( \{(x, y): -2 \leq x \leq 5, -2 \leq y \leq 2\} \).

Recovery of pentagon \( G_1 \) (with a disk lake \( K \)) and disk \( G_2 \).
A very suggestive example...

Zeros of $p_{40}$, $p_{60}$ and $p_{80}$, for pentagon $G_1$, disk lake $K$, disk $G_2$

$$d\mu = dA|_{G_1} + dA|_{G_2} - dA|_K$$

$$d\mu = dA|_{G_1} + dA|_{G_2} + dA|_K$$
Let $\mu_1$ and $\mu_2$ be positive finite (non-trivial) Borel measures with compact supports $S_1 := \text{supp}(\mu_1)$ and $S_2 := \text{supp}(\mu_2)$ in $\mathbb{C}$ and set

$$\mu := \mu_1 + \mu_2.$$ 

Let $\{p_n(\mu, z)\}_{n=0}^{\infty}$ and $\{p_n(\mu_1, z)\}_{n=0}^{\infty}$ denote the two sequences of orthonormal polynomials defined, respectively, by the inner products

$$\langle f, g \rangle_{\mu} := \int f(z)\overline{g(z)}d\mu(z) \quad \text{and} \quad \langle f, g \rangle_{\mu_1} := \int f(z)\overline{g(z)}d\mu_1(z),$$

with

$$p_n(\mu, z) := \kappa_n(\mu)z^n + \cdots, \quad \kappa_n(\mu) > 0,$$

and

$$p_n(\mu_1, z) := \kappa_n(\mu_1)z^n + \cdots, \quad \kappa_n(\mu_1) > 0.$$
Comparable asymptotics for $\mu$ and $\mu_1$

**Theorem**

Assume that $\lim_{n \to \infty} \| p_n(\mu_1, \cdot) \|_{L^2(\mu_2)} = 0$. Then the following hold:

(i) $\lim_{n \to \infty} \kappa_n(\mu_1)/\kappa_n(\mu) = 1$.

(ii) $\lim_{n \to \infty} \| p_n(\mu_1, \cdot) - p_n(\mu, \cdot) \|_{L^2(\mu)} = 0$.

(iii) Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \text{Co}(\text{supp}(\mu))$,

$$\lim_{n \to \infty} p_n(\mu_1, z)/p_n(\mu, z) = 1.$$ (iii)

(iv) If, in addition, $\text{cap}(S_1) > 0$ and

$$\lim_{n \to \infty} \max_{z \in S_2} \sum_{j=n}^{\infty} |p_j(\mu_1, z)|^2 = 0,$$

then $\lim_{n \to \infty} \Lambda_n(\mu_1, z)/\Lambda_n(\mu, z) = 1$, uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (S_1 \cup S_2)$. 

Construction  Jordan  Archipelago  Lakes  General  Example  Theory