ON THE COMPUTATION OF PARAMETER DERIVATIVES
OF SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS

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Abstract. A method is given to compute the parameter derivatives of recessive solutions of second-order inhomogeneous linear difference equations. The case of difference equations in which all solutions have the same rate of growth is also discussed.

The method is illustrated by numerical computations of parameter derivatives of incomplete gamma functions and confluent hypergeometric functions.

1. Introduction and summary

Many special functions depend on arguments and parameters. Usually they satisfy differential equations with respect to the arguments and difference equations with respect to the parameters. Special functions are useful tools in many applications, and in these applications the exceptional cases often involve, what can be seen, parameter derivatives of the special functions. For example, one of the incomplete gamma function is defined as

$$\Gamma(a, z) = \int_z^\infty e^{-t}t^{a-1} \, dt.$$  \hspace{1cm} (1.1)

See section 11.2 in [3]. The parameter derivative is

$$\frac{\partial}{\partial a}\Gamma(a, z) = \int_z^\infty e^{-t}t^{a-1} \ln t \, dt,$$  \hspace{1cm} (1.2)

and these functions are needed in hyperasymptotics, see appendix A3 in [1]. From this example it is also obvious that known integrals with additional logarithmic factors can often be seen as parameter derivatives. Another example is

$$\frac{\partial}{\partial c}U(a, c, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}t^{a-1} (1 + t)^{c-a-1} \ln(1 + t) \, dt,$$  \hspace{1cm} (1.3)

where $U(a, c, z)$ is one of the confluent hypergeometric functions. See section 7.2 in [3].

The numerical computation of recessive solutions of second-order linear difference equations is well understood. See [2], or section 2 of this paper where we summarise the results of [2]. We want to compute the parameter derivatives of the recessive solutions of these second-order inhomogeneous linear difference equation. Our method is based on the simple observation, made in section 3, that the parameter derivative of the difference equation itself is of the same form as the original difference equation, that is, only the inhomogeneous term has changed. Hence, the methods of [2] can also be used to compute the parameter derivatives.

2000 Mathematics Subject Classification. Primary: 39A11, 33C15; secondary: 33B20, 65Q05.

Key words and phrases. confluent hypergeometric functions, difference equations, incomplete gamma functions, parameter derivatives.