Trig. Problems

Evaluating Trig. Functions

1. Calculate \( \tan(-\frac{\pi}{6}) \) and \( \sec(\frac{5\pi}{6}) \).

By definition, \( \tan \theta = \frac{\sin \theta}{\cos \theta} \). Now \( \tan(-\frac{\pi}{6}) = \frac{\sin(-\frac{\pi}{6})}{\cos(-\frac{\pi}{6})} = -\frac{\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})} = -\frac{\sin(\frac{\pi}{6})}{\sin(\frac{\pi}{6})} \). To calculate the values of the sine and cosine functions at \( \theta = \frac{\pi}{6} \) (i.e., 30°), we construct a 30°-60°-90° right triangle i.e., a triangle with sides 1, \( \sqrt{3} \), and 2. Then \( \sin(\frac{\pi}{6}) = \frac{1}{2} \) and \( \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \) which implies that \( \tan(-\frac{\pi}{6}) = -\frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}} \).

By definition, \( \sec \theta = \frac{1}{\cos \theta} \). Hence, \( \sec(\frac{5\pi}{6}) = \frac{1}{\cos(\frac{5\pi}{6})} \). To calculate \( \sec(\frac{5\pi}{6}) \), we use the trig. identity:

\[
\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi.
\]

Chose \( \theta = \pi \) and \( \phi = \frac{\pi}{6} \). Then \( \cos(\frac{5\pi}{6}) = \cos(\pi - \frac{\pi}{6}) = \cos(\pi) \cos(\frac{\pi}{6}) + \sin(\pi) \sin(\frac{\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2} \). Therefore, \( \sec(\frac{5\pi}{6}) = \frac{1}{\sqrt{3}/2} = -\frac{2}{\sqrt{3}} \).

2. Calculate \( \sin^2(-\frac{\pi}{12}) \).

We first note that \( \sin^2(-\frac{\pi}{12}) = \sin(-\frac{\pi}{12}) \cdot \sin(-\frac{\pi}{12}) = (-1)^2 \sin^2(\frac{\pi}{12}) = \sin^2(\frac{\pi}{12}) \). We use a trig. identity. Recall the \( \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \). Choosing \( \phi = \theta \), we have

\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta \implies \cos(2\theta) = 1 - \sin^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta \implies \sin^2 \theta = \frac{1}{2} [1 - \cos(2\theta)].
\]

Using this identity, with \( \theta = \frac{\pi}{12} \), \( \sin^2(\frac{\pi}{12}) = \frac{1}{2} [1 - \cos(2 \cdot \frac{\pi}{12})] = \frac{1}{2} [1 - \cos(\frac{\pi}{6})] = \frac{1}{2} [1 - \frac{\sqrt{3}}{2}] \).

3. Express 30° in radians. Express \( \frac{3\pi}{4} \) radians in terms of degrees.

The formula for conversion is: \( \Theta = \frac{180 \theta}{\pi} \implies \theta = \frac{\pi \Theta}{180} \). Hence, \( \theta = \frac{\pi \cdot 30}{180} = \frac{\pi}{6} \).

On the other hand, \( \Theta = \frac{180 \theta}{\pi} = \frac{180 (3 \pi/4 \pi)}{540} = \frac{540}{4} = 135 ^\circ \).

4. Prove that \( (\sin x + \cos x)^2 = 1 + \sin 2x \).

\[
(\sin x + \cos x)^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 + 2 \sin x \cos x = 1 + \sin 2x
\]
Solving Trig. Equations

1. Find all of the values of \( x \) in the interval \([0, 2\pi]\) for which \( 4\sin^2 x = 1 \).

We rewrite the equation \( \sin^2 x = \frac{1}{4} \) as \( \sin x = \pm \frac{1}{2} \). Suppose \( \sin x = \frac{1}{2} \). Looking at the graph of the sine function on \([0, 2\pi]\), we can find this value. There are two values of \( x \) that satisfy this condition: \( x = \frac{\pi}{6} \) and \( x = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \).

\[
\begin{array}{c|c|c|c|c|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
-1 & -0.5 & 0 & 0.5 & 1 & 0.5 & -0.5
\end{array}
\]

Suppose \( \sin x = -\frac{1}{2} \). Again from the graph \( x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \) and \( x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \).

2. Solve the inequality \(-1 < \tan x < 1\) for values of \( x \in [0, \pi] \).

The best way to solve this problem is to plot \( \tan x \) on \([0, \pi]\).
We look at the values where \( \tan x = 1 \) \( i.e., \ x = \frac{\pi}{4} \) and \( \tan x = -1 \) \( i.e., \ x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} \). We see from the graph that \( |\tan x| < 1 \) when \( x \in [0, \frac{\pi}{4}) \) or \( x \in (\frac{3\pi}{4}, \pi] \).

3. For \( 0 \leq x \leq 2\pi \), solve the equation \( \sin x = \tan x \).

Rewriting \( \tan x \) in terms of \( \sin x \) and \( \cos x \), we have \( \sin x = \frac{\sin x}{\cos x} \implies \sin x = 0 \) or \( \cos x = 1 \). Hence, \( x = 0, \pi, 2\pi \) or \( x = 0, 2\pi \). Therefore, there are three solutions: \( x = 0, \pi, 2\pi \). Here is a plot of the functions \( y = \sin x \) and \( y = \tan x \).

4. Given that \( \sin \theta = \frac{3}{5}, 0 < \theta < \frac{\pi}{2} \), calculate \( \cos \theta \) and \( \tan \theta \).

Here, \( \sin \theta = \frac{3}{5} \). Consider the 3-4-5 triangle. We then know that the right triangle has a hypothenuse of length 5, one side of length 3, and one side of length 4.

From the triangle, we see that \( \cos \theta = \frac{4}{5} \) and \( \tan \theta = \frac{3}{4} \).

5. Find all values of \( x \) in the interval \( [0, 2\pi] \) that satisfy \( 2\cos x - 1 = 0 \).

We rewrite the equation \( 2\cos x - 1 = 0 \) as \( \cos x = \frac{1}{2} \). Therefore, we want to find the values of \( x \) in \( [0, 2\pi] \) for which \( \cos x = \frac{1}{2} \). We plot the function \( \cos x \).
We see that the graph of \( \cos x \) intersects the line \( y = \frac{1}{2} \) at two places. One place is \( x = \frac{\pi}{3} \) since \( \cos \left( \frac{\pi}{3} \right) = \frac{1}{2} \). The other intersection occurs at \( x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \). Therefore, the two solutions of \( 2\cos x - 1 = 0 \) that lie in \([0, 2\pi]\) are \( x = \frac{\pi}{3}, \frac{5\pi}{3} \).

**Deriving Trig. Identities**

1. Given \( \sin^2 x + \cos^2 x = 1 \), prove that \( 1 + \cot^2 x = \csc^2 x \).

\[
\sin^2 x + \cos^2 x = 1 \implies \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \implies 1 + \cot^2 x = \frac{1}{\sin^2 x} \implies 1 + \cot^2 x = \csc^2 x.
\]

2. Given \( \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \) and \( \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \), prove that

\[
\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.
\]

\[
\tan(x \pm y) = \frac{\sin(x \pm y)}{\cos(x \pm y)} = \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y \mp \sin x \sin y} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \tan x \pm \tan y = \frac{1 \mp \tan x \tan y}{1 \mp \tan x \tan y}.
\]

3. Given \( \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \), prove that \( \sin 2x = 2 \sin x \cos x \).

\[
\sin(x + x) = \sin x \cos x + \cos x \sin x \implies 2 \sin x \cos x.
\]