Trig. Problems

Evaluating Trig. Functions

1. Calculate $\tan(-\frac{\pi}{6})$ and $\sec(\frac{5\pi}{6})$.

By definition, $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Now $\tan(-\frac{\pi}{6}) = \frac{\sin(-\frac{\pi}{6})}{\cos(-\frac{\pi}{6})} = \frac{-\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})} = -\frac{\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})}$. To calculate the values of the sine and cosine functions at $\theta = \frac{\pi}{6}$ (*i.e.*, $30°$), we construct a 30°-60°-90° right triangle *i.e.*, a triangle with sides 1, $\sqrt{3}$, and 2. Then $\sin(\frac{\pi}{6}) = \frac{1}{2}$ and $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ which implies that $\tan(-\frac{\pi}{6}) = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$.

By definition, $\sec\theta = \frac{1}{\cos\theta}$. Hence, $\sec(\frac{5\pi}{6}) = \frac{1}{\cos(\frac{5\pi}{6})}$. To calculate $\cos(\frac{5\pi}{6})$, we use the trig. identity:

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Chose $\theta = \pi$ and $\phi = \frac{\pi}{6}$. Then $\cos(\frac{5\pi}{6}) = \cos(\pi - \frac{\pi}{6}) = \cos(\pi)\cos(\frac{\pi}{6}) + \sin(\pi)\sin(\frac{\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$. Therefore, $\sec(\frac{5\pi}{6}) = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$.

2. Calculate $\sin^2(-\frac{\pi}{12})$.

We first note that $\sin^2(-\frac{\pi}{12}) = \sin(-\frac{\pi}{12})\cdot\sin(-\frac{\pi}{12}) = (-1)^2\sin^2(\frac{\pi}{12}) = \sin^2(\frac{\pi}{12})$. We use a trig. identity. Recall the $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$. Choosing $\theta = \phi$, we have

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \Longrightarrow \cos(2\theta) = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta \Longrightarrow \sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)].$$

Using this identity, with $\theta = \frac{\pi}{12}$, $\sin^2(\frac{\pi}{12}) = \frac{1}{2}[1 - \cos(2\cdot\frac{\pi}{12})] = \frac{1}{2}[1 - \cos(\frac{\pi}{6})] = \frac{1}{2}\left[1 - \frac{\sqrt{3}}{2}\right]$.

3. Express $30°$ in radians. Express $\frac{3\pi}{4}$ radians in terms of degrees.

The formula for conversion is: $\Theta = \frac{180\,\theta}{\pi} \Longrightarrow \theta = \frac{\pi\Theta}{180}$. Hence, $\theta = \frac{\pi\cdot 30}{180} = \frac{\pi}{6}$.

On the other hand, $\Theta = \frac{180\,\theta}{\pi} = \frac{180\,(3\pi/4)}{\pi} = \frac{540}{4} = 135°$.

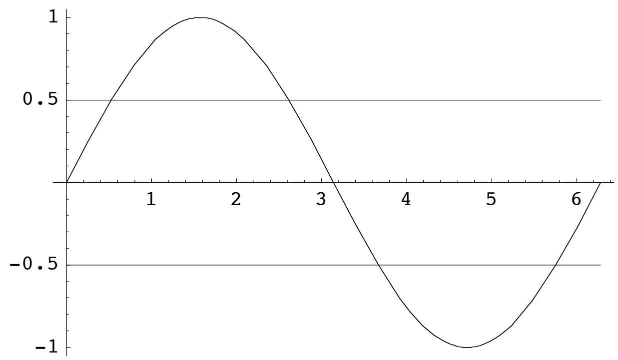
4. Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x$.

$$(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + 2\sin x\cos x = 1 + \sin 2x$$

Solving Trig. Equations

1. Find all of the values of x in the interval $[0, 2\pi]$ for which $4\sin^2 x = 1$.

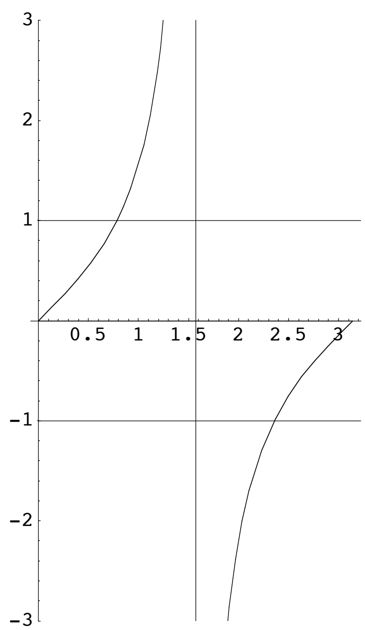
We rewrite the equation $\sin^2 x = \frac{1}{4}$ as $\sin x = \pm\frac{1}{2}$. Suppose $\sin x = \frac{1}{2}$. Looking at the graph of the sine function on $[0, 2\pi]$, we can find this value. There are two values of x that satisfy this condition: $x = \frac{\pi}{6}$ and $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.



Suppose $\sin x = -\frac{1}{2}$. Again from the graph $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ and $x = 2\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

2. Solve the inequality $-1 < \tan x < 1$ for values of $x \in [0, \pi]$.

The best way to solve this problem is to plot $\tan x$ on $[0, \pi]$.



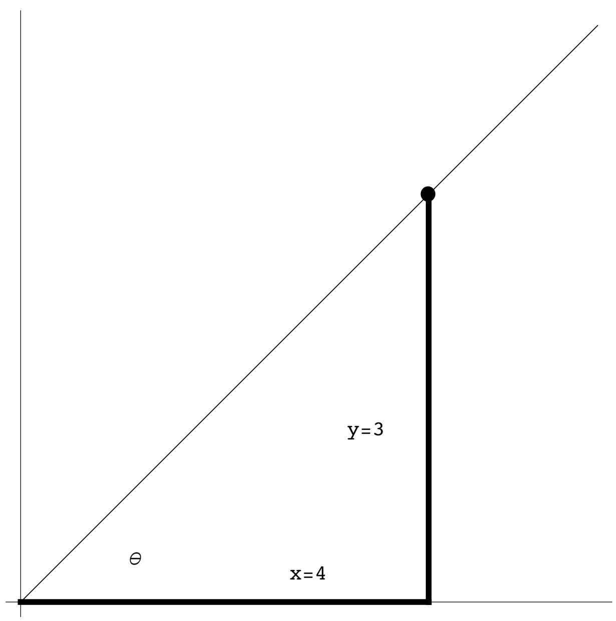
We look at the values where $\tan x = 1$ *i.e.*, $x = \frac{\pi}{4}$ and $\tan x = -1$ *i.e.*, $x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$. We see from the graph that $|\tan x| < 1$ when $x \in [0, \frac{\pi}{4})$ or $x \in (\frac{3\pi}{4}, \pi]$.

3. For $0 \leq x \leq 2\pi$, solve the equation $\sin x = \tan x$.

Rewriting $\tan x$ in terms of $\sin x$ and $\cos x$, we have $\sin x = \frac{\sin x}{\cos x} \Longrightarrow \sin x = 0$ or $\cos x = 1$. Hence, $x = 0, \pi, 2\pi$ or $x = 0, 2\pi$. Therefore, there are three solutions: $x = 0, \pi, 2\pi$. Here is a plot of the functions $y = \sin x$ and $y = \tan x$

4. Given that $\sin\theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$, calculate $\cos\theta$ and $\tan\theta$.

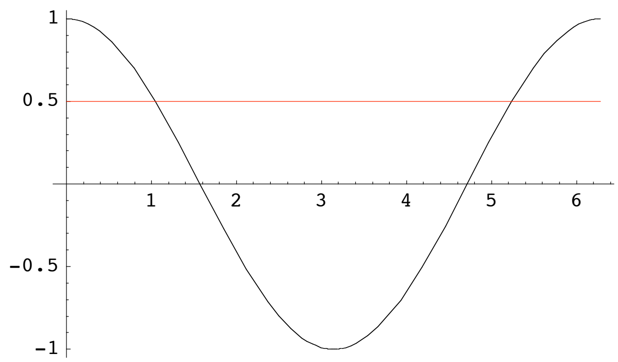
Here, $\sin\theta = \frac{3}{5}$. Consider the 3-4-5 triangle. We then know that the right triangle has a hypothenuse of length 5, one side of length 3, and one side of length 4



From the triangle, we see that $\cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$.

5. Find all values of x in the interval $[0, 2\pi]$ that satisfy $2\cos x - 1 = 0$.

We rewrite the equation $2\cos x - 1 = 0$ as $\cos x = \frac{1}{2}$. Therefore, we want to find the values of x in $[0, 2\pi]$ for which $\cos x = \frac{1}{2}$. We plot the function $\cos x$.



We see that the graph of $\cos x$ intersects the line $y = \frac{1}{2}$ at two places. One place is $x = \frac{\pi}{3}$ since $\cos(\frac{\pi}{3}) = \frac{1}{2}$. The other intersection occurs at $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$. Therefore, the two solutions of $2\cos x - 1 = 0$ that lie in $[0, 2\pi]$ are $x = \frac{\pi}{3}, \frac{5\pi}{3}$.

Deriving Trig. Identities

1. Given $\sin^2 x + \cos^2 x = 1$, prove that $1 + \cot^2 x = \csc^2 x$.

$$\sin^2 x + \cos^2 x = 1 \Longrightarrow \frac{\sin^2 + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Longrightarrow 1 + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Longrightarrow 1 + \cot^2 x = \csc^2 x.$$

2. Given $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ and $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$, prove that

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

$$\tan(x \pm y) = \frac{\sin(x \pm y)}{\cos(x \pm y)} = \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y \mp \sin x \sin y} = \frac{\frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y \mp \sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 \mp \frac{\sin x}{\cos y}\frac{\sin y}{\cos y}} = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

3. Given $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$, prove that $\sin 2x = 2 \sin x \cos x$.

$$\sin(x + x) = \sin x \cos x + \cos x \sin x \Longrightarrow \sin 2x = 2 \sin x \cos x.$$