# Typical Problems for Algebra

## Simplifying Algebraic Expressions

1. Simplify $8^{-4/3}$.

$$8^{-4/3} = \frac{1}{8^{4/3}} = \frac{1}{(8^{1/3})^4} = \frac{1}{2^4} = \frac{1}{16}$$

2. Simplify $\left(\frac{9^{-3}.9^5}{9^{-2}}\right)^{-1/2}$.

$$\left(\frac{9^{-3}.9^5}{9^{-2}}\right)^{-1/2} = \frac{1}{\left(\frac{9^{-3}.9^5}{9^{-2}}\right)^{1/2}} = \frac{1}{(9^{-3+5-(-2)})^{1/2}} = \frac{1}{(9^4)^{1/2}} = \frac{1}{9^2} = \frac{1}{81}$$

3. Simplify $\sqrt{81\, x^6\, y^{-4}}$.

$$\sqrt{81\, x^6\, y^{-4}} = \sqrt{\frac{81\, x^6}{y^4}} = \frac{\sqrt{81}\ \sqrt{x^6}}{\sqrt{y^4}} = \frac{9\, x^3}{y^2}$$

4. Simplify $\sqrt{x^{-1}} \cdot \sqrt{9\, x^{-3}}$.

$$\sqrt{x^{-1}} \cdot \sqrt{9\, x^{-3}} = \sqrt{x^{-1} \cdot 9\, x^{-3}} = \sqrt{9\, x^{-4}} = 3\, \sqrt{x^{-4}} = 3\, \sqrt{\frac{1}{x^4}} = 3\, \frac{1}{\sqrt{x^4}} = \frac{3}{x^2}$$

5. Simplify $\sqrt[3]{x^{3\,a+b}}$.

$$\sqrt[3]{x^{3\,a+b}} = (x^{3\,a+b})^{1/3} = (x^{3\,a})^{1/3}\,(x^b)^{1/3} = x^a\, x^{b/3} = x^{a+b/3}$$

6. Simplify $(2\, a^{1/6}\, b^{5/6})^{-6}$.

$$(2\, a^{1/6}\, b^{5/6})^{-6} = \frac{1}{(2\, a^{1/6}\, b^{5/6})^6} = \frac{1}{2^6\, a^{6/6}\, b^{30/6}} = \frac{1}{64\, a\, b^5}$$

7. Simplify $\sqrt[3]{\frac{20\, x}{y^2}\ \sqrt[3]{\frac{50\, x^2}{y^4}}}$.

$$\sqrt[3]{\frac{20\, x}{y^2}\ \sqrt[3]{\frac{50\, x^2}{y^4}}} = \left(\frac{20\, x}{y^2}\right)^{1/3} \left(\frac{50\, x^2}{y^4}\right)^{1/3} = \left(\frac{20\, x}{y^2} \cdot \frac{50\, x^2}{y^4}\right)^{1/3} = \left(\frac{1000\, x^3}{y^6}\right)^{1/3} = \frac{1000^{1/3}\,(x^3)^{1/3}}{(y^6)^{1/3}} = \frac{10\, x}{y^2}$$

8. Simplify $\frac{2a^2-3ab-9b^2}{2ab^2+3b^3}$.

$$\frac{2a^2-3ab-9b^2}{2ab^2+3b^3} = \frac{(2a+3b)(a-3b)}{b^2(2a+3b)} = \frac{a-3b}{b^2}$$

9. Simplify $\frac{3(x^3-1)(x^2+x-1)(2x+1)-(x^2+x-1)(3x^2)}{(x^3-1)^2}$.

$$\frac{3(x^3-1)(x^2+x-1)(2x+1)-(x^2+x-1)(3x^2)}{(x^3-1)^2} = \frac{3(x^2+x-1)[(x^3-1)(2x+1)-x^2]}{(x^3-1)^2} = \frac{3(x^2+x-1)(2x^4+x^3-x^2-2x-1)}{(x^3-1)^2}$$

10. Simplify $\frac{\frac{1}{x}+\frac{1}{y}}{1-\frac{1}{xy}}$.

$$\frac{\frac{1}{x}+\frac{1}{y}}{1-\frac{1}{xy}} = \frac{\frac{x+y}{xy}}{\frac{xy-1}{xy}} = \frac{x+y}{xy-1}$$

11. Simplify $\frac{x^{-3}-y^{-3}}{x^{-1}-y^{-1}}$.

$$\frac{x^{-3}-y^{-3}}{x^{-1}-y^{-1}} = \frac{\frac{1}{x^3}-\frac{1}{y^3}}{\frac{1}{x}-\frac{1}{y}} = \frac{x^3y^3\left(\frac{1}{x^3}-\frac{1}{y^3}\right)}{x^3y^3\left(\frac{1}{x}-\frac{1}{y}\right)} = \frac{y^3-x^3}{x^2y^3-x^3y^2} = \frac{y^3-x^3}{x^2y^2(y-x)} = \frac{(y-x)(y^2+xy+x^2)}{x^2y^2(y-x)} = \frac{y^2+xy+x^2}{x^2y^2}$$

## Factoring, Roots, and Completing the Square

1. Simplify $(3x^2+5xy+2y)+(4-3xy-2x^2)$.

$$(3x^2+5xy+2y)+(4-3xy-2x^2) = (3-2)x^2+(5-3)xy+2y+4 = x^2+2xy+2y+4$$

2. Simplify $3x^2-(x^2+1-x(x-(2x-1)))+2$.

$$3x^2-(x^2+1-x(x-(2x-1)))+2 = 3x^2-x^2-1+x(x-(2x-1))+2 = 2x^2-1+x^2-(2x^2-x)+2$$

$$= x^2+x+1$$

3. Factor $3x^3-x^2+3x-1$.

$$3x^3-x^2+3x-1 = 3(x^3+x)-(x^2+1) = 3x(x^2+1)-(x^2+1) = (x^2+1)(3x-1)$$

4. Factor $8a^2-2ab-6b^2$.

$$8a^2-2ab-6b^2 = 2(4a^2-ab-3b^2) = 2(4a+3b)(a-b)$$

5. Find the greatest common factor of $4x^2y^2z - 2x^5y^2 + 6x^3y^2z^2$.

$$4x^2y^2z - 2x^5y^2 + 6x^3y^2z^2 = 2x^2y^2(2z - x^3 + 3xz^2)$$

6. Find the real roots of $3x^2 - x - 4 = 0$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-4)}}{2(3)} = \frac{1 \pm \sqrt{1+48}}{6} = \frac{1 \pm 7}{6} = -1, \frac{4}{3}$$

7. Complete the square of $5x^2 - 3x + 1$.

$$5x^2 - 3x + 1 = 5\left(x^2 - \frac{3}{5}x + \frac{1}{5}\right) = 5\left(x^2 - \frac{3}{5}x + \frac{1}{5} - \left(\frac{3}{10}\right)^2 + \left(\frac{3}{10}\right)^2\right) = 5\left(\left(x - \frac{3}{10}\right)^2 + \frac{1}{5} - \left(\frac{3}{10}\right)^2\right)$$

$$= 5\left(\left(x - \frac{3}{10}\right)^2 + \frac{11}{100}\right)$$

8. Find all of the roots of the polynomial: $x^3 + 8x^2 - 11x - 18$.

By inspection we note that $x = -1$ is a root. We can then divide this root from the polynomial:

$$\frac{x^3 + 8x^2 - 11x - 18}{x+1} = x^2 + 7x - 18 = (x-2)(x+9).$$

Hence, the roots are: $x = -1, 2, -9$.

## Straight Lines ($\alpha x + \beta y = \gamma$ or $y = mx + b$)

1. Find the equation of the line that passes through the points (0, 1) and (-2,3).

The slope of the line is $m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{3-1}{-2-0} = -1$. The equation is then $-1 = m = \frac{y - y_0}{x - x_0} = \frac{y-1}{x-0} \Longrightarrow -x = y - 1$ or

$y = -x + 1$.

2. Find the equation of the line that is perpendicular to the line with a slope of 2 and passes through the point $(0, -1)$.

The slope of the desired line is $m = -\frac{1}{-2} = -\frac{1}{2}$. It passes through the point $(2, 1)$ and hence,

$$-\frac{1}{2} = \frac{y+1}{x-0} \Longrightarrow y + 1 = -\frac{1}{2}x \Longrightarrow y = -1 - \frac{1}{2}x.$$

3. Find the equation of the line with slope -3 and intercepts the $y$-axis at $(0, 2)$.

We use $y = m\,x + b$ with $m = -3$ and $b = 2$. Hence, $y = -3\,x + 2$.

4. Find the equation of the line that is parallel to the line $y = -5\,x + 9$ and passes through the point $(1, 1)$.

The slope of the line is $-5$. Hence,

$$-5 = m = \frac{y - y_0}{x - x_0} = \frac{y-1}{x-1} \Longrightarrow -5\,x + 5 = y - 1 \Longrightarrow y = -5\,x + 6.$$

## Inequalities

1. Express $[-\frac{6}{5}, -\frac{1}{2})$ in terms of inequalities.

$$[-\tfrac{6}{5}, -\tfrac{1}{2}) = \{x \in \mathbb{R} : -\tfrac{6}{5} \le x < -\tfrac{1}{2}\}$$

2. Solve the inequality $(2\,x - 4)(x + 2) \ge 0$.

We set up our sign table for the two factors $2\,x - 4 = 2\,(x - 2)$ and $x + 2$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 x - 4 | − | − | − | − | − | − | 0 | + | + |
| x + 2 | − | − | 0 | + | + | + | + | + | + |
| x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |

.

The product of the two factors must be nonnegative (*i.e.* $\ge 0$). This will occure when both are nonpositive ( $\le 0$) or nonnegative ( $\ge 0$). From our table, this is when $x \le -2$ or $x \ge 2$. Therefore, the solution is $(-\infty, -2] \bigcup [2, \infty)$.

3. Solve the inequality $\frac{x-1}{x+2} \le 4$.

We first rewrite the inequality as $\frac{x-1}{x+2} - 4 \le 0 \Longrightarrow \frac{x-1-4\,(x+2)}{x+2} \le 0 \Longrightarrow \frac{-3\,x-9}{x+2} \le 0 \Longrightarrow \frac{x+3}{x+2} \ge 0$. Now we setup our sign table.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| x + 3 | − | 0 | + | + | + | + | + | + | + |
| x + 2 | − | − | 0 | + | + | + | + | + | + |
| x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |

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We want the ratio $\frac{x+3}{x+2}$ to be nonnegative and hence the numerator and denominator have the same sign. Note that we don't allow $x = -2$ where the ratio is undefined. The top and bottom have the same sign when $x \le -3$ and $x > -2$. Therefore, the solution is $(-\infty, -3] \bigcup (-2, \infty)$.

# Domains of Functions

1. Find the domain of the function $f(x) = \frac{x^2 + 9x - 1}{x^2 - 9}$.

$D_f = \{x \in \mathbb{R} : x^2 - 9 \neq 0\} = \{x \in \mathbb{R} : x^2 \neq 9\} = \{x \in \mathbb{R} : x \neq \pm 3\} = \mathbb{R} - \{\pm 3\}$.

2. Find the domain of the function $f(x) = \sqrt[4]{x^2 - 1}$.

$D_f = \{x \in \mathbb{R} : x^2 - 1 \geq 0\} = \{x \in \mathbb{R} : x^2 \geq 1\} = \{x \in \mathbb{R} : |x| \geq 1\} = (-\infty, -1] \bigcup [1, \infty)$.

3. Find the domain of the function $f(x) = \frac{3x + |x|}{x}$.

$D_f = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \bigcup (0, \infty)$.