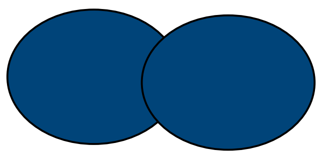
Algebra and Trigonometry

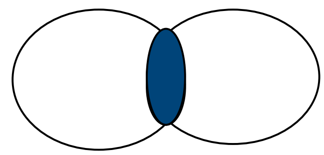
Interval notation:

Closed interval	$[a,b] = \{x \mid a \le x \le b\}$
Open interval	$(a,b) = \{x \mid a < x < b\}$
Half-open interval	$[a,b) = \{x \mid a \le x < b\}$
Half-open interval	$(a,b] = \{x \mid a < x \le b\}$

Set notation: If A is a set of objects, we write $x \in A$ if x is an object belonging to the set A. If A and B are sets, the <u>union</u> of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
The <u>intersection</u> of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.



$A \cup B$



$A \cap B$

Absolute value: The <u>absolute value</u> of a real number x is defined by $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$.

Rules for absolute values:

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
4. $|a+b| \le |a| + |b|$

Exponents: For a positive integer n, $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$ and $a^{1/n} = \sqrt[n]{a}$.

If $a \ne 0$, $a^{-n} = \dfrac{1}{a^n}$ and $a^0 = 1$.

Rules for exponents:

1. $a^n \cdot a^m = a^{n+m}$
2. $\dfrac{a^n}{a^m} = a^{n-m}$
3. $a^{n/m} = \sqrt[m]{a^n}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Rules for radicals:

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Logarithms: If $a > 0$ and $a^y = x$, we say that $\log_a x = y$.

Rules for logarithms:

1. $\log_a(xz) = \log_a x + \log_a z$
2. $\log_a\left(\dfrac{x}{z}\right) = \log_a x - \log_a z$
3. $\log_a x^z = z \log_a x$

Conjugates: The conjugate of an expression $a + b\sqrt{c}$ containing a radical is $a - b\sqrt{c}$.

Multiplying an expression by its conjugate eliminates the radical:

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2 c.$$

Polynomials: A polynomial of degree n in x has the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Factors of a polynomial are two or more polynomials of lower degree which multiply together to yield the given polynomial.

Rules for factoring:

1. $a^2 - b^2 = (a - b)(a + b)$
2. $a^2 + 2ab + b^2 = (a + b)^2$
3. $a^2 - 2ab + b^2 = (a - b)^2$
4. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
5. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Roots: A number r is a root of a polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ provided that $a_n r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0 = 0$. An n^{th} degree polynomial can have at most n roots. Also, if a number r is a root of a polynomial, then $x - r$ is a factor of the polynomial. The roots of a quadratic polynomial $ax^2 + bx + c$ can always be found from the

quadratic formula: $\boxed{r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

Rules for fractions:

1. $\dfrac{A}{B} \cdot \dfrac{C}{D} = \dfrac{AC}{BD}$
2. $\dfrac{A/B}{C/D} = \dfrac{A}{B} \div \dfrac{C}{D} = \dfrac{AD}{BC}$
3. $\dfrac{P}{Q} + \dfrac{R}{Q} = \dfrac{P + R}{Q}$
4. $\dfrac{P}{Q} - \dfrac{R}{Q} = \dfrac{P - R}{Q}$
5. $\dfrac{A}{B} + \dfrac{C}{D} = \dfrac{AD + BC}{BD}$
6. $\dfrac{A}{B} - \dfrac{C}{D} = \dfrac{AD - BC}{BD}$

Rules for inequalities: Let a, b, and c be any real numbers.

1. If $a < b$ and $b < c$, then $a < c$.
2. If $a < b$, then $a + c < b + c$.
3. If $a < b$ and c is positive, then $ac < bc$.
4. If $a < b$ and c is negative, then $ac > bc$.

Inequalities and absolute values: Suppose a is a real number and $f(x)$ is an expression involving x.

1. If $|f(x)| < a$, then $-a < f(x) < a$.
2. If $|f(x)| > a$, then $f(x) > a$ or $f(x) < -a$.

Functions: A function f is a rule that assigns to each element x in a set A one and only one element y in a set B, with notation $y = f(x)$. The set A is the domain of f, and the range of f is the set $\{y \in B \mid y = f(x) \text{ for some } x \in A\}$.

Lines: A vertical line has an equation of the form $x = k$ for some constant k. A line that is not vertical must contain points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ with $x_1 \neq x_2$; the slope of this line is the number $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. A horizontal line has slope 0. Two lines with slopes m_1 and m_2 are parallel if they have the same slope ($m_2 = m_1$); the lines are perpendicular if their slopes are negative reciprocals ($m_2 = -\dfrac{1}{m_1}$).

Equations for lines: 1. If a line has slope m and $P_1(x_1, y_1)$ is a point on the line, the point-slope form of the equation of the line is $y - y_1 = m(x - x_1)$.

2. If a line has slope m and y-intercept b (*i.e.*, it crosses the y-axis at $y = b$), the slope-intercept form of the equation of the line is $y = mx + b$.
3. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on the line, the two-point form of the equation of the line is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Circles, parabolas, ellipses and hyperbolas: A circle is the set of all points $P(x, y)$ that are a fixed distance r from a fixed point $C(h,k)$. The equation of a circle with radius r and center $C(h,k)$ is $(x - h)^2 + (y - k)^2 = r^2$.

A parabola is the set of all points $P(x, y)$ that are equidistant from a fixed point F, called the focus, and a fixed line ℓ, called the directrix. The point halfway between the focus and the directrix is the vertex of the parabola.

An equation for a parabola with vertex (h,k) and a horizontal directrix is

$(x - h)^2 = 4p(y - k)$. (Parabola opens up or down)

A parabola with vertex (h,k) and a vertical directrix has the equation

$(y - k)^2 = 4p(x - h)$. (Parabola opens right or left)

The sign on the constant p determines which way the parabola opens.

An ellipse is the set of all points $P(x, y)$ with the property that the sum of the distances from $P(x, y)$ to the fixed points F_1 and F_2 is a fixed constant value. The points F_1 and F_2 are the foci of the ellipse, and the point midway between the foci is the center. An equation for an ellipse with center (h, k) is

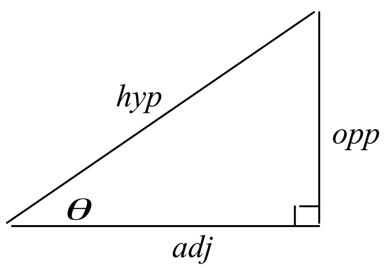
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

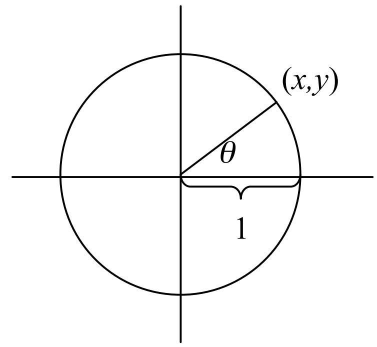
A hyperbola is the set of all points $P(x, y)$ with the property that the difference of the distances from $P(x, y)$ to the foci F_1 and F_2 is a fixed constant value. A hyperbola with center (h,k) has equation

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \qquad \text{or} \qquad \frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1$$

Trigonometry: Trigonometric functions can be defined from an acute angle of a right triangle or a central angle of the unit circle. Angles may be expressed in terms of <u>degrees</u> or <u>radians</u>. The conversion formula for changing from one to the other is

$$\pi \text{ radians} = 180° .$$





$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp} \qquad \sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj} \qquad \cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{adj}{opp} \qquad \tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Trigonometric functions of important angles:

θ	*radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	*undefined*
180°	π	0	-1	0
270°	$3\pi/2$	-1	0	*undefined*

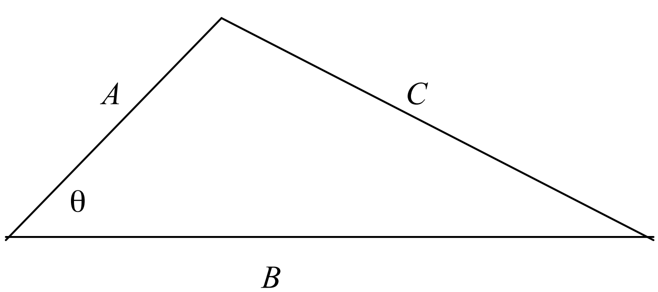
Trigonometric identities: The following equations are relationships that hold for all trigonometric functions:

$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

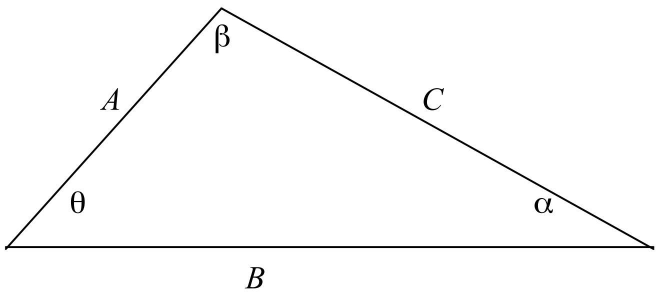
Two useful properties related to trigonometric functions are the **Law of Cosines** and the **Law of Sines**.

The familiar Pythagorean Theorem says that if *A*, *B*, and *C* are the sides of a right triangle and *C* is the side opposite the right angle, then $C^2 = A^2 + B^2$. The Law of Cosines is a generalization of the Pythagorean Theorem. If, as shown in the triangle below, θ is the angle opposite side *C*, then the **Law of Cosines** states that



$$\boxed{C^2 = A^2 + B^2 - 2AB\cos\theta}$$

The Law of Sines gives a relationship between the three sides and the three angles of a general triangle. If α is the angle opposite side *A*, β is the angle opposite side *B*, and θ is the angle opposite side *C*, then the **Law of Sines** states that



$$\boxed{\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\theta}}$$