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Free rat'l G-equiv. spectra. jt. w/ John Greenlees

Thrm: (Schwede-Shipley '03; Shipley '07)

Any (nice) rational stable homotopy theory with a (set of) generators is Quillen equivalent to d.g. modules over a $\mathbb{Q}$-DGA (DG-category)

(problem: not very explicit in general!)

Want: small, explicit algebraic models.

§2: Examples.

Free G-Sp — G-spectra generated by the free cell, G_+

$\simeq_{\mathbb{Q}}$ G_+-mod spectra.

Free $\mathbb{Q}$-GSp $\simeq_{\mathbb{Q}}$ $H\mathbb{Q}[G]$-mod spectra, where $H\mathbb{Q}[G] = H\mathbb{Q} \wedge G_+$

$\simeq_{\mathbb{Q}}$ d.g ($\Theta H\mathbb{Q}[G]$)-mod

→ how to identify this dga.

Ex1: $G = F$ finite. $\pi_* H\mathbb{Q}[F] = \mathbb{Q}[F]$ in degree 0

so dga has homology only in deg. 0 so

free-$\mathbb{Q}$-F-Sp $\simeq_{\mathbb{Q}}$ d.g $\mathbb{Q}[F]$-mod $\simeq_{\mathbb{Q}}$ chain complexes w/ F-action

Ex2: $G = T^r$ torus of rank r $\pi_* H\mathbb{Q}[T^r] = H\mathbb{Q}_*(T^r) = \Lambda_{\mathbb{Q}}[x_1, \dots, x_r]$ $x_i \in \deg 1$.

Q: does the homology determine type of dga? (ie is it formal?)

A: It depends. $\Lambda_{\mathbb{Q}}[x_1, \dots, x_r]$ not formal in assoc. dgas, but since T^r abelian, $H\mathbb{Q}[T^r]$ commutative, + this ring is intrinsically formal in commutative dgas.

- free-$\mathbb{Q}$-T^r-Sp $\underset{\text{formality}}{\simeq_{\mathbb{Q}}}$ d.g $H\mathbb{Q}_*(T^r)$-mod

Koszul duality: $\Lambda_{\mathbb{Q}}[x_i] \rightsquigarrow P_{\mathbb{Q}}[c_i]$.

- d.g. $H\mathbb{Q}_*(T^r)$-mod $\overset{\text{torsion}}{\simeq_{\mathbb{Q}}}$ d.g. $H\mathbb{Q}^*(BT^r)$-mod ← injective dim. r

$\mathbb{Q}$ generator ⟼ $\mathbb{Q}$ generator (of torsion modules)

Ex 3: G connected compact Lie. $\pi_* H\mathbb{Q}[G]$ is not intrinsically formal, since not necessarily ~~gr~~ commutative

Koszul duality in spectra: $H\mathbb{Q}[G] \rightsquigarrow F(BG_+, \mathbb{Q})$ ← here htpy comm. (in even degrees)?

then formality gives:

<u>Thrm</u> (Greenlees–Shipley) For any connect compact Lie group

$$\text{free-}\mathbb{Q}\text{-}G\text{-Sp} \simeq_{\mathbb{Q}} \text{torsion d.g. } H\mathbb{Q}^*(BG)\text{-modules}$$

→ Koszul duality doesn't play well with monoidal structures

injective dimension here is rank of the group

NB. $H^* = H\mathbb{Q}^*$ a lot.

§3. Nonconnected case

Let $N = G_1$ the identity component of G and $W = G/N$ the component group

$$\underset{}{1} \longrightarrow \underset{\text{connected}}{N} \longrightarrow G \longrightarrow \underset{\text{finite}}{W} \longrightarrow 1$$

Naive guess: $\text{free-}\mathbb{Q}\text{-}G\text{-Sp} \overset{?}{\simeq}_{\mathbb{Q}} \text{torsion } H^*(BN)\text{-modules w/ } W\text{-action}$

Define: $\tilde{B}N = EG/N \circlearrowleft G/N = W$ action

Naive guess #2: $\text{free-}\mathbb{Q}\text{-}G\text{-Sp} \overset{?}{\simeq}_{\mathbb{Q}} \text{torsion } \underbrace{H^*(\tilde{B}N)[W]}_{\text{twisted group ring}}\text{-modules}$

Ex: $1 \to S^1 \to O(2) \to \mathbb{Z}/2 \to 1$

$$H^*(\tilde{B}S^1) = \mathbb{Q}[c] \circlearrowleft \mathbb{Z}/2 \quad c \mapsto -c$$

$1 \to S^1 \to Pin(2) \to \mathbb{Z}/2 \to 1$ has same twisted group ring

<u>Main Thrm</u> Naive guess #2 is true!

For any compact Lie group G,

$$\text{free-}\mathbb{Q}\text{-}G\text{-Sp} \simeq_{\mathbb{Q}} \text{torsion } H^*(\tilde{B}N)[W]\text{-modules}$$

Proof outline:

Free G-Sp = G-Sp, underlying w. equiv (detected by G/e_+)

$= G_+\text{-cell-}G\text{Sp}$.

<u>Defn</u>: For $\mathcal{C}$ stable model cat, $A \in \mathcal{C}$, A-cell $\mathcal{C}$ is a model structure on $\mathcal{C}$ with:
- f is a w. equiv if $[A, f]_*$ is iso.
- f is a fib. if fib in $\mathcal{C}$

NB: <u>stable</u> cellularization → is $\{\Sigma^n A\}_{n \in \mathbb{Z}}$-cell-$\mathcal{C}$ in Hirschhorn)

$\mathrm{Ho}(A\text{-cell-}\mathcal{C})$ = localizing subcat. of $\mathrm{Ho}(\mathcal{C})$ containing A

<u>Cellularization Principle</u>: For $\mathcal{C}, \mathcal{D}$ stable model categories, given $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$

a Quillen adjunction and A in $\mathcal{C}$, define $B = \bar{L}A$ in $\mathcal{D}$.

If $A \simeq \bar{R}B \simeq \bar{L}\bar{R}A$, then L, R induces a Quillen equiv. on

$$A\text{-cell-}\mathcal{C} \simeq_Q B\text{-cell-}\mathcal{D}$$

NB other direction works as well

Rest of talk is over $\mathbb{Q}$

Step 1: $EG_+ \to S^0$ induces $\mathbb{S} \to F(EG_+, \mathbb{S}) =: DEG_+$ map of ring spectra.

Induced Quillen adjunction:

$$\mathbb{S}\text{-mod}/G\text{-sp} \underset{}{\overset{DEG_+ \wedge -}{\rightleftarrows}} DEG_+/G\text{-sp}$$

$$G_+ \longmapsto DEG_+ \wedge G_+$$ is weak equiv b/c G_+ is free

(continue to call $DEG_+ \wedge G_+$)

cellularization: G_+-cell-$\mathbb{S}$-mod/G-sp $\overset{①}{\simeq}_Q$ G_+-cell-DEG_+-mod/G-sp

free G-sp.

Step 2: Take fixed points

$D\tilde{B}N_+ := (DEG_+)^N \circlearrowleft G/N = W$ · taking fixed pt on ring & G-Spectra.

Have a fixed point adjunction

$$DEG_+\text{-mod}/G\text{-sp} \underset{\Psi^N}{\overset{(DEG_+ \wedge_{D\tilde{B}N} -)\circ \inf}{\rightleftarrows}} D\tilde{B}N_+\text{-mod}/W\text{-sp}$$

$$\begin{tikzcd} G_+ \arrow[r] & (G_+)^N \arrow[dl] \\ DEG_+ \wedge_{(DEG_+)^N} (G_+)^N \arrow[u, "\simeq"] & \end{tikzcd}$$

EMSS $\to$ $\simeq$

$(G_+)^N = \Sigma^d W_+$ by Wirthmüller

cellularization princ. applies:

$$G_+\text{-cell-}DEG_+\text{-mod}/G\text{-sp} \overset{②}{\simeq}_Q W_+\text{-cell-}D\tilde{B}N\text{-mod}/W\text{-sp}$$

Step 3: Move to algebra.

Since $D\tilde{B}N$ is commutative, corresponding comm. DGA $C^*\tilde{B}N$ ← cohomology even degrees

$$W_+\text{-cell-}D\tilde{B}N\text{-mod}/W\text{-sp} \overset{③}{\simeq}_Q W_+\text{-cell-dg } C^*\tilde{B}N\text{-mod}/\text{dg } \mathbb{Q}[W]\text{-mod}$$

Step 4: $H^*(C^*\tilde{B}N) = H^*(\tilde{B}N)$ poly. on even degree gen.

W-homomorphism $H^*\tilde{B}N \xrightarrow{\simeq} C^*\tilde{B}N$ so formality gives

$$\overset{④}{\simeq}_Q W_+\text{-cell-}H^*\tilde{B}N/\mathbb{Q}[W]\text{-mod}$$

$$\cong W_+\text{-cell-}(H^*\tilde{B}N)[W]\text{-mod}.$$

Step 5: small algebraic model.

Up to cellular equivalence, all objects have torsion H^*, so one (eventually) derives $\underset{Q}{\simeq}^{⑤}$ torsion - $(H^*\tilde{B}N)[W]$ - modules

G compact Lie group	Ho($\mathbb{Q}$-G-Sp)	$\mathbb{Q}$-G-Sp model cat	free $\mathbb{Q}$ G-Sp model cat	monoidal
F finite	Greenlees-May '95	Schwede-S '03		Barnes '09
S^1	Greenlees '99	S. '02		
O(2)	Greenlees '98			← Barnes (120). -)
SO(3)	Greenlees '01			
connected				
T^r torus		Greenlees - S (1101.2-)		
any			Greenlees - S (1101.4-)	