Stefan Schwede: 2/14/2012

$Sp^n(X) = X^n/\Sigma_n$, $x_0 \in X$ base pt

$Sp^n(X) \longrightarrow Sp^{n+1}(X) \quad [x_1,\dots,x_n] \longmapsto [x_0, x_1, \dots, x_n]$

$Sp^\infty(X) = \bigcup_{n\geq 0} Sp^n(X)$

Dold-Thom Thm: $Sp^\infty(S^m) \simeq K(\mathbb{Z}, m) \quad m \geq 1$

$Sp^\infty = \{Sp^\infty(S^m)\}_{m\geq 0} \simeq H\mathbb{Z}$

Filtration: $\mathbb{S} = Sp^1 \hookrightarrow Sp^2 \hookrightarrow Sp^3 \quad \cdots \quad \longrightarrow Sp^\infty = H\mathbb{Z}$

Facts: $Sp^n/Sp^{n-1} = \{Sp^n(S^m)/Sp^{n-1}(S^m)\}_m = \{(S^m)^n/\Sigma_n\}$

- $Sp^m/Sp^{m-1} \simeq_{\mathbb{Q}} *$ for $m \geq 2$ } Nakaoka
- $m \neq p^n \quad Sp^m/Sp^{m-1} \simeq *$ } Nakaoka
- $Sp^{p^n}/Sp^{p^{n}-1}$ is p-local
- $p=2 \quad Sp^{2^k}/Sp^{2^k-1} \simeq \Sigma^k L(k) \quad L(k)$ is a stable summand of $\Sigma^\infty_+ B\mathbb{Z}/2^k$

Priddy-Mitchell

- $Sp^n/Sp^{n-1} \simeq \Sigma^\infty B\mathcal{F}_n^\diamond \quad \mathcal{F}_n$ family of nontrans subgp of Σ_n (Lesh)

unred. susp.

- $Sp^n/Sp^{n-1} \simeq \Sigma^\infty(P_n^\diamond \wedge S^n) \quad P_n$ = partition complex (Arone-Dwyer)

G finite group $\quad \{Sp^n(S^V)\}_{V \in \mathcal{U}_G} \quad G$ acts by action on V + functoriality

this gives G-spectrum

$\pi_0^G(Sp^n) = ?$

$n=1 \quad Sp^1 = \mathbb{S}$ equiv. sphere spectrum

$n=\infty \quad Sp^\infty = \{Sp^\infty(S^V)\}_V \simeq H\underline{\mathbb{Z}}$

on π_0^G level:

$$A(G) = \pi_0^G(Sp^1) \to \pi_0^G Sp^2 \to \cdots \to \pi_0^G Sp^n \to \cdots \to \pi_0^G Sp^\infty = \mathbb{Z}$$

Thm For all $n \geq 1$, $\underline{\pi}_0 Sp^n \cong A(-)/\langle \{1,\dots,n\} - n \cdot 1/\Sigma_n \rangle$ ← global subfunctor

global functor

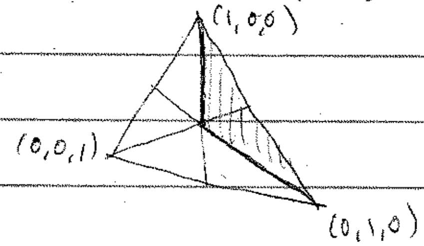
$\Longrightarrow_{\text{algebra}} \quad \pi_0^G(Sp^n) = A(G)/\ [G/K] - [H:K][G/H]$ for all $K \leq H \leq G$, $[H:K] \leq n$

Basic homotopy: $\bar{\nu}_n$ = red. nat. rep. on $\mathbb{R}^n \quad \bar{\nu}_n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid \sum x_i = 0\}$

Prop: The composite $S^{\bar{\nu}_n} \xrightarrow{tr^{\Sigma_n}_{\Sigma_{n-1}}} S^{\bar{\nu}_n} \hookrightarrow Sp^n(S^{\bar{\nu}_n})$ is homotopic to

$$\Delta : S^{\bar{V}_n} \longrightarrow Sp^n(S^{\bar{V}_n})$$
$$x \longmapsto [x, \dots, x]$$

$\Delta = (n-1)\text{-simplex} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \Sigma x_i = 1,\ x_i \geq 0\}$ $\quad S^{\bar{V}_n} \cong \Delta / \partial\Delta$



- move barycenter to opposite side, fix shaded side + extend linearly.
- since equiv, only describe on one side

- barycenter is isolated fixed point

Schwede's approach to global homotopy theory:

- global spectrum is just orthog. spectrum
  - but all compact Lie groups embed in orthog. groups.

X orthog spectrum, V fin dim. inner product space

$X(V) \circlearrowleft O(V)$.

V G rep $\rightarrow X(V)$ is a G-space via $O(V) \circlearrowright X(V)$

<u>Thm</u>: There is a model structure on the category of orthogonal spectra with weak equiv. the "global equiv." $f : X \rightarrow Y$ s.t.

$$\pi_n^G f : \pi_n^G X \longrightarrow \pi_n^G Y \quad \text{is an isom. for all } G \text{ + all } n \in \mathbb{Z}$$

(in fact are several choices of model structure.)

Here $\quad \pi_0^G X = \operatorname{colim}_{V \subseteq \mathcal{U}_G} [S^V, X(V)]_*^G$

$\rightarrow$ finer equivalence relation

traditional htpy types "bifurcate" for different point-set models.

<u>Defn</u>: <u>Burnside category</u> has obj finite groups

$A(K, G)$ = Groth grp of right free K-G-bisets

A <u>global functor</u> is an additive functor $F : A^{op} \longrightarrow Ab$

"inflation functor" in Webb.

Ex: $\{\pi_0^G X\}_G = \underline{\pi}_0 X$ is a global functor $\quad X \in Sp^O$

- can construct global Eilenberg-MacLane spectrum
  - restriction along arbitrary homomorphisms; transfers for inclusions.

<u>Thrm</u>: $Sp^n / Sp^{n-1} \underset{gl}{\simeq} \Sigma^\infty B_{gl} \mathcal{F}_n$ $\quad$ $\mathcal{F}_n$ family of nontrans. subgrps of Σ_n

$\quad$ ^ global classifying space

$B_{gl}\mathcal{F}_n\langle K\rangle = E\mathcal{F}_n(K)/\Sigma_n$ $\quad$ $\mathcal{F}_n(K)$ is family of subgrps Γ of $K \times \Sigma_n$ s.t. $\Gamma \cap 1 \times \Sigma_n \in \mathcal{F}_n$

then $\Sigma^\infty B_{gl}\mathcal{F}_n$ at K is susp. spectrum of $B_{gl}\mathcal{F}_n\langle K\rangle$

<u>Pf</u>: $Sp^n/Sp^{n-1}(S^V) = (S^V)^{\wedge n}/\Sigma_n = S^{\nu_n \otimes V}/\Sigma_n = S^V \wedge (S^{\bar{\nu}_n \otimes V}/\Sigma_n)$

$= S^V \wedge (S(\bar{\nu}_n \otimes V)/\Sigma_n)^\diamond$

$S(\bar{\nu}_n \otimes \mathcal{U}_K)$ is an $E\mathcal{F}_n(K)$.

<u>Thrm</u> $Sp^n/Sp^{n-1} \underset{gl_K}{\overset{\langle K\rangle}{\simeq}} \Sigma^\infty S^1 \wedge (E\mathcal{P}_n(K)^\diamond \wedge S^{\bar{\nu}_n})_{h_{gl}\Sigma_n}$ $\quad$ $E\mathcal{P}_n(K)$ = univ. space for collection of $\Gamma \leq K \times \Sigma_n$ s.t. $\Gamma \cap 1 \times \Sigma_n$ is nontriv, proper, conj $\overset{?}{=} \Sigma_{i_1} \times \Sigma_{i_2} \times \cdots \Sigma_{i_k}$

$i_1 + \cdots + i_k = n$ (partition)

$\Rightarrow \underline{\pi}_0(Sp^n/Sp^{n-1}) = 0$

$\bigoplus_{(H)\in P_n} A(-, N_{\Sigma_n}H) \to A(-, \Sigma_n) \to \underline{\pi}_1(Sp^n/Sp^{n-1}) \to 0$

- Cofib. seq. of orthog. spectra:

$Sp^{n-1} \hookrightarrow Sp^n \to Sp^n/Sp^{n-1}$ $\quad$ $\rightsquigarrow$ long exact seq. for each G.

$\bigoplus_{(H)\in P_n} A(-, N_{\Sigma_n}H) \to A(-, \Sigma_n) \to \underline{\pi}_0 Sp^{n-1} \to \underline{\pi}_0 Sp^n \to 0$

$\mathrm{Id}_{\Sigma_n} \longmapsto tr_{\Sigma_{n-1}}^{\Sigma_n} - n \cdot 1$

- Note: rationally things not product of Eilenberg-MacLane spectra.

$n = 2$. $\quad$ $\mathcal{F}_2 = \langle e\rangle$ $\quad$ $B_{gl}\mathcal{F}_2 = B_{gl}\Sigma_2$

- maps out of $B_{gl}\Sigma_2$ rep. $\underline{\pi}_0^{C_2}$.

$$\begin{array}{ccc} \Sigma^\infty_+ \mathbb{R}P^\infty & \xrightarrow{tr} & \mathbb{S} \\ {\scriptstyle 2}\downarrow & & \downarrow \\ \mathbb{S} & \longrightarrow & Sp^2 \end{array}$$

$$0 \to \pi_1 Sp^? \longrightarrow A(-, \Sigma_2) \to A(-, e) \longrightarrow \pi_0 Sp^2 \longrightarrow 0$$

$$Id_{\Sigma_2} \to t_e^{\Sigma_2} - 2I$$

gives rational k-invariant.

note: since have cx generators, so Morita theory implies nontriv. ext groups...