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$G$-discrete group $R$-ring all spectra naive.

Want to study $\text{K}_*(R[\mathbb{G}])$ via assembly map.

**Assembly (old day, '70s)**

\[ \text{H} \longrightarrow \text{GL}_n(R[\mathbb{G}]) \longrightarrow \text{BH} \longrightarrow \text{BGL}_n(R[\mathbb{G}]) \]

\[ \text{H}_* (\text{BH}; R[\mathbb{G}]) \longrightarrow \text{K}_* R[\mathbb{G}] \]

Integral Novikov conjecture: $\alpha$ is injective for $G$ torsion-free.

Geometric description of $\alpha$:

$x$-metric space $(E_G)$

$\mathcal{B}_p(x)$ - cat of locally finite, $R$-free $R$-mod. over $x$ w/ bounded morphisms

$M \longrightarrow \mathcal{B}_p(x) \longrightarrow X$

$\text{morf. } \phi \longrightarrow \text{locally finite}$

For each $x \in \mathcal{B}_p(x)$ bounded:

$\text{dist}(x) > 0 \Rightarrow \phi(x) = 0$

Notation: $\text{K}_*(\mathcal{B}_p(x)) = \text{K}_* (x)$

Observation: If $X/\alpha$ is finite diameter ($G$ acting isometrically on $X$)

then $\mathcal{B}_p(x)^{\alpha} \cong \mathcal{B}_{R[\mathbb{G}]} (X/\alpha) \cong \text{fin gen. free } R[\mathbb{G}]-\text{mod.}$

$X \longrightarrow \mathcal{B}_p(x)^{\alpha} \cong R(\mathbb{G})$

To get homology theory, need a local version.

$\mathcal{C}_*(x) \cong B (x \times [0,1])$

morphism $\mathcal{C}_*(x)$ into $R(\mathbb{G})$ have propagation $\to 0$ as $t \to 1$.
Have \( eB(X) \leq e(X) \leq e(X) \rightarrow e(X)/e(X)_{e} \) Kanubqi quotient/pan category

Thm: The boundary map in K-theory for this sequence agrees with Toda's x when \( X = EG \), \( B \alpha \) fin. (W = cx)

Carlson's descent principle:

If \( e(EG) \) has trivial K-theory, then \( \alpha \) is (split) injective:

\[
\begin{align*}
\Omega(K\mathbb{Z}^{\infty}(EG)^{k}) & \xrightarrow{\partial} K^{0}(EG)^{k} \\
\Omega(K\mathbb{Z}^{\infty}(EG)^{k}) & \xrightarrow{\Omega(\partial)} K^{0}(EG)^{k}
\end{align*}
\]

Thm (Bartels, Carlsson-Goldfarb)

\( \alpha \) is inj. for \( h \) a geom. finite gap with finite asymptotic dim.

finite asymptotic dimension (Amarev)

\( X \) has a.d. \( \leq n \) if \( \forall r \exists i \exists X_{i} = \bigcup X_{i} \), \( X_{i} = \{ x_{i} \} \), \( d(x_{i}, x_{i}) \geq r \)

+ \( \forall X_{i} \) uniformly bounded, \( \forall \)

Ex: \( R \) has asympt. dim. 1: \( r > 0 \)

\( R^{2} \) has asympt. dim. 2: \( r > 0 \)

Today: \( \text{ft w/ Tessera-Yu} \) Extend to finite dec. complexity

Ex: \( R^{2} \) has FDC, \( r > 0 \)

Step 2: decompose previous strips

Step 1

Thm (Amarev-Tessera-Yu): A large linear gap \( G \in GL_{n}R \) have FDC with word metric
FDC: \( C \) set of metric families

\[ \forall x, y \in C \text{ if } \forall \chi, \eta \in \chi \cup \eta, \chi = \chi' \cup \eta', \eta = \chi'' \cup \eta'' \]

\[ \forall \chi, \eta, \chi', \eta' \in C \]

\[ \exists \chi'' \in C \]

\[ \exists \eta'' \in C \]

\[ \forall \chi, \eta \in C \text{ uniformly bounded} \]

For \( \chi \) an ordinal,

\[ \chi = \beta + 1 \Rightarrow \chi' = \chi \cup \{ \eta \} \]

\[ \chi' \text{ decomposes over } \chi'' \]

\[ \exists \chi'' \in C \]

\[ \forall \chi, \eta \leq \chi' \]

Defn: \( X \) has FDC if \( \{ \chi, \eta \} \in \chi'' \).

\[ \text{iff uniform actually get FAD} \]

Thm (Ramos-Tessera-Yan): \( \alpha \) is injective for \( \alpha \) w FDC \( \Rightarrow \text{Ba. Fin. Cov.} \).

More specifically \( K^c(\text{Ba}) = * \) \( \Rightarrow \text{R}(\text{EG}) = * \).

Basic principle: \( K^c(\text{EG}) = * \) \( \Rightarrow \text{EG} \) is good, convex.

\[ s: \mathbb{G}(x) \rightarrow \mathbb{L}(x) \]

need good model of \( \text{EG} \):

\[ \mathbb{P}_x \text{ has vertex set } \mathbb{G} \text{ and simplexes } \varpi \leq \mathbb{G} \]

\[ \text{Rips complex with } d(\varpi_i, \varpi_j) < s \]

\[ \text{Rips complex approach to } \mathbb{G} \text{ as } s \rightarrow \infty \text{ capture large-scale features} \]

Idea of proof that \( K^c(\mathbb{G}) = * \)

\[ \text{Use M-VReduce to decompose } K^c(\mathbb{P}_x \mathbb{G}) \text{ has } \mathbb{P}_x \mathbb{G} \text{ decompose } \mathbb{G} \]

\[ x \in K^c(\mathbb{P}_x \mathbb{G}) \text{ comes from morph of length } R \text{ (say)} \]

Decompose \( \mathbb{G} \):

\[ \mathbb{G} = \mathbb{U} \cup \mathbb{V} \]

\[ \mathbb{U} = \mathbb{U}_1 \cup \mathbb{U}_2 \]

\[ \mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2 \]

\[ \mathbb{U}_1 \cap \mathbb{V}_2 = \emptyset \]

\[ \mathbb{U}_2 \cap \mathbb{V}_1 = \emptyset \]

\[ \mathbb{U}_2 \text{ and } \mathbb{V}_1 \text{ lower-capacity} \]

so stuck in the lower level piece.

Induction, can kill classes on \( \mathbb{P}_x \mathbb{U}_1 \) by increasing \( s \).

(in bounded case, make pieces convex)

Assemble via Rips complexes:

\[ \text{colim } K^c(\mathbb{P}_x \mathbb{G})^s \rightarrow \text{colim } K^c(\mathbb{P}_x \mathbb{G})^s \]
\[ \operatorname{colim} K^0(P^6) \to \operatorname{colim} K^0(P^6)^\mathbb{A} \]
\[ \downarrow \text{co} \mathfrak{P}_{\mathbb{A}} \text{hocompact} \]
\[ \operatorname{colim} (K^0(P^6))^\mathbb{A} \to \operatorname{colim} (K^0(P^6))^{\mathbb{A}} \]
\[ \downarrow \text{?} \]
\[ \operatorname{colim} K^0(P^6)^{\mathbb{A}} \to \operatorname{colim} K^0(P^6)^{\mathbb{A}} \]

? interchanges htpy limit/colimit. When $BA$ is finite, $SS$ converge and have same $E^2$-terms.

$BA$ levelwise finite: $SS$ converge after restricting to $EG^{(k)}$ in $\mathbf{sk}$. Thus kernel of assembly are maps that are sort of "phantoms".