Dan Ramras: 2/13/2012

G - discrete group R - ring all spectra naive

Want to study $K_*(R[G])$ via assembly map.

<u>Assembly</u> (Loday, '70s)

$G \longrightarrow GL_\infty(\mathbb{Z}[G]) \quad \rightsquigarrow \quad BG \longrightarrow BGL_\infty(\mathbb{Z}[G])$

$g \longmapsto \begin{bmatrix} g & \\ & 1 \\ & & \ddots \end{bmatrix}$

$BG_+ \wedge BGL(R)^+ \longrightarrow BGL(\mathbb{Z}[G])^+ \wedge BGL(R)^+ \longrightarrow BGL\,R[G]^+$

$\rightsquigarrow BG_+ \wedge K(R) \xrightarrow{\alpha} KR[G]$

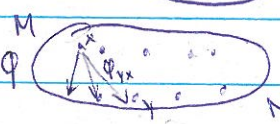
$H_*(BG; K(R)) \longrightarrow K_*R[G]$.

<u>Integral Novikov Conj</u>: α is injective for G torsion-free

Geometric description of α:

X - metric space (EG)

$\mathcal{B}_R(X)$ - cat. of locally finite, f.g. free R-mod. over X w/ bounded morphisms

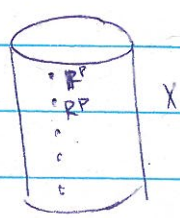
· morph. φ


locally finite
For each $\varphi \in \mathcal{B}_R(X)$
bounded: $\exists\, s > 0$ s.t. $d(x,y) > s \Rightarrow \varphi_{yx} = 0$

Notation: $\mathbb{K}(\mathcal{B}_R(X)) = K^b(X)$.

Observation: If X/G is finite diameter (G acting isometrically on X)

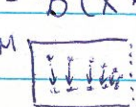
then $\mathcal{B}_R(X)^G \simeq \mathcal{B}_{R[G]}(X/G) \simeq$ fin. gen. free $R[G]$-modules



$\rightsquigarrow K^b(X)^G \simeq K(R[G])$

To get homology theory, need a local version.

$\mathcal{C}(X) \subseteq \mathcal{B}(X \times [0,1))$



morphisms in $\mathcal{C}(X)$ have propagation $\longrightarrow 0$ as $t \longrightarrow 1$

Have $cB(X) \subseteq \mathscr{C}(X)_{<1} \subseteq \mathscr{C}(X) \longrightarrow \mathscr{C}(X)/\mathscr{C}(X)_{<1}$ $\leftarrow \mathscr{C}^\infty(X)$ Karubi quotient / germ category.

Thrm: The boundary map in K-theory for G fixed pts of this sequence agrees with Loday's α when $X = EG$, BG fin. CW-cx.

Carlsson's descent principle:

BG fin. CW cx

If $\mathscr{C}(EG)$ has trivial K-theory, then α is (split) injective:

$$\begin{array}{ccc} \Omega(K\mathscr{C}^\infty(EG))^G & \xrightarrow{\partial=\alpha} & K^b(EG)^G \\ \downarrow \simeq & \text{equiv. when parameter space } EG \text{ is cocompact} & \downarrow \\ \Omega(K\mathscr{C}^\infty(EG))^{hG} & \xrightarrow[\simeq]{\partial} & K^b(EG)^{hG} \end{array}$$

Thrm (Bartels, Carlsson-Goldfarb)

α is inj. for G a geom. finite grp with finite asymptotic dim. — includes Gromov hyp. grps; discrete subgrps of cmpt Lie (cocompact.)

finite asymptotic dimension (Gromov)

X has a.d. $\leq n$ if $\forall r$ $X = \bigcup_{i=1}^{n+1} X_i$ & $X_i = \coprod^{r\text{-disj}} X_{ij}$

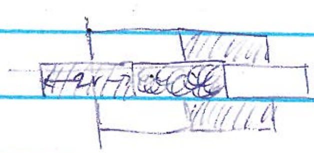
$d(X_{ij}, X_{ik}) \geq r$

+ $\{X_{ij}\}$ uniformly bounded.

Ex: $\mathbb{R}$ has asympt. dim. 1: $r > 0$

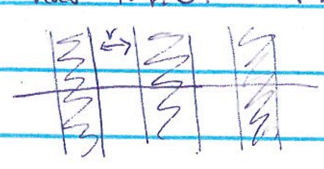
2 sets ▬ + —

$\mathbb{R}^2$ has asympt. dim 2: $r > 0$



Today: (jt w/ Tessera-Yu) Extend to finite decomp. complexity

Ex: $\mathbb{R}^2$ has F.D.C. $r > 0$



step 1

Step 2: decompose previous strips

now have uniformly bounded family.

Thrm (Guentner-Tessera-Yu): A fg linear grps $G \leq GL_n R$ have FDC wrt word metric.

FDC: $\mathcal{C}$ set of metric families

$\{X_i\}$ decomposes over $\mathcal{C}$ if $\forall r$ $X_i = \mathcal{U}_i \cup V_i$, $\mathcal{U}_i = \coprod^r \mathcal{U}_{ij}$ $V = \coprod^r V_{ij}$ & $\{\mathcal{U}_i\}, \{V_i\} \in \mathcal{C}$.

$\mathfrak{B}_0(X) = \{\mathcal{F} \subseteq \mathcal{P}(X) \mid \mathcal{F}$ is uniformly bounded

For γ an ordinal,

- $\gamma = \beta + 1$ $\mathfrak{B}_\gamma(X) = \{\mathcal{F} \subseteq \mathcal{P}(X) \mid \mathcal{F}$ decomp. over $\mathfrak{B}_\beta(X)\}$.
- γ limit $\mathfrak{B}_\gamma(X) = \bigcup_{\alpha < \gamma} \mathfrak{B}_\alpha(X)$.

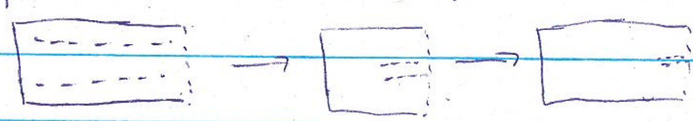
Defn: X has FDC if $\{X\} \in \mathfrak{B}_\gamma(X)$.

$\to$ if uniform (in r sequences) actually get FAD.

Thm (Ramras–Tessera–Yu) α is inj for G w/ FDC & BG fin. CW cx.

More specifically $K^c(EG) \simeq *$ (so $\mathcal{C}(EG) \simeq *$).

Basic principle: $K^c(X) \simeq *$ X bdd, convex.



$S : \mathcal{C}(X) \to \mathcal{C}(X)$

$S \oplus \mathrm{Id} \simeq S$ so Eilenberg swindle

need good model of EG:

$P_\infty G = \bigcup_s \underbrace{P_s G}_{\text{Rips cxs}}$ P_sG has vertex set G & a simplex $\forall \langle g_0, \dots, g_n \rangle$ with $d(g_i, g_j) < s$.

- cocompact approx to EG & as $s \to \infty$, capture large scale features

Idea of proof that $K^c(EG) \simeq *$

- use M-V seq's to decompose $K^c(P_sG)$ as we decompose G.

$x \in K^c P_s G$ x comes from morph. of length R (say)

Decompose G: $G = \mathcal{U} \cup V$ $\mathcal{U} = \coprod^{RS} \mathcal{U}_i$ $V = \coprod^{RS} V_i$ & $\mathcal{U}_i, V_i \in$ lower cxpity

$\boxed{\mathcal{U}_i} \leftarrow x \rightarrow \boxed{\mathcal{U}_j}$

so stuck in the lower level piece.

Induction, can kill classes on $P_s \mathcal{U}_i$ by increasing s.

(in bounded case, make pieces convex.)

Assembly via Rips complexes

$$\operatorname{colim}_s K^\infty(P_sG)^G \xrightarrow{\partial} \operatorname{colim}_s K^b(P_sG)^G$$

$$
\begin{tikzcd}
\operatorname{colim}_S K^\infty(P_S G)^G \arrow[r, "\partial"] \arrow[d, "\simeq\ P_S G \text{ cocompact}"] & \operatorname{colim}_S K^b(P_S G)^G \arrow[dd] \\
\operatorname{colim}_S (K^\infty(P_S G))^{hG} \arrow[d, "?"] & \\
(\operatorname{colim}_S K^\infty P_S G)^{hG} \arrow[r, "\simeq"] & (\operatorname{colim}_S K^b(P_S G))^{hG}
\end{tikzcd}
$$

? interchanges htpy limit / colimit. When BG is finite: SS converge & have same E^2-terms.

BG levelwise finite: SS converge after restricting to $EG^{(k)}$ fin. skeleton.

→ thus kernel of assembly are maps that are sort of "phantom"