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jt w/ Niles Johnson

Cat $\simeq$ Top    Not useful models for homotopical information
Perm $\simeq$ $\text{Spectra}_{\geq 0}$

Defn: A space $X$ is a homotopy $n$-type if for all choices of basepoint $x$ and all $i > n$, $\pi_i(X,x) = 0$

Work modelling $n$-types: Loday, Conduché, Baues.

Homotopy hypothesis: "weak $n$-groupoids model homotopy $n$-types"
→ this guides definitions of higher weak groupoids
For $n = 1, 2, 3$, have concrete defn of weak $n$-groupoid, and theorems of Lack, Paoli saying this is so.

$n=1$: connected 1-types.
$X \simeq K(\pi,1)$, $\pi = \pi_1(X)$
$n=1$, nonconnected
$X \simeq \coprod_{\pi_0} K(\pi_1(X,x),1)$  $x \in \pi_0$

$\text{Top} \xrightarrow{\Pi_1} \text{Grpd}$   fundamental groupoid functor induces equivalence of 1-types w/ groupoids on homotopy cat.

Thm: There is an equivalence of htpy categories
$$\text{Ho}(\text{Top}_{\leq 1}) \underset{B}{\overset{\Pi_1}{\rightleftarrows}} \text{Ho}(\text{Grpd})$$ (not equiv. before htpy categories)

Going to stable world:
Defn: A connective stable homotopy $n$-type is a spectrum $X$ with $\pi_i X = 0$ for $i < 0$, $i > n$.
Let $\mathcal{S}_0^n$ be the full subcategory of $\mathcal{S}$ given by the $n$-types
$n=0$. $\text{Ho}(\mathcal{S}_0^0) \simeq \text{Ho}(\text{Ab})$ ← isom. classes.
$n=1$. (unstable) homotopy 1-type ⟷ groupoid

$E_\infty$-structure $\longleftrightarrow$ symmetric monoidal

group like $\longleftrightarrow$ group-like

Symmetric monoidal grouplike groupoids are Picard groupoids.

We should also expect these at higher levels.

<u>Defn:</u> A <u>Picard</u> groupoid $\mathcal{C}$ is a symmetric monoidal groupoid s.t. $\pi_0\mathcal{C}$ is a group. (i.e. every object has an inverse up to isomorphism)

Want $\mathcal{S}^1_0 \underset{B}{\overset{\Pi_1}{\rightleftarrows}} \mathrm{Pic}$ ← obj. small Picard groupoids, morph. symm monoid functors giving equivalence on homotopy categories.

$\mathcal{O}$ operad $\mathcal{O}(j) = \tilde{\Sigma}_j$ alg. are perm. categories

$B\mathcal{O}$ is an $E_\infty$-operad in Top; $\mathcal{O}$ thus $E_\infty$-operad in Cat.

$\Pi_1 B\mathcal{O}$ is also an $E_\infty$-operad in Cat.

→ shows $\Pi_1$ lands in Pic. In fact, $\Pi_1$ only depends on lower htpy, so only need $E_3$ to get symm. monoidal structure on $\Pi_1 X$.

From unstable case, we have equivalences

$B\Pi_1 X \simeq X \qquad \Pi_1 B\mathcal{C} \simeq \mathcal{C}$

these equivalence will be $E_\infty$ / symmetric monoidal b/c $B/\Pi_1$ preserve products

This proves:

<u>Thm</u> There is an equivalence of homotopy categories

$\mathrm{Ho}(\mathcal{S}^1_0) \simeq \mathrm{Ho}(\mathrm{Pic})$

Try to understand homotopical information on topological side in categories.

<u>Understanding the k-invariant:</u>

Stable homotopy 1-types are classified by $\pi_0 X$, $\pi_1 X$ (ab. grps) + unique k-invariant $H\pi_0 \xrightarrow{k} \Sigma^2 H\pi_1$

Corresponds to $\eta^*: \pi_0 \to \pi_1$ stable quadratic map

(mult. by Hopf map $S^3 \xrightarrow{\eta} S^2$)

Given a Picard groupoid, we define $\pi_0\mathcal{C} = \mathcal{C}/\sim$ abelian grp

$\pi_1\mathcal{C} = \mathcal{C}(I, I)$ (endomorph. of identity) abelian grp. (from monoid structure by two mult. argument)

NB $\mathcal{C}(x,x) \cong \mathcal{C}(I,I)$ $\quad x \cong x\otimes I \xrightarrow{1\otimes f} x\otimes I \cong x$ (plus inverses)

$\pi_i(B\mathcal{C}) \cong \pi_i\mathcal{C}$; $\quad \pi_i(\Pi_1 X) \cong \pi_i(X)$

Thm Let $\mathcal{C}$ be a Picard groupoid. Then $\mathcal{C}$ is equivalent as a Picard groupoid to one that is skeletal and strict (associativitaly) with objects given by $\pi_0\mathcal{C}$ and automorphisms of an object given by $\pi_1\mathcal{C}$ and symmetry isomorphism induced by the quadratic map

$$c: \pi_0\mathcal{C} \longrightarrow \pi_1\mathcal{C}$$
$$x \longmapsto \gamma(x,x)$$

NB skeletal & strict at same time is hard.

Model the sphere spectrum (or rather, its one-truncation)

Let $S$ be the category with objects $:= \mathbb{Z}$

morph $S(m,n) = \begin{cases} \emptyset & \text{if } m\neq n \\ \mathbb{Z}/2 & m=n \end{cases}$

Take strict symmetric monoidal structure with addition in $\mathbb{Z}$ and symmetry isomorphism $\gamma_{n,m} = \begin{cases} 0 & \text{if } nm \text{ even} \\ 1\in\mathbb{Z}/2 & \text{if } nm \text{ odd} \end{cases}$

Thm: $S$ is the free Picard groupoid on one object.

Thm: $BS$ is the Postnikov 1-truncation of $QS^0$

Pf: Let $\mathcal{E}$ = (skeletal) cat. of finite sets & isom.

There is a sym. mon. functor $\mathcal{E} \longrightarrow S$

$$n \longmapsto n$$
$$\sigma\in\Sigma_n \longmapsto \mathrm{sgn}(\sigma)$$

induces

$$\begin{tikzcd} B\mathcal{E} \arrow[r] \arrow[dr, "\text{grp completion}"'] & BS & E_\infty\text{-map} \\ & QS^0 \arrow[u, "\text{grouplike}"'] & \text{iso on } \pi_0 \text{ and } \pi_1 \end{tikzcd}$$

Rmk: You infact get multiplicative structure.

Homotopy cofiber:

Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ map of Picard grpds. Want to identify htpy cofiber of

$$B\mathcal{C} \xrightarrow{BF} B\mathcal{D}$$

- htpy cofiber is a 2-type, so we must construct Picard bigrpd!

Define $\mathrm{Coker}\,F$ a Picard bigroupoid

bigroupoid symm. monoidal & $\pi_0$ grp, objects invertible up to equiv.

$\mathrm{Ob}\,\mathrm{Coker}\,F = \mathrm{ob}\,\mathcal{D}$

1-cells $(N,f): X \to Y$ $\quad X \xrightarrow{f} Y \otimes F(N)$, $N \in \mathrm{Ob}\,\mathcal{C}$

2-cells $\alpha : (N,f) \to (N',f')$

$\alpha : N \to N'$ mor. in $\mathcal{C}$ s.t.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Y\otimes F(N) \arrow[rr, "1\otimes F(\alpha)"'] & & Y \otimes F(N') \end{tikzcd}$$

One can show that this is a symm. monoid bigroupoid, with weakly invertible objects ⇝ i.e. Picard bigroupoid

Moreover, there is a symm. monoidal pseudofunctor

$$\mathcal{D} \longrightarrow \mathrm{Coker}\,F$$

Thus $B\mathcal{D} \longrightarrow B\,\mathrm{Coker}\,F$ is an $E_\infty$-map (Angélica's thesis!)

Get a sequence $B\mathcal{C} \xrightarrow{BF} B\mathcal{D} \longrightarrow B\,\mathrm{Coker}F$ which gives an exact sequence of homotopy groups.

Looking into the future:

$n=2$. Picard bigroupoids, idea of proving equivalences.
still need to understand $2^{nd}$ k-invariant.