John Lind: 2/16/2012

Explain: (principle Π-fibration when Π is an A_∞-space) 2G

There are "structured models" for Top w/ $\boxtimes$ analogous to $\wedge$; monoids A_∞ / E_∞.

- EKMM: $\mathcal{L}(1)$-spaces (Blumberg's thesis)
- diagram space.

$\mathbb{I}$: cat. of finite sets + injections

An $\mathbb{I}$-space is a functor $X : \mathbb{I} \longrightarrow \mathrm{Top}$

ex: if E is a symmetric spectrum, then $\Omega^\bullet E$.

$$\{1, \dots, n\} = \underline{n} \longmapsto \Omega^n E_n$$

$\mathcal{U}$: cat of real inner product spaces V and lin. isometries $V \to W$.

$\mathcal{U}$-space a functor $X : \mathcal{U} \longrightarrow \mathrm{Top}$

There is an adjunction $\mathcal{U}\text{-spaces} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \text{orthog. spectra}$

<u>Defn of $\boxtimes$</u>: X, Y $\mathcal{U}$-spaces.

$X \boxtimes Y$ = left Kan extension of $(V, W) \longmapsto X(V) \times Y(W)$ along $\oplus : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$.

A $\boxtimes$-monoid (= $\mathcal{U}$-FCP functor w/ cartesian prod.) X consists of

$$X(V) \times X(W) \longrightarrow X(V \oplus W) \qquad \text{(unital, assoc, ...)}$$

$$\underset{\text{symm. monoidal}}{\mathcal{U}\text{-spaces}} \simeq_Q \underset{\text{not sym. monoidal}}{\mathcal{L}(1)\text{-spaces}} \simeq_Q \mathrm{Top}$$

$\mathcal{U}$-FCPs $\longleftrightarrow$ A^∞-spaces

comm $\mathcal{U}$-FCPs $\longleftrightarrow$ E_∞-spaces

<u>Defn</u>: X is an $\mathcal{U}$-space. $\pi_* X = \pi_* \operatorname{hocolim}_{\mathcal{U}} X \simeq \pi_* \operatorname{hocolim}_{V \subset U} X(V)$.

ex: $V \longmapsto O(V)$ $\qquad V \longmapsto BO(V)$.

Let Π be a grouplike $\mathcal{U}$-FCP (ie model for a grouplike A_∞-space)

<u>Defn</u>. A principal Π-fibration is a map $p : E \longrightarrow B$ of $\mathcal{U}$-spaces w/ a Π-module structure $E \boxtimes \Pi \longrightarrow E$ over B s.t.

i.) $\forall\, b \in B$, there exists a w.e. $E_b \simeq \Pi$ of Π-modules

ii.) p is a Hurewicz fibration of Π-modules (lifts of Π-mod. htpy)

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ * \arrow[r, "b"] & B \end{tikzcd}$$

(interested in case where B constant $\mathcal{U}$-space)

We can construct classifying spaces for such objects:

$$\begin{tikzcd} E\Pi = B^{\boxtimes}(*, \Pi, \Pi) \arrow[d] \\ B\Pi = B^{\boxtimes}(*, \Pi, *) \end{tikzcd}$$

Given a map $f: X \to B\Pi$, define

$$\begin{tikzcd} P(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (E\Pi)^{\text{fibration}} \arrow[d] \\ X \arrow[r, "f"] & B\Pi \end{tikzcd}$$

(generally just quasifibration)

<u>Thm</u>: If Π is cofibrant, grouplike $\mathbb{L}$-FCP (w/ nondegenerate base point), then $f \mapsto P(f)$ induces

$$[X, B\Pi] \longrightarrow \{\text{equiv. classes of principal } \Pi\text{-fibrations}\}$$

(X CW-cx, say.)

Proof internal to $\mathbb{L}$-spaces, although could possibly lift top. monoid version.

Rmk: $E\Pi \to B\Pi$ quasifibration of $\mathbb{L}$-spaces is the heart of the matter.

$\Pi = GL_1 R$ R connective orthogonal ring spectrum.

Define, given $f: X \to BGL_1 R$, $\mathcal{L}f = \Sigma^\infty_X Pf \wedge_{\Sigma^\infty_+ GL_1 R} R$ is a parametrized spectrum over X w/ fiber R.

$Mf = r_! \mathcal{L}f = \Sigma^\infty_+ Pf \wedge_{\Sigma^\infty_+ GL_1 R} R$

This induces: $[X, BGL_1 R] \cong \{\text{eq. classes of parametrized } Sp/_X \text{ w/ fiber } R\}$.

When fiber is $R^{\vee n}$ we get an interpretations of $K(R)^0(X)$ in terms of "R-bundles w/ fiber $R^{\vee n}$"

G compact Lie group

Let $\mathbb{L}_G$ be the cat. of G-repns V that admit an iso to $V \subset \mathcal{U}_G$; linear isometries.

<u>Def</u> An $\mathbb{L}_G$-space is a Top_G-enriched functor $X: \mathbb{L}_G \to Top_G$.

$$\mathbb{L}_G(V, W) \times X(V) \xrightarrow{G\text{-map}} X(W)$$

<u>Unverified speculation</u>:

$(\mathbb{L}_G\text{-spaces}) \simeq (\mathcal{L}_G(1)\text{-spaces})$ Everything else depends on this.

~~All principal~~ Let Π be an $\mathcal{U}_G$-FCP, grouplike (on all fixed pt), cofibrant.

A principal (Π, G)-fibration is defined in the same way.

- total space, base space have action of G; p G-equivalence, Hurewicz fibration etc.

Recall: When Π compact Lie, define a family $\mathcal{F}$ of subgrps Λ in $\Gamma = \Pi \rtimes G$

$1 \to \Pi \to \Gamma \to G \to 1$ s.t. $\Lambda \cap \Pi = e$.

Equivalently, $\Lambda = (H, \alpha) = \{(\alpha(h), h) \in \Pi \times G \mid h \in H$ where $H < G$, $\alpha: H \to \Pi$ 1-cocycle$\}$

(NB α continuous 1 cocycle.)

Defn $E_G \Pi = E\mathcal{F} = B(*, \mathcal{O}_{\mathcal{F}}, \mathcal{O})$ where $\mathcal{O}_{\mathcal{F}}$ is full subcat of orbit cat $\mathcal{O}_\Gamma$ on Γ/Λ, $\Lambda \in \mathcal{F}$.

$\downarrow$

$B_G \Pi = B(*, \mathcal{O}_{\mathcal{F}}, *)$ $\quad \mathcal{O}: \mathcal{O}_{\mathcal{F}} \to \mathrm{Top}_\Gamma \quad \Gamma/\Lambda \mapsto \Gamma/\Lambda$

Mimic this construction when Π A_∞-space.

Define an $\mathcal{U}_G$-FCP $\Gamma = \Pi \rtimes G$ by $\Gamma(V) = \Pi(V) \times G^{\mathcal{Q}G}$ (G acts by conj.)

multiplication $\Pi(V) \times G \times \Pi(W) \times G \longrightarrow \Pi(V \oplus W) \times G$.

$(x, g) \quad (y, h) \longmapsto (x \cdot {}^g y, gh)$.

Define a 1-cocycle $\alpha: H \to \Pi$ to be a map of $\mathcal{U}$-spaces satisfying

$\alpha_{V \oplus W}(h_1 \cdot h_2) = \alpha_V(h_1)\, {}^{h_1} \alpha_W(h_2)$

$\alpha_V: H \longrightarrow \Pi(V)$. only depends on levels.

Form $\Gamma = \Pi \rtimes G$ modules of the form $\Gamma/\Lambda = \Pi \rtimes G/(H, \alpha) = (\Pi \rtimes G) \boxtimes_{(H,\alpha)} *$

where $\Lambda = (H, \alpha)$ is the sub $\mathcal{U}_G$-FCP given by

$(\alpha_V(h), h) \in \Pi(V) \times G$.

(Γ-modules) is enriched in itself, so bar construction

$E_G \Pi = B(*, \mathcal{O}_{\mathcal{F}}, \mathcal{O})$ makes sense. ($\mathcal{O}_{\mathcal{F}}$, $\mathcal{O}$ as above)

$\downarrow$

$B_G \Pi = B(*, \mathcal{O}_{\mathcal{F}}, *)$

Need to show $E_G \Pi \to B_G \Pi$ appropriate G-quasifibration $\rightsquigarrow$ classifies.

If successful, we can define equivariant twisted R-theory for any G-ring spectrum R.