John Greenlees: 2/15/2012

Algebraic model for rat'l G-spectra.

· globally apply $\mathbb{Q}\otimes$

G compact Lie.

Conjecture: G-spectra$/\mathbb{Q} \simeq_Q$ DG-$\mathcal{A}(G)$ for some nice abelian cat $\mathcal{A}(G)$, inj. dim = rnk G

Example: $\underset{\text{free}}{\text{G-spectra}}$, $\simeq$ DG-$\mathcal{A}(G)$, where $\mathcal{A}(G)$ = tors-$H^*BG_1[\pi_0 G]$-mod.

Why? Calculations – Adams short exact seq.

Construction of G-spectra algebraically: G-equiv. elliptic cohom.

G = circle (Topology) or G = torus

(cf. equiv σ-genus Ando – G)

Elliptic cohom: elliptic curve C, G circle $V^G = 0$.

$$EC_G^1(S^V) = H^1(C; \mathcal{O}(DV))$$

$V = \sum_{n\in\mathbb{Z}} a_n z^n$

$D(V) = \sum_{n\in\mathbb{Z}} a_n C[n]$

Idea: $\mathcal{A}(G)$ category of sheaves over Sub(G)

with fiber over H capturing H-geometric isotropy information.

$$\mathcal{GI}(X) := \{H \mid X^{gH} \not\simeq *\}$$ ← geometric isotropy.

(← nonequivariantly essential.)

X, Y have geom. isotropy over a single conj. class (H).

$$\underset{\text{obstruction theory}}{[X,Y]^G} = [X,Y]^{N_G H} = [\underset{\text{free } W_G H\text{-spectra}}{X^{gH}, Y^{gH}}]^{N_G(H)/H}$$

Hence controlled by $H^*(BW_G^1(H))[\pi_0 W_G H]$ (from Brooke's talk).

Hence conj. is: $\mathcal{A}(G)$ should by sheaves of $\mathcal{O}$-modules over Sub(G)

where $\mathcal{O}_H = H^*(BW_G^1 H)[\pi_0 W_G^1 H]$, with additional structure for $K \trianglelefteq_T H$

reflecting localization theory.

1. The circle, G.

Sub(G)

· discrete space

· cotoral incl. for morph.

<u>sheaf of rings $\mathcal{O}$</u>

$H^*B(G/G) = \mathbb{Q}$

$$\begin{tikzcd} & & G & & \\ C_1 \arrow[urr] & C_2 \arrow[ur] & C_3 \arrow[u] & C_4 \arrow[ul] & \end{tikzcd}$$

$H^*B(G/C_1)$ $H^*B(G/C_2)$

<u>module over sheaf N</u>

$N(G) = V$

$N(1)$ $N(2)$ $N(3)$

modules over a $\mathbb{Q}[c]$ $|c| = -2$.

· using idempotents in rat'l Burnside ring to separate out the fin. subgroups

+ add'l cat. structure:

$$V \longrightarrow \mathcal{E}_G^{-1} \prod_{i\geq 1} N(i)$$

$$\mathcal{O}_{\mathcal{F}} = \prod_{n\geq 1} H^* B(G/C_n) \qquad \mathcal{F} = \text{family of finite grps}$$

α S^1 rep. $\quad c(\alpha) \in \mathcal{O}_{\mathcal{F}} \quad c(\alpha)(F) = c_1(\alpha^F) \in H^2(B(G/F))$

$\mathcal{E}_G = \{c(\alpha) \mid \alpha^G = 0\}$ — each $c(\alpha)$ finitely supported (1 except finitely many places)

call the category of N's $\mathcal{A}^{sheaf}(G)$

Repackage: $\mathcal{A}(G) = \left\{ \begin{array}{c} V \\ \downarrow \\ N \xrightarrow{h} P \end{array} \right.$ over $\begin{array}{c} \mathbb{Q} \\ \downarrow \\ \mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \end{array} \left| \begin{array}{l} h \,\&\, v \text{ extensions of scalars} \\ \text{i.e. } \mathcal{E}_G^{-1}N \cong P \cong \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \end{array} \right\}$

triv. of $\mathcal{E}_G^{-1}N$.

Relation b/w $\mathcal{A}(G)$ & $\mathcal{A}^{sheaf}(G)$

$$N \in \mathcal{A}^{sheaf}(G) \dashrightarrow \begin{array}{ccc} N & \to & \mathcal{E}_G^{-1}V \\ \downarrow & & \downarrow \\ \prod_{i\geq 1} N(G_i) & \to & \mathcal{E}_G^{-1}\prod_{i\geq 1} N(G_i) \end{array} \in \mathcal{A}(G).$$

<u>Theorem</u> (Greenlees–Shipley): There is a Quillen equivalence

$$G\text{-spectra}/\mathbb{Q} \simeq DG\text{-}\mathcal{A}(G) \quad \text{for } G \text{ a torus.}$$

2. <u>Diagrams of rings & modules</u>: (towards Euler-Hasse-Tate square!)

$$\mathbb{Z}^{\square} := \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \hat{\mathbb{Z}}_p & \longrightarrow & (\prod_p \hat{\mathbb{Z}}_p)\otimes\mathbb{Q} \end{array}$$

$\longleftarrow$ understand ab. grps via

Abgrps $\simeq \mathbb{Z}$-mod-abgrps

extend scalars to get mod. over $\mathbb{Z}^{\square}$ diagram

$\simeq \mathbb{Z}^{\square}$-cell $\mathbb{Z}^{\square}$-mod-abgrps

$$\mathbb{Z}^{\lrcorner} = \begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow \\ \prod_p \hat{\mathbb{Z}}_p & \longrightarrow & (\prod_p \hat{\mathbb{Z}}_p)\otimes\mathbb{Q} \end{array}$$

$\simeq \mathbb{Z}^{\lrcorner}$-cell - $\mathbb{Z}^{\lrcorner}$-mod-abgrps

Generally: $\mathbb{D} \hookrightarrow \mathbb{E}$ diagrams, $R: \mathbb{E} \to \mathcal{C}$, $R|_{\mathbb{D}}: \mathbb{D} \to \mathcal{C}$

Get restriction $i^*: R\text{-mod-}\mathcal{C} \to R|_{\mathbb{D}}\text{-mod-}\mathcal{C}$

<u>Case 1</u>: $(\cdot \hookrightarrow \square)$ i^* is right adjoint with left adjoint i_* (ext. of scalars).

$i_*\mathbb{D}$-cell$R$-mod-$\mathcal{C} \simeq \mathbb{D}$-cell-$R|_{\mathbb{D}}$-mod-$\mathcal{C}$ $\quad (\mathbb{D} \simeq R|_{\mathbb{D}}\text{-mod})$

<u>Case 2</u>: $(\lrcorner \hookrightarrow \square)$ i^* left adj. w/ right adj $i_!$

$i_!\mathbb{D}$-cell-R-mod-$\mathcal{C} \simeq \mathbb{D}$-cell-$R|_{\mathbb{D}}$-mod-$\mathcal{C}$

Proof of thm for G = circle grp:

$\tilde{E}\mathcal{F} \simeq S^{\infty V(G)} = \bigcup_{W^G = 0} S^W$

$$\begin{array}{ccc} \mathbb{S}^{\square} = & \mathbb{S} & \longrightarrow S^{\infty V(G)} \\ & \downarrow & \downarrow \\ & F(E\mathcal{F}_+, \mathbb{S}) & \longrightarrow DE\mathcal{F}_+ \wedge S^{\infty V(G)} \end{array}$$

$F(E\mathcal{F}_+, \mathbb{S}) = DE\mathcal{F}_+ = \prod_{F \in \mathcal{F}} DE\langle F \rangle \qquad E\langle 1 \rangle = EG$

G-spectra = $\mathbb{S}$-mod-G-spectra $\overset{\text{(ase 1)}}{\simeq} \mathbb{S}^{\square}$-mod-G-spectra-cell (cell. w/r/t images of all G/H_+'s.)

$\overset{\text{(ase 2)}}{\simeq}$ cell-$\mathbb{S}^{\perp}$-mod-G-spectra

fixed pts $\simeq$ cell-$(\mathbb{S}^{\perp})^G$-mod-spectra

$\simeq$ cell-$\mathcal{O}(\mathbb{S}^{\perp})^G$-mod-$\mathbb{Q}$-mod (algebraicization)

NB. $\pi_*(\mathbb{S}^{\perp})^G = \mathbb{Q} = \mathcal{O}$ $\simeq$ cell-$\mathcal{O}$-mod-$\mathbb{Q}$-mod

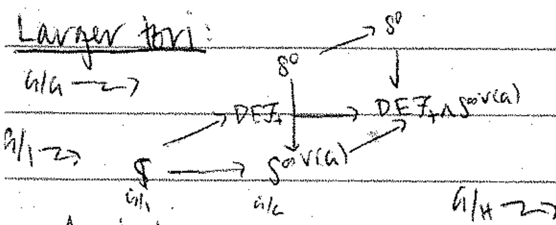
$\prod_{F \in \mathcal{F}} H^*(BG/F_+) \to \mathcal{E}^{-1}_G \prod_{F \in \mathcal{F}} H^*(BG/F_+)$ $\simeq \mathcal{A}(G)$ ~~torsion~~

, can make intrinsically formal

Lies/Warnings: $S^{\infty V(G)}$ not commutative ring. (no mult. norm maps)

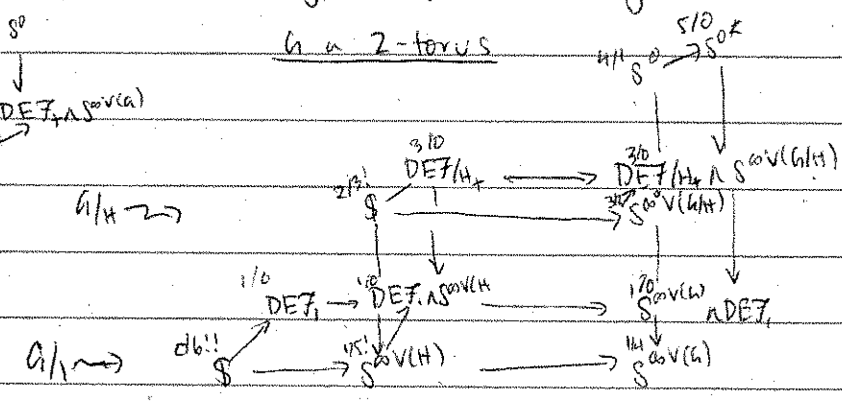
Replace "$S^{\infty V(G)}$-mod" by $L_{S^{\infty V(G)}}$(G-spectra) + model structure on suitable diagrams of model categories related by left Quillen functors.

Larger tori:



G circle:

G a 2-torus



1st increase from 0 to whole diagram (top numbers)

then go from back face to G!!

H a circle in G
secretly countably many H's