A MODEL STRUCTURE ON $G\mathcal{C}at$

ANNA MARIE BOHMANN, KRISTEN MAZUR, ANGÉLICA M. OSORNO, VIKTORIYA OZORNOVA, KATE PONTO, AND CAROLYN YARNALL

Abstract. We define a model structure on the category $G\mathcal{C}at$ of small categories with an action by a finite group G by lifting the Thomason model structure on $\mathcal{C}at$. We show there is a Quillen equivalence between $G\mathcal{C}at$ with this model structure and $G\mathcal{T}op$ with the standard model structure.

Introduction

There are familiar adjunctions

$$\mathcal{C}at \underset{c}{\overset{N}{\rightleftarrows}} s\mathcal{S}et \underset{|-|}{\overset{S_\bullet(-)}{\leftrightarrows}} \mathcal{T}op$$

between the categories of categories, simplicial sets, and topological spaces, and for the standard model structure on $s\mathcal{S}et$ and the Quillen model structure on $\mathcal{T}op$ the adjunction on the right is a Quillen equivalence. In [10] Thomason defined a model structure on $\mathcal{C}at$ and showed that the adjunction

$$\mathcal{C}at \underset{cSd^2}{\overset{Ex^2N}{\rightleftarrows}} s\mathcal{S}et$$

is a Quillen equivalence. In Thomason's model structure a functor $F\colon \mathcal{A} \to \mathcal{B}$ is a weak equivalence if $Ex^2N(F)$ is a weak equivalence in $s\mathcal{S}et$ or, equivalently, BF is a weak equivalence of topological spaces. A functor F is a fibration if $Ex^2N(F)$ is a fibration in $s\mathcal{S}et$. As shown in [2], this model structure is cofibrantly generated.

In this paper we use results by Stephan [9] to extend Thomason's model structure to the category of categories with an action by a finite group G. We let BG be the category with one object and endomorphisms given by the group G and define the **category of G objects** in a category $\mathcal{C}$, denoted by $G\mathcal{C}$, to be the category of functors

$$BG \to \mathcal{C}$$

and natural transformations. If $\mathcal{C}$ is a model category we can define a model structure on $G\mathcal{C}$ where the fibrations and weak equivalences are maps that are fibrations or weak equivalences in $\mathcal{C}$. Unfortunately, this perspective does not capture the desired homotopy theory. This is perhaps most familiar in the case of $G\mathcal{T}op$, where the desired notion of G-weak equivalence is a map that induces a non-equivariant weak equivalence on fixed point spaces for all subgroups of G.

Given a subgroup H of G, we have a functor $(-)^H\colon G\mathcal{C} \to \mathcal{C}$ defined by $X^H = \lim_{BH} X$. This notion coincides with the usual definition of the fixed point functor in the case that $\mathcal{C}$ is any of $\mathcal{S}et$, $\mathcal{T}op$, $s\mathcal{S}et$ or $\mathcal{C}at$. Let $\mathscr{O}_G$ be the orbit category of G; it has objects the orbits

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G/H for all subgroups H and morphisms all equivariant maps. Then an object $X \in G\mathcal{C}$ defines a functor

$$\Phi(X)\colon \mathscr{O}_G^{op} \to \mathcal{C}$$

by $\Phi(X)(G/H) = X^H$. If we let $\mathscr{O}_G$-$\mathcal{C}$ be the category of functors

$$\mathscr{O}_G^{op} \to \mathcal{C}$$

we can define a functor $\Phi\colon G\mathcal{C} \to \mathscr{O}_G\text{-}\mathcal{C}$ as above. The functor Φ has a left adjoint

$$\Lambda\colon \mathscr{O}_G\text{-}\mathcal{C} \to G\mathcal{C},$$

defined by $\Lambda(Y) = Y(G/e)$, where the G-action is inherited from the automorphisms of the object G/e in $\mathscr{O}_G$.

If $\mathcal{C}$ is a cofibrantly generated model category, such as $\mathcal{T}op$, $s\mathcal{S}et$ or Thomason's model structure on $\mathcal{C}at$, there is a model structure on $\mathscr{O}_G$-$\mathcal{C}$ where the fibrations and weak equivalences are defined levelwise. This is the **projective model structure** on the category $\mathscr{O}_G$-$\mathcal{C}$. For the category of topological spaces, or simplicial sets, this model structure captures the desired equivariant homotopy type.

For some categories $\mathcal{C}$ we can use the functor Φ to lift the projective model structure from $\mathscr{O}_G$-$\mathcal{C}$ to $G\mathcal{C}$. Then a map in $G\mathcal{C}$ is a fibration or weak equivalence if it is one after applying Φ. In the case of topological spaces this is the usual model structure on $G\mathcal{T}op$ [7, III.1.8]. In [1], Elmendorf constructed a functor $\mathscr{O}_G\text{-}\mathcal{T}op \to G\mathcal{T}op$ that was an inverse of Φ up to homotopy, thus showing that the homotopy categories of $G\mathcal{T}op$ and $\mathscr{O}_G$-$\mathcal{T}op$ were equivalent. Later Piacenza [8] showed that the adjunction given by Φ and Λ is a Quillen equivalence if $G\mathcal{T}op$ has this model structure and $\mathscr{O}_G$-$\mathcal{T}op$ has the projective model structure. Note that Elmendorf's functor can be thought of as the composition of the cofibrant replacement in $\mathscr{O}_G$-$\mathcal{T}op$ followed by Λ.

In this paper we prove a similar result for $\mathcal{C}at$.

Theorem A. *If G is a finite group there is a model structure on $G\mathcal{C}at$ where a functor is a fibration or weak equivalence if it is so after applying Φ. Using this model structure the Λ–Φ adjunction is a Quillen equivalence between $G\mathcal{C}at$ and $\mathscr{O}_G$-$\mathcal{C}at$.*

More can be said about this model structure. Since $\mathscr{O}_G$-$\mathcal{C}$ and $G\mathcal{C}$ are both diagram categories, an adjunction $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ defines adjunctions

$$L_*\colon \mathscr{O}_G\text{-}\mathcal{C} \rightleftarrows \mathscr{O}_G\text{-}\mathcal{D}\colon R_* \text{ and } L_*\colon G\mathcal{C} \rightleftarrows G\mathcal{D}\colon R_*$$

and so the classical adjunctions relating $\mathcal{C}at$, $s\mathcal{S}et$, and $\mathcal{T}op$ define adjunctions

$$\begin{tikzcd}
G\mathcal{C}at \arrow[r, shift left, "Ex^2N"] \arrow[d, shift left, "\Phi"] & Gs\mathcal{S}et \arrow[l, shift left, "cSd^2"] \arrow[r, shift right, "|-|"'] \arrow[d, shift left, "\Phi"] & G\mathcal{T}op \arrow[l, shift right, "S_\bullet(-)"'] \arrow[d, shift left, "\Phi"] \\
\mathscr{O}_G\text{-}\mathcal{C}at \arrow[r, shift left, "Ex^2N"] \arrow[u, shift left, "\Lambda"] & \mathscr{O}_G\text{-}s\mathcal{S}et \arrow[l, shift left, "cSd^2"] \arrow[r, shift right, "|-|"'] \arrow[u, shift left, "\Lambda"] & \mathscr{O}_G\text{-}\mathcal{T}op. \arrow[l, shift right, "S_\bullet(-)"'] \arrow[u, shift left, "\Lambda"]
\end{tikzcd}$$

The usual Quillen equivalences between $\mathcal{C}at$, $s\mathcal{S}et$ and $\mathcal{T}op$ are known to induce Quillen equivalences between $\mathscr{O}_G$-$\mathcal{C}at$, $\mathscr{O}_G$-$s\mathcal{S}et$ and $\mathscr{O}_G$-$\mathcal{T}op$.

Theorem B. *The adjunctions in the top row of the diagram above are Quillen equivalences.*

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1. Model structures on G-categories

Let $\mathcal{C}$ be a cofibrantly generated model category. To lift the model structure from $\mathcal{C}$ to the category $G\mathcal{C}$ we need some compatibility between the model structure on $\mathcal{C}$ and the group action. The relevant notion of compatibility is captured using the fixed point functors.

Definition 1.1. A fixed point functor $(-)^H\colon G\mathcal{C} \to \mathcal{C}$ is **cellular** if

(1) it preserves directed colimits of diagrams where each arrow is a non-equivariant cofibration after applying the forgetful functor $G\mathcal{C} \to \mathcal{C}$,
(2) it preserves pushouts of diagrams where one leg is given by
$$G/K \otimes f\colon G/K \otimes A \to G/K \otimes B$$
for some closed subgroup K of G and a cofibration $f\colon A \to B$ in $\mathcal{C}$, and
(3) for any closed subgroup K of G and any object A of $\mathcal{C}$ the induced map
$$(G/K)^H \otimes A \to (G/K \otimes A)^H$$
is an isomorphism in $\mathcal{C}$.

Note that since $\mathcal{C}$ is cocomplete, for a G-set X and an object A of $\mathcal{C}$ we have the categorical tensor $X \otimes A$ which is the G-object $\coprod_X A$ with G-action induced by the G-action on X.

In [9], Stephan gives conditions to lift a model structure from $\mathscr{O}_G$-$\mathcal{C}$ to $G\mathcal{C}$.

Theorem 1.2. [9, Theorem 1.2] *Let G be a discrete group, $\mathcal{C}$ be a model category which is cofibrantly generated and assume*

- *for any subgroup $H \leq G$ the H-fixed point functor $(-)^H\colon G\mathcal{C} \to \mathcal{C}$ is cellular and*
- *for all $H, K \leq G$ the functor $(G/K)^H \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations.*

Then there is a ***fixed point model structure*** *on $G\mathcal{C}$ where a map f in $G\mathcal{C}$ is a fibration or weak equivalence if and only if $\Phi(f)$ is a fibration or weak equivalence in the projective model structure on $\mathscr{O}_G$-$\mathcal{C}$. Additionally, there is a Quillen equivalence*
$$\Lambda\colon \mathscr{O}_G\text{-}\mathcal{C} \rightleftarrows G\mathcal{C} : \Phi$$
between $\mathscr{O}_G$-$\mathcal{C}$ with the projective model structure and $G\mathcal{C}$ with this model structure.

This theorem can be made functorial with respect to Quillen adjunctions.

Theorem 1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be cofibrantly generated model categories satisfying the hypotheses of Theorem 1.2. If*
$$L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$$

is a Quillen adjunction (resp. Quillen equivalence) then there is an induced Quillen adjunction (resp. Quillen equivalence)

$$L_*\colon G\mathcal{C} \rightleftarrows G\mathcal{D} : R_*$$

where $G\mathcal{C}$ and $G\mathcal{D}$ have fixed point model structures.

Proof. To show we have a Quillen adjunction it is enough to show that $R_*\colon G\mathcal{D} \to G\mathcal{C}$ is a right Quillen functor, that is, to show that R_* preserves fibrations and acyclic fibrations. We will show R_* preserves fibrations; the case for acyclic fibrations is similar.

Let $f\colon X \to Y$ be a fibration in $G\mathcal{D}$. Since $G\mathcal{D}$ has the fixed point model structure, fibrations are created in $\mathscr{O}_G$-$\mathcal{D}$. Thus $\Phi f\colon \Phi X \to \Phi Y$ is also a fibration in $\mathscr{O}_G$-$\mathcal{D}$. By assumption, $R\colon \mathscr{O}_G$-$\mathcal{D} \to \mathscr{O}_G$-$\mathcal{C}$ is right Quillen and thus $R\Phi f\colon R\Phi X \to R\Phi Y$ is a fibration in $\mathscr{O}_G$-$\mathcal{C}$.

As a right adjoint, R commutes with limits. Thus for any $H \le G$ and $X\colon BG \to \mathcal{D}$,

$$\Phi(R_*X)(G/H) = (R_*X)^H = \lim_{BH} RX = R\lim_{BH} X = R(\Phi(X)(G/H)).$$

By definition we then have $R(\Phi X(G/H)) = (R_*\Phi X)(G/H)$. This means that $\Phi RX \xrightarrow{\Phi Rf} \Phi RY$ is a fibration, and thus, since fibrations in $G\mathcal{C}$ are created under Φ, $RX \xrightarrow{Rf} RY$ is a fibration.

Suppose $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a Quillen equivalence. To show the adjunction $G\mathcal{C} \rightleftarrows G\mathcal{D}$ is a Quillen equivalence, we apply the 2-out-of-3 property for Quillen equivalences [5, Corollary 1.3.15]. We then have a diagram of Quillen adjunctions, in which both the diagrams of the left adjoints and the right adjoints commute,

$$\begin{tikzcd} G\mathcal{C} \arrow[r, shift left, "L_*"] \arrow[d, shift left, "\Phi"] & G\mathcal{D} \arrow[l, shift left, "R_*"] \arrow[d, shift left, "\Phi"] \\ \mathscr{O}_G\text{-}\mathcal{C} \arrow[r, shift left, "L_*"] \arrow[u, shift left, "\Lambda"] & \mathscr{O}_G\text{-}\mathcal{D} \arrow[l, shift left, "R_*"] \arrow[u, shift left, "\Lambda"] \end{tikzcd}$$

such that bottom and two side adjunctions are Quillen equivalences. Thus the top adjunction must be a Quillen equivalence as well. □

After we verify that $\mathcal{C}at$ satisfies the conditions of Theorem 1.2 in the next section, Theorem 1.3 completes the proof of Theorem B.

We now record that Stephan's construction preserves right properness.

Proposition 1.4. *Let $\mathcal{C}$ be a cofibrantly generated model category that is right proper and satisfies the conditions of Theorem 1.2. Then the fixed point model structure on $G\mathcal{C}$ is right proper.*

Proof. Suppose $\mathcal{C}$ is right proper and consider a pullback diagram in $G\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d] & Y \arrow[d, "h"] \\ Z \arrow[r, "f"'] & W \end{tikzcd}$$

where h is a fibration and f is a weak equivalence. We must show that f' is also a weak equivalence. Since weak equivalences and fibrations in $G\mathcal{C}$ are created by the functor $\Phi\colon G\mathcal{C} \to \mathscr{O}_G\text{-}\mathcal{C}$ and $\mathscr{O}_G\text{-}\mathcal{C}$ is right proper [4, Thm. 13.1.14] this follows from the fact that Φ is a right adjoint and thus commutes with pullbacks. $\square$

To apply Theorem 1.2 to the category $\mathcal{C}at$ and a finite group G, we will show this category and its fixed point functors satisfy conditions that imply the fixed point functors are cellular.

Proposition 1.5. *Let G be a discrete group, H be a subgroup, and $\mathcal{C}$ be a cofibrantly generated model category. Assume the H-fixed point functor $(-)^H\colon G\mathcal{C} \to \mathcal{C}$*

(1) *preserves all filtered colimits,*
(2) *preserves pushouts of diagrams where one leg is given by*
$$G/K \otimes f\colon G/K \otimes A \to G/K \otimes B$$
for a subgroup K of G and a generating cofibration $f\colon A \to B$ in $\mathcal{C}$, and
(3) *for any subgroup K of G and any object A of $\mathcal{C}$ the induced map*
$$(G/K)^H \otimes A \to (G/K \otimes A)^H$$
is an isomorphism in $\mathcal{C}$.

Then the H-fixed point functor is cellular.

We postpone the proof to §3, but first observe that it allows us to prove a dual result to Proposition 1.4. This proof is also postponed to §3.

Proposition 1.6. *Let $\mathcal{C}$ be a cofibrantly generated model category that is left proper and satisfies the second condition of Theorem 1.2 and the conditions of Proposition 1.5. Then the fixed point model structure on $G\mathcal{C}$ is left proper.*

2. The model category $G\mathcal{C}at$

In this section we will show that $\mathcal{C}at$ satisfies the hypotheses of Proposition 1.5 and Theorem 1.2 proving Theorem A. We start by giving an explicit description of the cofibrations in Thomason's model structure on $\mathcal{C}at$.

Theorem 2.1. [2, Thm. 6.3] *The Thomason model structure on $\mathcal{C}at$ is cofibrantly generated with generating cofibrations*
$$\{c\mathrm{Sd}^2\partial\Delta[m] \to c\mathrm{Sd}^2\Delta[m] \mid m \geq 0\}$$
and generating acyclic cofibrations
$$\{c\mathrm{Sd}^2\Lambda^k[m] \to c\mathrm{Sd}^2\partial\Delta[m] \mid m \geq 1 \text{ and } 0 \leq k \leq m\}.$$
Here c is the left adjoint of the nerve functor and Sd *is barycentric subdivision.*

To verify the conditions of Theorem 1.2 and Proposition 1.5 we will consider a more general collection of maps, the *Dwyer maps*, rather than working directly with these generating cofibrations and acyclic cofibrations.

Recall that subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is a **sieve** if for every morphism $\beta\colon b \to a$ in $\mathcal{B}$ with target a in $\mathcal{A}$, both the object b and the morphism β lie in $\mathcal{A}$. A **cosieve** is defined dually.

Definition 2.2. [10] A sieve inclusion $\mathcal{A} \to \mathcal{B}$ is a **Dwyer map** if there is a cosieve $\mathcal{W}$ in $\mathcal{B}$ containing $\mathcal{A}$ so that the inclusion functor $i\colon \mathcal{A} \to \mathcal{W}$ admits a right adjoint $r\colon \mathcal{W} \to \mathcal{A}$ satisfying $ri = \mathrm{id}_{\mathcal{A}}$ and the unit of this adjunction is the identity.

Note that the generating cofibrations and acyclic cofibrations are Dwyer maps of posets.

We consider the conditions of Theorem 1.2 for the case $\mathcal{C} = \mathcal{C}at$ and G a finite group. For $\mathcal{C} = \mathcal{C}at$, the first condition will follow from Proposition 1.5. The second condition holds automatically: note first that $G/K \otimes A$ is simply the product of the discrete category G/K and A, with G-action concentrated in the G/K factor. Direct product with any discrete category preserves weak equivalences and Dwyer maps, and thus tensoring with $(G/K)^H$ preserves cofibrations and acyclic cofibrations.

We next show $\mathcal{C}at$ satisfies the conditions of Proposition 1.5. Condition 3 of Proposition 1.5 is satisfied because the action of G on $G/K \otimes A$ is entirely through the action of G on G/K. Conditions 1 and 2 are proved in the next two propositions.

Proposition 2.3. *Let H be a subgroup of G. The fixed point functor $(-)^H$ preserves all filtered colimits.*

Proof. Let I be a filtered category and F be a functor from I to $G\mathcal{C}at$. First note that $N\,\mathrm{colim}_I(F(i)^H) \cong \mathrm{colim}_I\left(N\left(F(i)^H\right)\right)$ since the nerve commutes with filtered colimits [6]. The nerve is a right adjoint and taking fixed points is a limit, so we have an isomorphism $N\left(F(i)^H\right) \cong (NF(i))^H$. Together these give an isomorphism

$$N\,\mathrm{colim}_I(F(i)^H) \cong \mathrm{colim}_I\left((NF(i))^H\right).$$

Finite limits and filtered colimits commute in $\mathcal{S}et$ and this extends to $s\mathcal{S}et$ since limits and colimits in $s\mathcal{S}et$ are computed levelwise. We thus have

$$\mathrm{colim}_I\left((NF(i))^H\right) \cong (\mathrm{colim}_I\, NF(i))^H \cong (N\,\mathrm{colim}_I\, F(i))^H.$$

Finally, we have an isomorphism $(N\,\mathrm{colim}_I\, F(i))^H \cong N\big(\mathrm{colim}_I\, F(i)\big)^H$ since the nerve is a right adjoint. Together this gives an isomorphism

$$N\,\mathrm{colim}_I(F(i)^H) \cong N\big(\mathrm{colim}_I\, F(i)\big)^H.$$

The nerve is fully faithful, so the result above implies $\mathrm{colim}_I(F(i)^H) \cong \big(\mathrm{colim}_I\, F(i)\big)^H$, completing the proof. $\square$

We will verify the second condition in Proposition 1.5 for Dwyer maps of posets since they allow simple descriptions of pushouts of categories.

Proposition 2.4. *Let $\mathcal{A} \to \mathcal{B}$ be a Dwyer map of posets and suppose the diagram*

$$\begin{tikzcd} G/K \times \mathcal{A} \arrow[r] \arrow[d, "F"'] & G/K \times \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

is a pushout diagram in $G\mathcal{C}at$. Then this diagram remains a pushout after taking H-fixed points.

The proof of this proposition is based on a very explicit description of the morphisms in $\mathcal{D}$. We give that description first and then continue to the proof of the proposition.

Lemma 2.5. *Let $i\colon \mathcal{A} \to \mathcal{B}$ be a Dwyer map between posets with cosieve $\mathcal{W}$ and retraction r, and let $F\colon \mathcal{A} \to \mathcal{C}$ be any functor. If $\mathcal{D}$ is the pushout of i and F, the set of objects of $\mathcal{D}$ can be identified with*

$$ob(\mathcal{C}) \amalg (ob(\mathcal{B}) \setminus ob(\mathcal{A}))\,.$$

If c is an object of $\mathcal{C}$ and b is an object of $\mathcal{B}$ that is not an object of $\mathcal{A}$, then

$$\mathcal{D}(c,b) \cong \mathcal{C}(c, F(r(b))),$$

if b is in $\mathcal{W}$, and is otherwise empty.

Proof. The proof of [3, Proposition 5.2] gives a simple description for the pushout $\mathcal{D}$ of a full inclusion $i\colon \mathcal{A} \to \mathcal{B}$ and a functor $F\colon \mathcal{A} \to \mathcal{C}$. In the case when i is a sieve, the description is as follows. The objects of $\mathcal{D}$ are $\mathrm{ob}(\mathcal{C}) \amalg (\mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}))$ and some of the morphisms are given by

$$\mathcal{D}(d,d') = \begin{cases} \mathcal{B}(d,d') & \text{if } d,d' \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}), \\ \mathcal{C}(d,d') & \text{if } d,d' \in \mathrm{ob}(\mathcal{C}), \\ \emptyset & \text{if } d \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}) \text{ and } d' \in \mathrm{ob}(\mathcal{C}). \end{cases}$$

For an object c of $\mathcal{C}$ and an object b of $\mathcal{B}$ not in $\mathcal{A}$, the morphisms from c to b in $\mathcal{D}$ are equivalence classes of pairs (β, γ) where β is a morphism $a \to b$ in $\mathcal{B}$ for some $a \in \mathcal{A}$ and γ is a morphism $c \to F(a)$. The equivalence relation on these pairs is generated by $(\beta\alpha, \gamma) \sim (\beta, F(\alpha)\gamma)$ for α in $\mathcal{A}$, whenever the compositions in question are defined. The equivalence relation is compatible with composition.

Now assume that $\mathcal{A} \to \mathcal{B}$ is a Dwyer map between posets. We denote the counit of the adjunction between the inclusion $\mathcal{A} \to \mathcal{W}$ and the retraction r by ε. If (β, γ) is a pair of morphisms as above, then β is in $\mathcal{W}$ by the definition of cosieve, and

$$\beta = \varepsilon_b r(\beta)$$

since the source of β is in $\mathcal{A}$. Since $r(\beta) \in \mathcal{A}$, (β, γ) is equivalent to $(\varepsilon_b, F(r(\beta))\gamma)$ and, as the reader can check, every equivalence class has a unique representative of the form (ε_b, γ). $\square$

Proof of Proposition 2.4. We must show that if the diagram on the left is a pushout and $\mathcal{A} \to \mathcal{B}$ is a Dwyer map of posets then the diagram on the right is also a pushout.

$$\begin{tikzcd} G/K \times \mathcal{A} \arrow[r] \arrow[d, "F"'] & G/K \times \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} (G/K)^H \times \mathcal{A} \arrow[r] \arrow[d, "F"'] & (G/K)^H \times \mathcal{B} \arrow[d] \\ \mathcal{C}^H \arrow[r] & \mathcal{D}^H \end{tikzcd}$$

First observe that since $G\mathcal{C}at$ is a diagram category, the pushout is computed in the underlying category $\mathcal{C}at$. The objects of $\mathcal{D}$ are

$$\mathrm{ob}(\mathcal{C}) \amalg \big(G/K \times \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A})\big)$$

so the objects of $\mathcal{D}^H$ are given by $\mathrm{ob}(\mathcal{C})^H \amalg \big((G/K)^H \times (\mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}))\big)$. The objects of the pushout $\mathcal{P}$ of $(G/K)^H \times \mathcal{A} \to (G/K)^H \times \mathcal{B}$ and F^H are identical to those in $\mathcal{D}^H$ and the induced map from this pushout to $\mathcal{D}^H$ is an isomorphism on objects.

For morphisms, observe that $G/K \times \mathcal{A} \to G/K \times \mathcal{B}$ and $(G/K)^H \times \mathcal{A} \to (G/K)^H \times \mathcal{B}$ are Dwyer maps between posets, so we can apply Lemma 2.5. The morphisms of $\mathcal{D}$ are

$$\mathcal{D}(d,d') = \begin{cases} \{\mathrm{id}_{gK}\} \times \mathcal{B}(b,b') & \text{if } d=(gK,b), d'=(gK,b') \text{ for} \\ & \ b,b' \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}), \\ \mathcal{C}(d,d') & \text{if } d,d' \in \mathrm{ob}(\mathcal{C}), \\ \{\mathrm{id}_{gK}\} \times \mathcal{C}(d, F(gK, r(b))) & \text{if } d \in \mathrm{ob}(\mathcal{C}) \text{ and } d' = (gK,b) \text{ for} \\ & \ b \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}), \\ \emptyset & \text{if } d=(gK,b) \text{ and } d' \in \mathrm{ob}(\mathcal{C}) \text{ for} \\ & \ b \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}). \end{cases}$$

and so for objects d and d' in $\mathcal{D}^H$ we have

$$\mathcal{D}^H(d,d') = \begin{cases} \{\mathrm{id}_{gK}\} \times \mathcal{B}(b,b') & \text{if } d=(gK,b), d'=(gK,b') \text{ for} \\ & \ gK \in (G/K)^H \text{ and } b,b' \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}), \\ \mathcal{C}^H(d,d') & \text{if } d,d' \in \mathrm{ob}(\mathcal{C}^H), \\ \{\mathrm{id}_{gK}\} \times \mathcal{C}^H(d, F(gK, r(b))) & \text{if } d \in \mathrm{ob}(\mathcal{C}^H) \text{ and } d' = (gK,b) \text{ for} \\ & \ gK \in (G/K)^H \text{ and } b \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}). \\ \emptyset & \text{if } d=(gK,b) \text{ and } d' \in \mathrm{ob}(\mathcal{C}^H) \text{ for} \\ & \ gK \in (G/K)^H \text{ and } b \in \mathrm{ob}(\mathcal{B}) \setminus \mathrm{ob}(\mathcal{A}). \end{cases}$$

For the pushout $\mathcal{P}$, the analogous statement holds, and thus we have the same description for the morphism sets of $\mathcal{P}$ and $\mathcal{D}^H$ and the induced map $\mathcal{P} \to \mathcal{D}^H$ is an isomorphism on morphism sets. □

3. Cellular functors and left proper model structures

We now return to the proof of Proposition 1.5. We only need to show that condition (2) in Proposition 1.5 can be extended from generating cofibrations to all cofibrations. This is a direct consequence of the following lemma.

Lemma 3.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between cocomplete categories and I a set of morphisms of $\mathcal{C}$. If F preserves filtered colimits and pushouts along morphisms in I then F preserves pushouts along all retracts of transfinite compositions of pushouts of morphisms in I.*

Proof. Suppose we have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[r] \arrow[d, "i'"] & C' \arrow[d] \\ B \arrow[r] & D \arrow[r] & D' \end{tikzcd}$$

where both small squares are pushouts and i is in I. Then the exterior is a pushout. Applying F we see that both the left square and the outside rectangle remain pushouts. This implies the right square is a pushout, so F preserves pushouts along pushouts of morphisms in I.

Now suppose we have a λ-sequence $X\colon \lambda \to \mathcal{C}$ for some ordinal λ so that F preserves pushouts along all of $X_i \to X_{i+1}$. We will show that F preserves pushouts along the transfinite composition $X_0 \to \mathrm{colim}_\lambda X_\beta$.

We proceed by transfinite induction. Assume the claim is already proven for all ordinals smaller than λ. Recall that for any limit ordinal $\beta < \lambda$, the induced map $\operatorname{colim}_{i<\beta} X_i \to X_\beta$ is an isomorphism by the definition of a λ-sequence. Note furthermore that for a non-limit ordinal, say, $\beta + 1$, the indexing category $i < \beta + 1$ has the terminal object β, so $\operatorname{colim}_{i<\beta+1} X_i \to X_\beta$ is an isomorphism.

Assume first that $\lambda = \beta + 1$ is not a limit ordinal. Then we have a diagram of pushouts

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & \operatorname{colim}_{i<\beta} X_i \arrow[r] \arrow[d] & \operatorname{colim}_{i<\beta+1} X_i \arrow[d] \\
C \arrow[r] & D \arrow[r] & D'.
\end{tikzcd}$$

Since the map $\operatorname{colim}_{i<\beta} X_i \to \operatorname{colim}_{i<\beta+1} X_i$ is either an isomorphism or the given map $X_{\beta-1} \to X_\beta$ (where $\beta - 1$ denotes the predecessor of β in this case), the functor F preserves the smaller pushouts squares (for the left one, we use the induction hypothesis), and thus also the outer pushout rectangle.

Now assume that λ is a limit ordinal. Since colimits commute with each other we have

$$\operatorname{colim}(C \leftarrow X_0 \to \operatorname{colim}_\lambda X) \cong \operatorname{colim}_\lambda \operatorname{colim}(C \leftarrow X_0 \to X_\beta).$$

Using this observation and the assumption that F commutes with filtered colimits we obtain

$$F(\operatorname{colim}(C \leftarrow X_0 \to \operatorname{colim}_\lambda X)) \cong \operatorname{colim}_\lambda F(\operatorname{colim}(C \leftarrow X_0 \to X_\beta)).$$

The induction hypothesis allows us to replace the right hand side by

$$\operatorname{colim}_\lambda \operatorname{colim}(FC \leftarrow FX_0 \to FX_\beta)$$

and we can exchange the colimits to replace the colimit above by

$$\operatorname{colim} \operatorname{colim}_\lambda (FC \leftarrow FX_0 \to FX_\beta) \cong \operatorname{colim}(FC \leftarrow FX_0 \to \operatorname{colim}_\lambda FX_\beta).$$

Finally we observe that F preserves filtered colimits to see

$$\operatorname{colim}(FC \leftarrow FX_0 \to \operatorname{colim}_\lambda FX_\beta) \cong \operatorname{colim}(FC \leftarrow FX_0 \to F \operatorname{colim}_\lambda X_\beta).$$

For the last condition, suppose that $i'\colon B \to B'$ is a retract of a map $i\colon A \to A'$, and F preserves pushouts along i. Then we have a diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "i'"] & A \arrow[r] \arrow[d, "i"] & B \arrow[d, "i'"] \\
B' \arrow[r] & A' \arrow[r] & B'
\end{tikzcd}$$

where both horizontal composites are the identity. If we take the pushout of this diagram along a map $f\colon B \to C$ we obtain pushout squares

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "i'"] & C \arrow[d] \\
B' \arrow[r] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & C \arrow[d] \\
A' \arrow[r] & P
\end{tikzcd}$$

and the left hand square is a retract of the right hand square. Applying F to both squares preserves the retraction and the right pushout square. Since a retract of a pushout square is a pushout, F applied to the left pushout square is a pushout. $\square$

We also use this lemma in the proof of Proposition 1.6.

Proof of Proposition 1.6. First observe that since $\mathcal{C}$ is left proper $\mathscr{O}_G$-$\mathcal{C}$ is also left proper [4, Theorem 13.1.14].

Since $\mathcal{C}$ is cofibrantly generated both $\mathscr{O}_G$-$\mathcal{C}$ and $G\mathcal{C}$ are cofibrantly generated and generating cofibrations for both $G\mathcal{C}$ and $\mathscr{O}_G$-$\mathcal{C}$ can be defined in terms of the generating cofibrations of $\mathcal{C}$. In fact, we can choose generating cofibrations I for $G\mathcal{C}$ so that ΦI is a collection of generating cofibrations for $\mathscr{O}_G$-$\mathcal{C}$ [9]. Since Φ preserves retracts, filtered colimits, and pushouts along generating cofibrations, we see that Φ preserves cofibrations.

By assumption, the fixed point functor $(-)^H$ preserves pushouts along generating cofibrations in $G\mathcal{C}$, so by Lemma 3.1, it also preserves pushouts along all cofibrations. It follows that the functor Φ also preserves pushouts along cofibrations.

Consider a pushout diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ Z \arrow[r] & W \end{tikzcd}$$

in $G\mathcal{C}$ where f is a weak equivalence and h is a cofibration. Applying Φ we have a pushout diagram in $\mathscr{O}_G$-$\mathcal{C}$ and by construction of the model structure on $G\mathcal{C}$, $\Phi(f)$ is a weak equivalence. By the observations above $\Phi(h)$ is a cofibration. It follows that $\Phi(f')$ is a weak equivalence in $\mathscr{O}_G$-$\mathcal{C}$, so by definition f' is a weak equivalence in $G\mathcal{C}$. $\square$

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Department of Mathematics, Northwestern University, Evanston, IL 60208, USA
E-mail address: bohmann@math.northwestern.edu

Department of Mathematics, Lafayette College, Easton, PA 18042, USA
E-mail address: mazurk@lafayette.edu

Department of Mathematics, Reed College, Portland, OR 97202, USA
E-mail address: aosorno@reed.edu

Fachbereich Mathematik, Universität Bremen, 28359 Bremen, Germany
E-mail address: ozornova@math.uni-bremen.de

Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
E-mail address: kate.ponto@uky.edu

Department of Mathematics, Wabash College, Crawfordsville, IN 47933, USA
E-mail address: yarnallc@wabash.edu