Bjorn Dundas: 2/13/2012

1.) "Why does the standard smash product fail to give a model for THH?"

2.) "What part of S^1 is important in the eyes of K-theory?"

3.) "What are the slices of THH?"

$$\begin{tikzcd} K \arrow[r] \arrow[d] & TC^\wedge \arrow[d] \\ THH^{hS^1} \arrow[r] & THH^{hS^1 \wedge} \end{tikzcd}$$

← genuine structure of the finite subgrps of S^1

↑ naive S^1

homotopy Cartesian in some important cases

A commutative orthogonal ring spectrum (connective)

$$\begin{tikzcd} K(A) \arrow[r] & TC(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\cdot\!\lrcorner"] & THH(A)^{hS^1} \arrow[r] \arrow[d] & THH(A) \\ & TC(A)^\wedge \arrow[r] & THH(A)^{hS^1 \wedge} & \end{tikzcd}$$

} maps of comm. orthog. ring spectra

Rognes' redshift conjecture:

What are the conditions on A so that

telecom $K(A)$ = telecom A +1 (telecom = telescopic complexity)

Ex: $K(\mathbb{F}_p)_p \overset{\text{Quill}}{\simeq} H\mathbb{Z}_p$ $\mathbb{Z}_p$ telecom 0

$K(K(\mathbb{F}_p)_p)_p \overset{\text{BM}}{\simeq} J_p \times BJ_p \times SU_p$ $\supseteq v_1$ telecom 1

$K(K(K(\mathbb{F}_p)_p)_p)_p$ $\supseteq v_2$ Ausoni theorem telecom 2

} come from TC

$\pi_0 A = \mathbb{Z}_p$

$KA_p \to TCA_p \to \Sigma^{-1} H\mathbb{Z}_p$

Hesselholt–Madsen.

$TC \cdots TC$ bad functor (mixes limits & colimits :)) ← only for detecting!

pull out all the limits ; $((A \wedge A)^{C_2} \wedge (A \wedge A)^{C_2})^{C_2}$ vs. $(A \wedge A \wedge A \wedge A)^{C_2 \times C_2}$

$THH(A) = A \otimes S^1$

$THH \cdots THH(A) = (A \otimes S^1) \otimes \cdots \otimes S^1 = A \otimes T^n$

$$\begin{tikzcd} K \cdots K(A) \arrow[r] \arrow[d] & TC \cdots TC(A) \arrow[d] \\ ? \arrow[r] & GL_n\mathbb{Q}_p/_{(p)} \circlearrowright T^n C(A, P) \end{tikzcd}$$

covering homotopy finite subgrps of T^n

$P : T^n \to T^n$

constructing $T^n C(A,P)$:

$$\underset{\substack{\nearrow \\ \text{space}}}{X} \longmapsto \wedge_X A \qquad G \text{ finite, } X \text{ free } G\text{-space}$$

Bö-type construction.

$[\wedge_X A]^G$. built from ~~{~~ $\{(\wedge_{X_H} A)_{hW_G H}\}$ conj. classes $H \leq G$.

How are these assembled?

"Ghost map" $[\wedge_X A]^G \longrightarrow \prod_{\substack{H \leq G \\ \text{conj}}} \wedge_{X/H} A$

$$\underset{X \text{ connected}}{\pi_0 [\wedge_X A]^G} = W_G \pi_0 A \qquad \text{(Burnside-Witt)}$$

Often compared via the Hasse square:

$$\begin{array}{ccc} [\wedge_X A]^G & \longrightarrow & \wedge_X A^{\,hG} \\ \downarrow r & & \\ [\wedge_X A]^{\,hG} & \longrightarrow & \wedge_X A^{tG} \end{array}$$

Ex: 1.) $A = \mathbb{Z}_p$, $X = S^1$ BM & Tsalidis Γ peq. in pos. degrees

2.) SW + R: X finite, A finite type Γ peq.

3.) $(\wedge_{T^2} H\mathbb{F}_2)^{hT^2}$ detects v_1 Rognes

$(\wedge_{T^2} H\mathbb{Z}_p)^{hT^2}$ " v_2 Veen

$\rightarrow$ maybe there is a next step.

Categorical smash ~~product~~ powers: [Stolz]

G compact Lie.

$$\text{orth. } G\text{-spectra} \xrightarrow[\;ev_V\;]{\;\mathcal{G}_V\;} O_V \rtimes G\text{-space}$$

$$\underset{X}{\overset{\cup}{}} \longmapsto X_V \qquad O_V \rtimes G \text{ space, since } G \to O_V.$$

- want to lift model structure from space level & stablize

$\rightarrow$ need to take care about what equivalences to use on the spaces.

Choose a family of subgroups of G, $\mathcal{H}$.

$$\begin{array}{ccc} \mathcal{F} \subseteq \{P \subseteq O_V \rtimes G\} & & O_V \rtimes H \\ \downarrow \qquad \text{inj} \searrow \; \downarrow & & \text{sect.} \uparrow \\ \qquad\quad G & & H \\ \mathcal{H} \subseteq \{\text{subgrp of } G\}. & & \end{array}$$

$O_V \rtimes G$-spaces: model structure

Weak eq. = $\mathcal{F}$-w.e. equiv. cof. with all orbit types.

cof = genuine cof ← mixed model structure. (in sense of Shipley)

fib = RLP

Gives level structure for orthog. G-spectra: (nb doesn't depend on universe yet.)

we/fib = we/fib in each level

cof = LLP.

compactly gen. model cat.

Stabilize: w.e. π_*-isomorphism: π_*^H-isos $H \in \mathcal{H}$.

cof = level cof

fib = RLP.

Theorem (Stolz) positive (wrt fixed points?)

Commutative orthogonal G-ring spectra $\underset{\text{forget}}{\overset{\text{free}}{\rightleftarrows}}$ orthog. G-spectra

w.e./fib = w.e/fib in orthog. G-spectra

· Get compactly generated model structure.

· forget preserves cofibration.

Filtration of smash powers:

G finite. S finite free G-set

M cellular orthog. spectrum.

$M^{\wedge S}$ cellular G-orthog. spectrum.

$G_+ \wedge_{H_*} g_V((S^{n-1} \subseteq D^n) \times O_V)^{\square H}$ ← give the type of the cells.

$H \leq G$ V $P \leq O_V$

so: $M^{\wedge S/H} \cong (M^{\wedge S})^{\Phi H}$

Enough functoriality to push this up to tori

X naively cof. T^n-space $H \leq T^n$ finite.

A cof. comm. orthog. ring spectrum. Then: nat. isom of T^n/H-comm. orth. ring sp

$A \otimes X/H \cong (A \otimes X)^{\Phi H}$ upshot: Bökstedt-type construction agrees w/ categorical.

isotropy separation sequence:

$$(\Lambda_X A)_{hG} \longrightarrow [\Lambda_X A]^G \longrightarrow \underset{\overleftarrow{G \supseteq H \neq 0}}{\mathrm{holim}} \; [\Lambda_{X/H} A]^{G/H}$$

$$[\Lambda_X A]^G \cong [A \otimes X]^G$$

theorem of prev. equiv. $A \otimes X/H \cong (A \otimes X)^{gH}$ + induction