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Quillen's algebraic K-theory of fields $K(F)$

G_F acts on $K(\bar{F})$ $(*)\ KF \xrightarrow{?} K(\bar{F})^{hG_F}$ (Quillen-Lichtenbaum)

Suslin: $K(\bar{F})^\wedge_\ell \simeq ku^\wedge_\ell$ (ℓ prime, away from ...)

Spectral sequence $H^*(G_F; M_\ell^{\otimes n}) \Rightarrow K(\bar{F})^{hG_F}$

G_F ~~...~~ $\mathbb{Z}_\ell$ $\quad \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ ⟵ already problems here

$(*)$ is false on the nose statement

Bott periodic theory version: $K^{per}(F)^{hG_F} \cong K^{per}(F)$

- For this talk, all fields contain alg closed subfields ⟹ Bott elt exists completed at ℓ
- Proved by Thomason.

Wanted to get non-periodic statement.

- Bloch-Lichtenbaum construct SS $\Rightarrow K_* F$
- Bloch-Kato conjecture, Beilinson-Lichtenbaum conjecture

Upshot: There is a spectral sequence with E_2 depending only on G_F, converging to completed K-theory (at some ℓ)

- Complete calculation of $(KF)^\wedge_{\ell *}$

Question: Is there a model for KF built entirely out of G_F? (together with ku, ..., $K\bar{F}$ ...)

How might such a model look?

- Completions can give interesting models

Noetherian rings – completion mostly boring (alg. completion)

Non Noetherian rings

There is a derived notion of completion of commutative ring spectra.

$f: A \to B$ homomorph. of comm ring spectra, M A-module

$M^\wedge_B :=$ Tot cosimplicial spectrum of the monad for $B \underset{A}{\otimes} M$

(Amitsur completion)

Consider $R[\mathbb{Z}_p]$ a repr. ring

$R[\mathbb{Z}_p] = \bigcup_n R[\mathbb{Z}/p^n] = \mathbb{Z}[\chi(\mathbb{Z}_p)]$ ⟵ cnts. character group

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathbb{Z}[\mathbb{Z}/p^\infty]}$

$$\begin{tikzcd} R[\mathbb{Z}_p] \arrow[r, "\text{aug}"] \arrow[rr, bend right, "\ell"'] & \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

What is $R[\mathbb{Z}_p]^\wedge_p$?

Homotopy SS has a simple form:

· Tensor prod. of E_2 term for $\mathbb{Z} \to \mathbb{F}_p$ with the mod-p homology Eilenberg-Moore SS for $B\mathcal{X}(\mathbb{Z}_p) \xrightarrow{i} BS^1$

· $H_*(i, \mathbb{F}_p)$ is iso

· same as Eilenberg-Moore SS for $\mathbb{C}P^\infty$ completed at p

· Hence: $\pi_*(R[\mathbb{Z}_p]^\wedge_p) \cong \mathbb{Z}_p$ for $* = 0, 1$; 0 otherwise.

$\mathcal{X}(\mathbb{Z}_p) \longrightarrow \mathcal{X}(\mathbb{Z})$

$\mathbb{Z}[\mathcal{X}(\mathbb{Z}_p)] \longrightarrow \mathbb{Z}[\mathcal{X}(\mathbb{Z})]$ is iso after completion at mod p augmentation

<u>Tyler's thesis</u>: Consider N fin. gen. nilpotent grp.

Form completion at p $N^\wedge_p$.

$N \longrightarrow N^\wedge_p$.

$R[N^\wedge_p] \longrightarrow R[N]$ ← topological ring — has stable info. about rep'n varieties.

Tyler shows: $R[N^\wedge_p]^\wedge_p \longrightarrow R[N]^\wedge_p$ is an equivalence.

<u>Representational Assembly</u>

Descent data

$$\begin{array}{c} \bar{F} \\ G_F \big| \\ F \end{array}$$

(V, α), where V is an $\bar{F}$-vector space

α is an action of G_F on V

$\alpha(g)(\bar{f} v) = \bar{f}^g \alpha(v)$ ($\bar{F}$ "semilinear")

Serre - Hilbert 90: Category of descent data $\simeq$ Cat of Vect_F

Call cat. of descent data $\mathcal{DD}_F$.

Thus $K\mathcal{DD}_F \simeq KF$

$F \supset k$ alg. closed. (k char. not dividing thing in G_F)

Consider k-linear rep'ns of G_F. $\mathrm{Rep}_k[G_F]$ (cont. rep'n)

$K_0(\mathrm{Rep}_k[G_F]) = \mathrm{Rep}[G_F]$

$\mathrm{Rep}_k[G_F] \longrightarrow \mathcal{DD}_F$

W with G_F-action $\rightsquigarrow \bar{F} \otimes_F W$ w/ diagonal action

$$\begin{tikzcd} K\mathrm{Rep}_k[G_F] \arrow[r] \arrow[d, "\rho"] & K\mathcal{DD}_F \cong KF \arrow[d, "\rho_F"] \\ H\mathbb{F}_p \arrow[r, "id"] & H\mathbb{F}_p \end{tikzcd}$$

p-adic completion.

complete: $KRep_k[G_F]^\wedge_p \xrightarrow{A_p^{rep}} KF^\wedge_{pF}$ Rep'n'l assembly.

<u>Q</u>: Is there a chance that A_p^{rep} is $\simeq$?
(Clark Barwick has thought about this.)

Universal Example: $k[t^{\pm 1}] \to k[t^{\pm 1/p^\infty}]$ k alg. closed.
$A^{rep}_{k[t^{\pm1}]}$ exists
<u>Claim</u>: $A^{rep}_{k[t^{\pm1}]}$ is $\simeq$
localization seq:

$$t\text{-Tor}_n \longrightarrow k[t^{1/p^n}] \longrightarrow k[t^{\pm 1/p^n}]$$

$$\begin{tikzcd} K(t\text{-Tor}_n) \arrow[r] & K(k[t^{1/p^n}]) \arrow[r] \arrow[d, "\text{htpy property}"] & K(k[t^{\pm 1/p^n}]) \\ & KRep_k[G_F] & \end{tikzcd}$$

modules over $KRep_k[G_F]$

$\bigcup_n K(k[t^{1/p^n}])$ completes to $KRep_k[G_F]^\wedge$ domain of assembly.

Aside: $A \xrightarrow{f} B$ $M \xrightarrow{\theta} N$ map of A modules.
When is $M^\wedge_B \xrightarrow{\theta^\wedge_B} N^\wedge_B$ iso? Suffices that $B \wedge_A M \xrightarrow{B\wedge\theta} B \wedge_A N$ is $\simeq$
Apply smash prod. to fiber sequence. (preserves fiber seq.)
Need to show $\bigcup_n K(t\text{-Tor}_n) \wedge_{KRep_k[G_F]} H\mathbb{F}_p \simeq *$

homotopy of $K(t\text{-Tor}_n)$ is a flat $K_*Rep_k[\mathbb{Z}_p]$-module. $\mathbb{F}_p \xrightarrow{\text{rep}} \mathbb{F}_p \xrightarrow{\text{rep}} \mathbb{F}_p$
Proves that assembly is $\simeq$ in this universal example.

KF, $u \in F^\times = K_1F$. Adjoin all p-power roots of u. $E = F(\sqrt[p^n]{u})$.
$k[t^{\pm 1/p^n}] \longrightarrow$ G^E_F = Galois group is $\mathbb{Z}_p$, quotient G_F.
$t \longmapsto u$ $t^{1/p^n} \longmapsto u^{1/p^n}$

$$\begin{tikzcd} KRep_k[G^E_F] \arrow[r] \arrow[rr, bend right] & KRep_k[G_F] \arrow[r, "A_F^{rep}"] & KF \end{tikzcd}$$

onto part. elt) from univ. example

Use this result to prove that A_F^{rep} iso for abelian absolute Galois grps.
(Key ingred: explicitly compute $\pi_*(Rep_k[\text{Ab profinite grp}])$)

Allows one to show that A_F^{rep} is surjective in general.
(Block-Kato: K-theory gen. in degrees 1 & 2; abelianization ...)

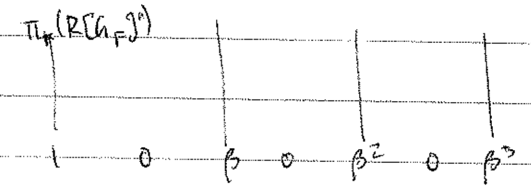
Ideas for injectivity:

$$\begin{tikzcd}
K\mathrm{Rep}_k[G_F] \arrow[r, "A"] \arrow[d, "\text{on htpy groups}"] & KF \arrow[d] \\
K^{per}\mathrm{Rep}_k[G_F] \arrow[r, "A^{per}"] \arrow[d] & K^{per}F \arrow[d, "\cong \text{ By Thomason}"] \\
K^{per}(k)^{hG_F} \arrow[r, "\text{equiv. by Suslin}"'] & K^{per}(\bar{F})^{hG_F}
\end{tikzcd}$$

equivalence by profinite Atiyah-Segal Thm

Alg to geometric SS: $A \to B$

$\pi_*(M^\wedge_\beta)$. each $\pi_i(M)$ is an A-module. Can be completed at $\pi_0 A \to \pi_0 B$



no differentials, so collapses & gives injection into periodic form.

NB: there is a condition ~~that~~ for iso which preserved by colimits.