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Goal: understand $K(MU)$ $\{K(\text{Thom spectrum})\}$

$f: X \longrightarrow BGL_1 S$

1.) Rich class of examples of nonalgebraic ring spectra

2.) Waldhausen: $K(S)$ related to manifold geometry

$S \longrightarrow MU$ Hopf-Galois extension ($\Sigma^\infty_+ BU$ co-Galois group)

MU connective, so attack $K(MU)$ via trace methods

$$K(MU) \longrightarrow TC(MU) \longrightarrow THH(MU)$$

- Discuss work in progress (Angeltveit, Gerhardt, Hill, Lawson - jt w/)
  to understand cyclotomic structure on $THH(MU)$
  Slogan: $THH(\text{Thom spectrum}) \simeq$ some S^1-equivariant Thom spectrum
- Recall THH of Thom spectrum nonequivariantly:
  $(C, \otimes, 1)$ symm. monoidal cat
  $N^{cyc}_{\otimes} M$, M monoid object, given by $|\{k\} \longrightarrow M^{\otimes k}|$, usual cyclic structure maps
  In spectra, $|N^{cyc}_{\wedge} R|$ is $THH(R)$
  In spaces, $|N^{cyc}_{\times} M|$ is ΛBM (free loop space)
- Thom spectra:

$$\mathcal{J}/BGL_1 S \longrightarrow \text{spectra}$$

1.) $BGL_1 S \simeq \operatorname{colim}_V BhAut(S^V) \simeq BF$

$X \to BF$, filtration on BF restricts one on X

$$\begin{tikzcd} Q_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B(S^V, hAut(S^V), *) \arrow[d] \\ X_V \arrow[r] & B(*, hAut(S^V), *) \end{tikzcd}$$

$\{Q_V / i\}$ is Thom prespectrum

↑ section; there's secret fibrant replacement

2.) "Mike Hopkins model"

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EGL_1 S \arrow[d] \\ X \arrow[r, "f"] & BGL_1 S \end{tikzcd}$$

$Mf = \Sigma^\infty_+ P \wedge_{\Sigma^\infty_+ GL_1 S} S$

(at least morally?)

$\mathcal{J}/BGL_1 S$ $\begin{matrix} X \longrightarrow BGL_1 S \\ Y \longrightarrow BGL_1 S \end{matrix}$ $\rightsquigarrow$ $\begin{matrix} X \times Y \longrightarrow BGL_1 S \times BGL_1 S \\ \downarrow \\ BGL_1 S \end{matrix}$

Observation: M is symmetric monoidal

$M(N_{*}^{cyc} f) \simeq N_{\wedge}^{cyc} Mf$

$N^{cyc} f : N^{cyc} X \longrightarrow N^{cyc} BGL_1 S \longrightarrow BGL_1 S$

$$\begin{array}{c} N^{cyc} BGL_1 S \\ \scriptstyle{\simeq} \\ BGL_1 S \otimes S^1 \longrightarrow BGL_1 S \end{array}$$

How to make this precise?

ingredient: setting for rigid infinite loop space theory

$\tilde{T}$ s.t. $\tilde{T}$ has a symm. mon. product $\boxtimes$

$\tilde{T} \simeq T$ (cat. of space) $\quad X \boxtimes Y \simeq X \times Y$

monoids for $\boxtimes$ are A_∞-spaces, comm. monoids E_∞-spaces

Two approaches:

1.) "EKMM" spaces {L(1)-spaces}

2.) diagram spaces {symmetric, orthogonal spaces} Schlichtkrull-Sagave (symmetric case)

$\rightarrow$ comparison: Lind

We'll use L(1)-spaces.

Idea for L(1)-spaces. $\mathcal{L}(1) = \mathcal{L}(\mathcal{U}, \mathcal{U})$ $\quad$ $\mathcal{U}$ universes.

$$X \longrightarrow \mathcal{L}(1) \times X$$

$\operatorname{coeq}(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times (X \times Y) \rightrightarrows \mathcal{L}(2) \times X \times Y)$ gives the product.

Monoid in $\mathcal{L}(1)$-spaces is A_∞-space over linear isom. operad

comm. monoid is E_∞-space

Defect here: hard to see how to understand the S^1-action on THH(Mf)

Goal: generalize this to S^1-equiv. setting

1.) Theory of equivariant $\mathcal{L}(1)$-spaces

2.) Construction of equivariant Thom spectra

3.) Use HHR norm to describe cyclotomic structure on THH.

3.) A model of the cyclotomic structure on THH(R) via the norm

$X \wedge_B Y$ model (Bökstedt) $\quad \operatorname*{hocolim}_{I \times I} \operatorname{Map}(S^{n_1+n_2}, X_{n_1} \wedge Y_{n_2} \wedge S^V)$

$(X \wedge_B X)^{gC_2} \simeq X$ $\quad$ (Shipley - think about this in symm. spectra) detection theorem

Observe that $N_e^{C_2} X$ has this property: $(N_e^{C_2} X)^{\Phi C_2} \xrightarrow{\simeq} X$

$sd_n\, THH(R)$

$N_e^{C_n}(R \wedge R \wedge R)$

$N_e^{C_n}(R \wedge R)$ (but ojo: cyclic structure maps different!)

$N_e^{C_n} R$

Defn: $N_e^{S^1}$: spectra $\longrightarrow$ S^1-spectra

if R is commutative, $N_e^{S^1}$: comm. S^1-alg $\longrightarrow$ comm. alg S^1-spectra is adjoint to the forgetful functor.

(NB. same as HHR on point-set, but homotopy is harder).

$R \longmapsto I_{\mathbb{R}^\infty}^{U} R \otimes S^1$ (for R commutative; otherwise N^{cyc} first)

Claim: This is cyclotomic, and if R is cofibrant, this is $THH(R)$ (both associative & commutative R)

Want to construct a Thom spectrum functor from S^1-spaces / $BGL_1^{S^1} S$ $\longrightarrow$ S^1-spectra

1.) For this, use $\mathcal{L}_{S^1}(1)$-spaces. Fixed universe $\mathcal{U}$

$\mathcal{L}_{S^1}(1) = \mathcal{L}_{S^1}(\mathcal{U}, \mathcal{U})$ 1st piece of the G-linear isom. operad

Repeat the story as before: $\boxtimes_{S^1}$ on $\mathcal{L}_{S^1}(1)$-spaces s.t.

monoids / comm. monoids are spaces for S^1-linear isometries operad

Warning: Don't have deloopings, but...

$\Sigma^\infty_{+,S^1}: \mathcal{L}_{S^1}(1)\text{-spaces} \longrightarrow S^1\text{-spectra}$

symmetric monoidal functor

operads enough for mult. but not additive structure for G not finite.

Given $BGL_1^{S^1} S$:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EGL_1^{S^1} S \arrow[d] \\ X \arrow[r] & BGL_1^{S^1} S \end{tikzcd}$$

$Mf = \Sigma^\infty_{S^1 +} P \wedge_{\Sigma^\infty_{S^1 +} GL_1^{S^1} S} S_{S^1}$

$X \to BGL_1 S$ noneq: $X \to \mathcal{U} BGL_1^{S^1} S$

forgetful functor.

$$N_e^{S^1} X \to N_e^{\mathcal{U}} \mathcal{U} BGL_1^{S^1} S \to BGL_1^{S^1} S$$

$$S^1 \overset{S^1}{\otimes} BGL_1 S$$

Apply equiv. Thom spectrum functor:

$$M(N_e^{S^1} X \to N_e^{S^1} \mathcal{U} BGL_1^{S^1} S \to BGL_1^{S^1} S) \cong N_e^{S^1} Mf$$ (which is THH(Mf) cyclo.)

(ASIDE: This model of cyclotomic structure on THH relativizes on pt/set level)

$TC_A(R)$, R A-algebra

→ actually we needed E_1 things.