Construction of Models in low-dimensional Quantum Field Theory using Operator Algebraic Methods

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CHAPTER 1

Introduction

In algebraic quantum field theory (AQFT) [Haa96] one studies local nets (e.g. von Neumann algebras) that assign to a space-time region the algebra of observables localized in it. These nets are asked to fulfill certain axioms coming from basic physical principles; we mention as examples the locality principle—which asks that the algebras assigned to causally disjoint regions should commute (local nets)—and the covariant assignment with respect to some “symmetry group” of the space-time. This approach brought many conceptional and model independent results, but recently it seems to be useful for construction and classification of models.

In the AQFT approach also conformal quantum field theory (CQFT) has been treated successfully by considering nets on two dimensional Minkowski space and its chiral parts, which can be regarded as nets on the real line or as nets on the circle. Some important achievements are classification results [KL04, Xu05] and new constructions [Xu07], which were not obtained in other approaches. An important tool is the relation to subfactors: inclusions of von Neumann algebras with trivial centers. For example there is a notion of index of subfactors giving relations to the statistical dimensions of representations. For finite index subfactors and there are several invariants with combinatorial nature making classifications possible.

A conformal net on the real line (or the circle) is a net (precosheaf), which assigns to each proper interval a von Neumann algebra on a fixed Hilbert space $\mathcal{H}$. Locality is encoded by asking that the algebras of disjoint intervals commute, covariance that there is a positive energy representation of the Möbius group (\(\cong \text{PSL}(2, \mathbb{R}) \cong \text{PSU}(1, 1)\)) such that $\mathcal{A}(gI) = U(g)\mathcal{A}(I)U(g)^*$. Often these nets turn out to be also covariant with respect to $\text{Diff}_+(\mathbb{S}^1)$ (the orientation preserving diffeomorphisms of the circle) and there is a subnet $\text{Vir}$ giving a representation of the Virasoro algebra and one can associates a central charge $c$. The von Neumann algebras turn out to be type III$_1$ factor in Connes classification. Superselection sectors or Doplicher–Haag–Roberts (DHR) representations can be described by localized endomorphisms giving a connection to Jones theory subfactors. In contrast to the result of higher dimensions, where just Fermions and Bosons can exist, in low dimensions more general so-called anyons exist and give rise to braid group statistics.

There is the notion of completely rational conformal nets and the category of DHR representation of such net turns out to be a modular tensor category (MTC) (cf. [KLM01]). This gives relations to other fields in Mathematics and Physics, for example MTCs give rise to 3D topological quantum field theory and 3 manifold invariants [Tur94]. They also play an important rôle in topological quantum computation (for a review see e.g. [NSS+08]).

Besides CQFT on the full Minkowski space also boundary conformal quantum field theory (BCFT) on the Minkowski half-plane $x > 0$ is described in the algebraic approach. More precisely, in the paper [LR04] Longo and Rehren associate with a local conformal net $\mathcal{A}$ on the real line a local conformal boundary net $\mathcal{A}_+$ (the trivial boundary net) on Minkowski half-plane and obtain more general boundary nets which are extending $\mathcal{A}_+$. The non-chiral extensions come from non-local extensions of the net on the boundary $\mathcal{A}$. 
Lately, in [LW11] Longo and Witten have given a framework to construct models in boundary quantum field theory (BQFT) by investigating into local nets on the Minkowski half-plane, which are in general only time-translation covariant and can be considered as a deformation of the net $\mathcal{A}_+$. Specifically, the construction starts with a conformal net $\mathcal{A}$ on the real line together with an element $V$ of a unitary semigroup $\mathcal{E}(\mathcal{A})$ associated with $\mathcal{A}$ to construct a net on Minkowski half-plane, where the special case $V = 1$ is the net $\mathcal{A}_+$.

Another successful example of AQFT is the construction of so-called factorizing models in 2D Minkowski space. This are models with a simple class of S-matrices which is factorizing which means that they are determined by the two-particle S-matrix. Such models were intensively studied in the form factor program, but a mathematically satisfying existence proof, that these models satisfy the basic principles of QFT (e.g. Wightman axioms) are just given for simplest examples. Due to an idea of Schroer and work of Buchholz and Lechner the models could be constructed in the framework of AQFT for a special class of S-matrices and they turn out to be asymptotically complete and have the scattering data [Lec08]. They are models of massive particles. Also some progress of constructing and understanding interacting models of massless particles in 2D where made (e.g. [DT11, Tan12a]).

Such models can be conveniently described by a Borchers triple $(\mathcal{M}, U, \Omega)$ consisting of a single von Neumann algebra (factor) and a unitary representation (with positive energy) of the space-time translations $\mathbb{R}^2$ denoted by $U$ and a cyclic and separating vector $\Omega$ for $\mathcal{M}$ such that the additional condition $U(x)\mathcal{M}U(x)^* \subset \mathcal{M}$ for $x \in W_L = \{ x \in \mathbb{R}^2 : x_1 > |x_0| \}$ is fulfilled. From this data a wedge-local net can be defined by $\mathcal{A}(x + W_L) = U(x)\mathcal{M}U(x)^*$ and $\mathcal{A}(x + W_R) = U(x)\mathcal{M}'U(x)$ where $W_L = \{ x \in \mathbb{R}^2 : -x_1 > |x_0| \}$ is the left-wedge. On can pass to algebra for double cones by taking relative commutants, but the obtained algebras can be even trivial. If the vector $\Omega$ is also cyclic for the algebra of bounded regions, that means there are many observables in bounded regions, we call the Borchers triple strictly local, because it gives rise to a local Poincaré covariant net, while in general a Borchers triple give just rise to a wedge-local Poincaré covariant net. In the massive case there are strictly local examples by Lechner, while in the massless case this problem is still open and need new techniques.

This thesis is organized as follows.

In Part I we collect some background material and some tools used later. This part contains in particular no new results.

In Chapter 4 conformal nets associated with lattices (which contain certain loop group models at level 1 as a subclass) are considered and Longo–Witten unitaries are constructed by extension of second quantization unitaries on a Bosonic Fock space.

In Chapter 5 the restriction of Longo–Witten unitaries which are second quantization unitaries on a Fermionic Fock space. A particular tool is the Boson–Fermion correspondence. The classes of conformal nets reachable by this construction has a overlap with the one in Chapter 4, but it is shown that the Longo–Witten unitaries which are obtained are different (besides the trivial cases).

In Chapter 6 we investigate the extension of Longo–Witten unitaries for finite index subnets.

In Chapter 7 it is looked into conformal nets arising from extensions of (subnets) of free Fermions by the so-called framed vertex operator construction and the construction Longo–Witten unitaries for this models.

In Part III it is considered the construction of models in 2D quantum field theory, i.e. (wedge-local) nets on 2D Minkowski space. Chapter 9 depends on Chapter 5 and establish a new family of
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massless interacting models which can be seen as a deformation of a Bosonic free field and has a S-matrix which do not factorize.

In the Chapter 10 is studied some relation between different nets and models. In particular, half-ray local models on a light ray are constructed by second quantization on a generalized Fock space and it is shown that these are building blocks for massive and massless models.

Some of the results I obtained for this thesis work are published in journals. The material in Chapter 4 come from [Bis12] and are published in Communications in Mathematical Physics (online first). The material of Chapter 5 and 9 was obtained in collaboration with Yoh Tanimoto and is based on [BT11] (accepted for Communications in Mathematical Physics).
Part I

Preliminaries
CHAPTER 2

Preliminaries on operator algebras

2.1. Operator algebras

2.1.1. Operators on a Hilbert space. Basically to fix some notations, we repeat some basics. We refer to [RS80 RS75 Tak02 Tak03] for details. Let $\mathcal{H} \equiv (\mathcal{H}, (\cdot, \cdot))$ be a Hilbert space with inner product $(\cdot, \cdot)$, which is here antilinear in the first and linear in the second argument. We denote by $x \mapsto \langle x \rangle$ the antilinear isomorphism from $\mathcal{H}$ to its continuous functionals $\mathcal{H} \to \mathbb{C}$, where $\langle x \rangle y := (x, y)$. Let $\mathcal{K}$ be another Hilbert space, then we denote by $B(\mathcal{H}, \mathcal{K})$ the bounded linear maps from $\mathcal{H} \to \mathcal{K}$ and for the *-algebra of bounded operators $B(\mathcal{H}, \mathcal{H})$ on $\mathcal{H}$ we simply write $B(\mathcal{H})$. A (possibly unbounded) operator $A$ on $\mathcal{H}$ is a linear map from a domain $\text{Dom}(A) \subset \mathcal{H}$ to $\mathcal{K}$. It is called densely defined if $\text{Dom}(A)$ is dense in $\mathcal{H}$. An unbounded operator $A : \text{Dom}(A) \subset \mathcal{H} \to \mathcal{K}$ is called closed if for every sequence $\{x_n\}$ in $\text{Dom}(A)$ with $x_n \to x$ such that $Ax_n \to y$ one has $x \in \text{Dom}(A)$ and $Ax = y$. Equivalently, the graph $\{(x, Ax) \in \text{Dom}(A) \oplus \mathcal{K}\}$ is closed in $\mathcal{H} \oplus \mathcal{K}$.

We remark that an operator defined on a dense domain $\text{Dom}(A)$ which is bounded, i.e. the operator norm

$$\|A\| := \sup_{x \in \text{Dom}(A) \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$$

is finite uniquely extends to an operator on $\mathcal{H}$ by continuity. We can assume from the outset it is defined on all $\mathcal{H}$. The adjoint of $A : \text{Dom}(A) \subset \mathcal{H} \to \mathcal{K}$ is the operator $A^* : \text{Dom}(A^*) \subset \mathcal{K} \to \mathcal{H}$ with domain

$$\text{Dom}(A^*) = \{x \in \mathcal{K} : \text{Dom}(A) \ni y \mapsto (x, Ay) \text{ is continuous} \}$$

defined by $A^*x = z$ with $(z, y) = (x, Ay)$, more precisely $\varphi_x : y \mapsto (x, Ay)$ extends uniquely to a continuous linear functional on $\mathcal{H}$ and therefore exists a unique $z \in \mathcal{H}$ such that $\langle z \rangle = \varphi_x$. If we write $A = B$ we mean that $\text{Dom}(A) = \text{Dom}(B)$ and $Ax = Bx$ for all $x \in \text{Dom}(A)$. Further we write $A \subset B$ if $\text{Dom}(A) \subset \text{Dom}(B)$ and $Ax = Bx$ for all $x \in \text{Dom}(A)$. It is always $A \subset A^{**} = (A^*)^*$.

A densely defined operator is called symmetric if $A \subset A^*$. An operator is called self-adjoint if $A^* = A$ and a self-adjoint operator is necessarily densely defined and closed, because $A^*$ is closed.

If the closure of the graph $\{(x, Ax) \in \text{Dom}(A) \oplus \mathcal{K}\}$ happens to be the graph of an operator denoted by $\tilde{A}$, then we say that $A$ is closable with closure $\tilde{A}$.

A symmetric operator $T$ is called essentially self-adjoint if its closure $\tilde{T}$ is self-adjoint. If $T$ is closed, a subspace $\mathcal{D} \subset \text{Dom}(T)$ is called a core for $T$ if $\overline{T \upharpoonright \mathcal{D}} = T$. A family of unitary operators $\{U(t)\}_{t \in \mathbb{R}}$ is called a strongly continuous one-parameter unitary group if $U(t + s) = U(t)U(s)$ for all $t, s \in \mathbb{R}$ and $t \to U(t)$ is strongly continuous. If $A$ is a self-adjoint operator then $U(t) = e^{itA}$ is a strongly continuous one-parameter unitary group (cf. [RS80, Thm VIII.7]). Also the converse is true.

**Theorem 2.1.1** (Stone’s Theorem [RS80, Thm VIII.8]). Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$. Then there is a self-adjoint operator $A$, such that $U(t) = e^{itA}$
The operator $A$ is called the (infinitesimal) generator of $U(t)$. In particular there is a one-to-one correspondence between unitary strongly continuous one-parameter groups $\{U(t)\}_{t \in \mathbb{R}}$ on $\mathcal{H}$ and self-adjoint operators $A$ on $\mathcal{H}$, given by $U_A(t) = e^{itA}$.

**Definition 2.1.2.** Let $A$ be an operator on $\mathcal{H}$. The set

$$C^\infty(A) := \bigcap_{n=1}^{\infty} \text{Dom}(A^n)$$

is called the $C^\infty$–vectors for $A$. A vector $\psi \in C^\infty(A)$ is called analytic for $A$ if

$$\sum_{n=0}^{\infty} \frac{\|A^n\psi\|}{n!} \cdot t^n < \infty$$

for some $t > 0$.

**Theorem 2.1.3** (Nelson’s analytic theorem cf. [RS75, Theorem X.39]). Let $A$ be a symmetric operator on $\mathcal{H}$. If $\text{Dom}(A)$ contains a total set of analytic vectors, then $A$ is essentially self-adjoint.

**Rank one, trace class and Hilbert–Schmidt operators.** Let $\mathcal{H}$ be a Hilbert space. For $a, b \in \mathcal{H}$ we denote by $|a\rangle \langle b|$ the rank one operator given by $|a\rangle \langle b| x = (b, x) \cdot a$. It holds

- $|a\rangle \langle b| = |b\rangle \langle a|$
- $|a\rangle \langle b|$ is a projection if and only if $a = b$ with $\|a\| = 1$.

**Definition 2.1.4.** Let $\{e_i\}$ be a orthonormal basis of a Hilbert space $\mathcal{H}$. The nuclear norm $\| \cdot \|_1$ and Hilbert–Schmidt norm $\| \cdot \|_2$ are given by

$$\|A\|_1 = \text{tr} |A| := \sum_i (e_i, |A| e_i), \quad |A| = (A^*A)^{\frac{1}{2}}$$

$$\|A\|_2^2 = \text{tr}(A^*A) := \sum_i \|A e_i\|^2,$$

respectively. An operator $A \in \mathcal{B}(\mathcal{H})$ is called nuclear or an operator of trace class if $\|A\|_1 < \infty$ and Hilbert–Schmidt if $\|A\|_2 < \infty$. The ideal of trace class operators in $\mathcal{B}(\mathcal{H})$ is denoted by $L^1(\mathcal{H})$ and the ideal of Hilbert–Schmidt operators in $\mathcal{B}(\mathcal{H})$ is denoted by $L^2(\mathcal{H})$.

$L^2(\mathcal{H})$ is itself a Hilbert space with the inner product $(A, B)_{L^2(\mathcal{H})} = \text{tr}(A^*B)$. There is a identification of the Hilbert space $\mathcal{H} \otimes \mathcal{H}^*$ with $L^2(\mathcal{H})$ given by the extension of the map:

$$\mathcal{H} \otimes \mathcal{H}^* \rightarrow L^2(\mathcal{H})$$

$$a \otimes b^* \mapsto |a\rangle \langle b|,$$

namely, one can check e.g.

$$(|a\rangle \langle b|, |c\rangle \langle d|)_{L^2(\mathcal{H})} = \text{tr}(|a\rangle \langle b|^* |c\rangle \langle d|) = (a, c)(d, b) = (a \otimes b^*, c \otimes d^*) = (a \otimes b^*, c \otimes d^*).$$

**2.1.2. $C^*$-algebras and categories.** A $C^*$-category is a $*$-category, where each set of arrows $\text{Hom}(\rho, \sigma)$ possess a norm $\| \cdot \|$ making it a Banach spaces such that the $C^*$-condition $\|S^* \otimes S\| = \|S\|^2$ and $\|T \cdot S\| \leq \|S\| \|T\|$, where the composition is defined, holds. It is a generalization of a $C^*$-algebra, namely a $C^*$-category with just one element is a $C^*$-algebra. As an example of a $C^*$-category we take as objects Hilbert spaces and as arrows $\mathcal{H} \rightarrow \mathcal{K}$ bounded linear maps in $\mathcal{B}(\mathcal{H}, \mathcal{K})$. We note that we can embed $\mathcal{B}(\mathcal{H}, \mathcal{K})$ in $\mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H} \oplus \mathcal{K})$ and in the same way one can basically treat $C^*$-categories as $C^*$-algebras.
A representation $\pi$ of a $C^*$-algebra is a unital $*$-homomorphism from $A$ to the bounded operators on a Hilbert space $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is called cyclic if $\pi(A)\Omega$ is dense in $\mathcal{H}$. A state $\omega$ is a positive linear functional on $A$ with $\omega(1) = 1$. If $\pi$ is a representation on $\mathcal{H}$ and $\Omega$ a cyclic vector, then $\omega(x) = (\Omega, \pi(x)\Omega)$ defines a space. Conversely, if $\omega$ is a state then there exist a triple $(\mathcal{H}_0, \pi_0, \Omega_0)$ of a Hilbert state $\mathcal{H}_0$, a representation $\pi_0$ of $A$ on $\mathcal{H}_0$ and a cyclic vector $\Omega_0$, which is called the Gelfand–Naimark–Segal (GNS) construction. Namely, on $A$ we define a positive semi-definite sesquilinear form $(x, y) = \omega(x^*y)$. Then $I = \{ x : (x, x) = 0 \}$ defines a left ideal and $\mathcal{H}_0$ is the closure of $A/I$ under the norm $\|x\|^2 = (x, x)$. The action of $A$ is defined by $\pi_0(x)[y] = [xy]$ where $[y] \in \mathcal{H}_0$ is the equivalence class of $y \in A$ in $\mathcal{H}$ and $\Omega_0 : = [1]$.

2.1.3. Operator topologies. The predual $B_*(\mathcal{H})$ of the algebra of bounded operators $B(\mathcal{H})$ can be identified with the trace class operators $L^1(\mathcal{H})$, namely for $\omega \in B_*(\mathcal{H})$ there is a corresponding $t_\omega \in L^1(\mathcal{H})$ and vice versa, such that $\omega(x) = \text{tr}(t_\omega x)$ for $x \in B(\mathcal{H})$. By $B_*(\mathcal{H})_+$ we denote the positive elements in $B_*(\mathcal{H})$, i.e. elements $\omega \in B_*(\mathcal{H})$ such that $\omega(x^*x) \geq 0$ for $x \in B(\mathcal{H})$. We sum up a list of operator topologies on $B(\mathcal{H})$, which are each induced by a family of seminorms and therefore give $B(\mathcal{H})$ the structure of a locally convex topological vector space.

**ultra weak**: Family of seminorms: $\{ \|x(\cdot)\| : x \in B_*(\mathcal{H}) \}$.

**ultra strong**: Family of seminorms: $\{ p_\omega : \omega \in B_*(\mathcal{H})_+ \}$ with $p_\omega(x) := \omega(x^*x)^{1/2}$ for $x \in B(\mathcal{H})$.

**ultra strong**$: Family of seminorms: $\{ \hat{p}_\omega : \omega \in B_*(\mathcal{H})_+ \}$, where

$$
\hat{p}_\omega(x) = \left( p_\omega(x^2) + p_\omega(x^*x) \right)^{1/2}
$$

for $x \in B(\mathcal{H})$.

**weak**: Family of seminorms: $\{ |(\xi, \cdot \cdot \eta)| : \xi, \eta \in \mathcal{H} \}$.

**strong**: Family of seminorms: $\{ \| \cdot \xi \| : \xi \in \mathcal{H} \}$.

**strong**$: Family of seminorms: $\{ \hat{p}_\xi : \xi \in \mathcal{H} \}$, where

$$
\hat{p}_\xi(x) := \left( \|x\xi\|^2 + \|x^*\xi\|^2 \right)^{1/2}
$$

where $\prec$ means that the topology on the left hand side is finer/stronger then the one on the right hand side.

2.1.4. Von Neumann algebras. For a set $\mathcal{M} \subset B(\mathcal{H})$ we define its commutant $\mathcal{M}'$ to be $\mathcal{M}' := \{ x \in B(\mathcal{H}) : [x, \mathcal{M}] = \{0\} \}$.

It is convenient to use the following definition:

**Definition 2.1.5.** A von Neumann algebra $\mathcal{M}$ (on $\mathcal{H}$) is a $*$-subalgebra of $B(\mathcal{H})$ with $\mathcal{M}'' \subset \mathcal{M}$.

The commutant $\mathcal{M}'$ of a von Neumann algebra is a von Neumann algebra, because $\mathcal{M}''' = \mathcal{M}'$. By the same reasoning follows directly: if a set $\mathcal{S} \subset B(\mathcal{H})$ is closed under adjoints then $\mathcal{S}'$ and $\mathcal{S}''$ are also von Neumann algebras. In particular $\mathcal{S}''$ is the smallest von Neumann algebra containing $\mathcal{S}$, the von Neumann algebra generated by $\mathcal{S}$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two von Neumann algebras on
2. PRELIMINARIES ON OPERATOR ALGEBRAS

\( \mathcal{H} \) then \( \mathcal{M}_1 \vee \mathcal{M}_2 := (\mathcal{M}_1 \cup \mathcal{M}_2)^\prime \) is the von Neumann algebra generated by \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). Also \( \mathcal{M}_1 \wedge \mathcal{M}_2 := \mathcal{M}_1 \cap \mathcal{M}_2 \) is a von Neumann algebra and the relations \((\vee, \wedge, \cdot)\) define a complemented lattice on the set of all von Neumann algebras on \( \mathcal{H} \).

While the definition we made for a von Neumann algebra is algebraic, it also has an analytic counterpart saying basically that a \( \ast \)-algebra is a von Neumann algebra if it is closed under one of the intruduced topologies but the the norm topology, precisely it holds:

**Theorem 2.1.6** (von Neumann’s bicommutant theorem). Let \( \mathcal{M} \subset B(\mathcal{H}) \) a \( \ast \)-algebra on \( \mathcal{H} \) containing the identity operator. Then the following statements are equivalent:

- \( \mathcal{M}'' = \mathcal{M} \)
- \( \mathcal{M} \) is closed under the weak topology
- \( \mathcal{M} \) is closed under the weak* topology
- \( \mathcal{M} \) is closed under the ultra weak topology
- \( \mathcal{M} \) is closed under the strong topology
- \( \mathcal{M} \) is closed under the strong* topology
- \( \mathcal{M} \) is closed under the ultra strong topology

**Definition 2.1.7.** Let \( \mathcal{M} \subset B(\mathcal{H}) \) a von Neumann algebra. A closed and densely defined operator \( A \) on \( \mathcal{H} \) is said to be **affiliated with** \( \mathcal{M} \) if \( A \) commutes with every unitary in \( \mathcal{M}' \).

**2.1.5. Tomita–Takesaki theory.** Let \( \mathcal{M} \subset B(\mathcal{H}) \) be a von Neumann algebra. A vector \( \Omega \in \mathcal{H} \) is called **cyclic** if \( \mathcal{M} \Omega \) is dense in \( \mathcal{H} \) and **separating** if for \( x \in \mathcal{M} \) with \( x \Omega = 0 \) it is \( x = 0 \). If \( \Omega \) is cyclic for \( \mathcal{M} \) then it is separating for \( \mathcal{M}' \), because for \( x \in \mathcal{M}' \) with \( x \Omega = 0 \) we have \( \mathcal{M} x \Omega = x \mathcal{M} \Omega = 0 \) and therefore \( x = 0 \), and also the converse is true by interchanging the rôle of \( \mathcal{M} \) and \( \mathcal{M}' \). In particular, if \( \Omega \) is cyclic and separating for \( \mathcal{M} \) then it is cyclic and separating for \( \mathcal{M}' \) and in this case we say that \( (\mathcal{M}, \Omega) \) is a von Neumann algebra in **standard form**.

For \( (\mathcal{M}, \Omega) \) a von Neumann algebra in standard form we can define an unbounded antilinear operator \( S_0 \) with \( \text{Dom}(S_0) = \mathcal{M} \Omega \) by \( S_0 x \Omega = x^* \Omega \) which is well defined because \( \Omega \) is separating and densely defined because \( \Omega \) is cyclic. Further it is closeable and we denote its closure by \( S_{(\mathcal{M}, \Omega)} \). The operator \( S_{(\mathcal{M}, \Omega)} \) possesses a unique **polar decomposition**

\[
S_{(\mathcal{M}, \Omega)} = J_{(\mathcal{M}, \Omega)} \Delta_{(\mathcal{M}, \Omega)}^{\frac{1}{2}}
\]

with \( J_{(\mathcal{M}, \Omega)} = J^*_{(\mathcal{M}, \Omega)} = J^\dagger_{(\mathcal{M}, \Omega)} \) an antiunitary operator called **modular conjugation** and \( \Delta_{(\mathcal{M}, \Omega)} \) a positive self-adjoint operator called **modular operator**. The **modular group** is the one-parameter group of unitaries \( \{\Delta^t\}_{t \in \mathbb{R}} \) defined by spectral calculus with generator \( \ln \Delta \). It is \( S_{(\mathcal{M}', \Omega)} = S^*_{(\mathcal{M}, \Omega)} \) and therefore \( J_{(\mathcal{M}', \Omega)} = J_{(\mathcal{M}, \Omega)} \) and \( \Delta_{(\mathcal{M}', \Omega)} = \Delta_{(\mathcal{M}, \Omega)}^{-1} \).

**Theorem 2.1.8** (Tomita–Takesaki theorem). Let \( (\mathcal{M}, \Omega) \) be a von Neumann algebra in standard form then

\[
J_{(\mathcal{M}, \Omega)} \mathcal{M} J_{(\mathcal{M}, \Omega)} = \mathcal{M}', \quad \Delta_{(\mathcal{M}, \Omega)}^t \mathcal{M} \Delta_{(\mathcal{M}, \Omega)}^{-it} = \mathcal{M}
\]

for all \( t \in \mathbb{R} \).

**Theorem 2.1.9** (Borchers theorem [Flo98]). Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \( \Omega \in \mathcal{H} \) with a cyclic and separating vector \( \Omega \in \mathcal{H} \) for \( \mathcal{M} \). Let \( T \) be a one-parameter group on \( \mathcal{H} \) fixing \( \Omega \), with positive generator \( P \), satisfying

\[
T(t) \mathcal{M} T(-t) \subset \mathcal{M}, \quad t \geq 0
\]
then following commutation relations hold
\[ \Delta_{M,\Omega}^t T(t) \Delta_{M,\Omega}^{-i\sigma} = T(e^{-2\pi i t}) \]
\[ J_{M,\Omega} T(t) J_{M,\Omega}^{-1} = T(-t). \]

**Definition 2.1.10.** Let \( N \subset M \) be an inclusion of von Neumann algebras. A **conditional expectation** \( E : M \to N \) is a completely positive, normalized map with the property
\[ E(n_1 m n_2) = n_1 E(m) n_2, \quad n_1, n_2 \in N, \ m \in M \]
and it is called **normal** if it is weakly continuous. It is called **faithful** if \( E(m^* m) = 0 \) implies \( m = 0 \). The set of faithful and weakly continuous conditional expectations \( M \to N \) is denoted by \( C(M,N) \).

The following fundamental theorem will be an important tool.

**Theorem 2.1.11** (Takesaki’s Theorem [Tak03, Theorem IX.4.2.]). Let \( N \subset M \) be an inclusion of von Neumann algebras on \( \mathcal{H} \) with \( \Omega \) a cyclic and separating vector for \( M \) and denote by \( \varphi \) the faithful and normal state \( \varphi = (\Omega, \cdot) \). The existence of a conditional expectation \( E : M \to N \) such that \( \varphi \circ E = E \) is equivalent with the global invariance \( \sigma_f^t (N) = N \) under the modular automorphism group \( \sigma_f^t = \text{Ad} \Delta_{M,\Omega}^t \).

In this case \( E(m) \Omega = em \Omega \) and \( e = E(m)e \) holds, where \( e \) is the **Jones projection**, i.e. the projection on \( \overline{N \Omega} \).

### 2.2. Subfactors

A von Neumann algebra \( M \subset B(\mathcal{H}) \) is called a **factor** if \( M \cap M' = \mathbb{C} \cdot 1 \). A subfactor is an inclusion \( N \subset M \) of two factors \( N, M \). It is called **irreducible** if \( N' \cap M = \mathbb{C} \cdot 1 \).

We say two projections \( e, f \) in a von Neumann algebra are equivalent if there is a partial isometry \( W \) such that \( e = W W^* \) and \( f = W^* W \) and we write \( e \sim f \). A projection \( e \) is **finite** if for \( f \sim e \) the condition \( e H \subset f H \) implies \( e = f \). Factors of von Neumann algebras can be classified in three types:

- **type I**: A factor is said to be type I, if there exists minimal projections, i.e. there is a projection \( 0 < e \) such that there are no other projections \( f \) with \( 0 < f < e \). Any type I factor is isomorphic to \( B(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \).
- **type II**: A factor is called type II if there are no minimal projections but nonzero finite projections.
- **type III**: A factor is type three if it does not contain any nonzero finite projection. In particular every projection is equivalent to 1.

**Definition 2.2.1.** Let \( N, M \) be type III factors. If \( N \subset M \) we define the **(Jones–Kosaki) index** or **minimal index** to be \([M : N] = \kappa^{-1} \in [1, \infty], \) where \( \kappa \geq 0 \) is the largest constant for which there exists a conditional expectation \( E \in C(M,N), \) such that
\[ E(m^* m) \geq \kappa \cdot m^* m \quad m \in M. \]

The index is multiplicative for \( N \subset M \subset L, \) i.e. \([L : N] = [L : M] \cdot [M : N].\) It takes values in \(\{4 \cos^2 \left( \frac{\pi}{8} \right) : n = 3, 4, \ldots \} \cup [4, \infty]\). We refer to [Kos98] for details. A homomorphism \( \sigma : N \to M \) gives rise to a subfactor \( \sigma(N) \subset M \) and every subfactor arises in this way, e.g. take the injection \( \iota : N \to M, \) so it will be convenient to write \( \iota(N) \subset M. \)

If \([M : N] < \infty \) then \( N' \cap M \) is finite dimensional. If \( N' \cap M = \mathbb{C} \) then there is only one conditional expectation and it is therefore the unique one that minimizes the index (cf. [Lon89, p. 230]).
2.2.1. Graphical notation / planar diagrammatic notation. We want to introduce some graphical notation. The idea is roughly the following: Subfactors can be described by so called Q-systems, which are Frobenius algebras in some tensor category. Frobenius algebras are related to 2D topological quantum field theories. By this all the laws of a Frobenius algebra are just homotopies of two manifolds. For tensor categories or more generally 2-categories (tensor category is a 2-category with just one 0-cell) exist so called string diagrams (cf. [Sel11, Section 4.8] and [BDH11]). One can perform calculation on such diagrams by certain allowed deformation which restricts the structure of the underlying category. In other words the free category of certain kind are given by combinations and manipulations of graphical elements. Further the diagrams give certain formulas a “topological meaning”.

We need some notions from category theory. We need the notion of \(2\text{-}\mathcal{C}^\ast\text{-category}\) denoted by \(\mathcal{C}\) (cf. [LR97]). We do not give the full definition here because we will just need a particular example and rather state the important properties to fix notation and give a graphical representation. A \(2\text{-}\mathcal{C}^\ast\)-category consists of the following data: a collection \(\text{Ob}(\mathcal{C})\) of objects or \(\text{0–cells}\). For our purpose they will be a finite set of von Neumann algebras \(\mathcal{M}, \mathcal{N}, \mathcal{P}, \cdots\) and they are represented by 2-dimensional shaded areas, e.g.

\[
\mathcal{M} = \begin{array}{c}
\end{array} \quad \mathcal{N} = \begin{array}{c}
\end{array} \quad \mathcal{P} = \begin{array}{c}
\end{array},
\]

where different shades denote different 0–cells. For every pair of objects \(\mathcal{M}, \mathcal{N}\) there is a category \(\text{Mor}(\mathcal{M}, \mathcal{N})\) of morphisms or \(1\text{-cells}\) and we denote \(\text{End}(\mathcal{M}) = \text{Mor}(\mathcal{M}, \mathcal{M})\). The morphisms can be composed and the composition is asked to be associative and there is a identity morphism \(\text{id}_\mathcal{M} \in \text{Mor}(\mathcal{M}, \mathcal{M}) \equiv \text{End}(\mathcal{M})\) for every 0–cell \(\mathcal{M}\). For us \(\text{Mor}(\mathcal{M}, \mathcal{N})\) will be unital (ultra weak continuous) \(\ast\)-morphism \(\mathcal{M} \to \mathcal{N}\). In the string diagrams they are displayed by 1-dimensional objects (vertical strings), e.g.

\[
\rho : \mathcal{N} \leftarrow \mathcal{M} = \begin{array}{c}
\end{array}.
\]

The compositions of \(\rho : \mathcal{M} \to \mathcal{N}\) and \(\sigma : \mathcal{P} \to \mathcal{M}\) is then represented by horizontally merging diagrams from the right to the left in the following way

\[
\sigma \circ \rho = \begin{array}{ccc}
\sigma & \circ & \rho \\
\rho & \circ & \rho \\
\sigma & \circ & \rho
\end{array},
\]

where we note that the shadings have to match.

For two morphisms \(\rho : \mathcal{M} \to \mathcal{N}\) and \(\sigma : \mathcal{M} \to \mathcal{N}\) the are \(2\text{-morphisms}\) or \(2\text{-cells}\) with an associative composition. The (\(\mathbb{C}\)-linear) vector space of 2–morphisms \(T : \rho \to \sigma\) is denoted by \(\text{Hom}(\rho, \sigma)\).

In our case they are intertwiners \((\rho, \sigma) = \text{Hom}(\rho, \sigma) = \{t \in \mathcal{N} : tp(m) = \sigma(M)t \text{ for all } m \in \mathcal{M}\}\).

The composition in this case is just given by the product \(T \cdot S\) and graphically by stacking the diagrams vertically

\[
T \cdot S = \begin{array}{ccc}
\sigma & \circ & \rho \\
\rho & \circ & \eta \\
\sigma & \circ & \rho
\end{array},
\]

where we note that the shadings have to match.
There is a unity $1_\rho \in \text{Hom}(\rho, \rho)$. For $T : \rho \to \sigma$ there is a $T^* : \sigma \to \rho$ which is given in the case of intertwiners by the adjoint. In diagrams

$$T = \begin{array}{c}
\sigma \\
\rho
\end{array} \quad T^* = \begin{array}{c}
\rho \\
\sigma
\end{array} ;$$

that means taking the adjoint on intertwiners corresponds to “reflecting” the diagram. For $\sigma, \sigma' : \mathcal{M} \to \mathcal{N}$ and $\rho, \rho' : \mathcal{N} \to \mathcal{P}$ and $T \in \text{Hom}(\rho, \rho')$ and $S \in \text{Hom}(\sigma, \sigma')$ there is a “tensor product of 2–morphisms” $T \otimes S$ graphically given by

![Diagram](image)

and in formulas in our particular case by

$$T \otimes S := \rho'(S) \cdot T \equiv T \cdot \rho(S) .$$

A morphism $\rho$ is called irreducible or simple if $\text{Hom}(\rho, \rho) = \mathbb{C} \cdot 1_\rho$. Two morphisms $\rho, \sigma : \mathcal{M} \to \mathcal{N}$ are equivalent if there exists a unitary intertwiner $S \in \langle \rho, \sigma \rangle$. The unitary equivalence class is denoted by $[\rho]$ and is called a sector. A morphism $\sigma : \mathcal{M} \to \mathcal{N}$ is a subobject of $\rho : \mathcal{M} \to \mathcal{N}$ if there exists a isometric intertwiner $S \in \langle \sigma, \rho \rangle$, i.e. isometric means that $S^*S \in \langle \sigma, \sigma \rangle = 1_\sigma$.

A morphism $\rho \in \text{Mor}(\mathcal{M}, \mathcal{N})$ is a direct sum of $\rho_i \in \text{Mor}(\mathcal{M}, \mathcal{N})$ if

$$\rho(\cdot) = \sum w_i \rho_i w_i^*$$

and isometries $w_i \in \text{Hom}(\rho_i, \rho)$ and we write $[\rho] = \bigoplus [\rho_i]$ and also $[\rho] = \bigoplus I N_i [\rho_i]$ if the sector $[\rho_i]$ comes with multiplicity $N_i \in \mathbb{N}$. We might ask semisimplicity, i.e. every morphism is a direct sum of irreducible ones.

A morphism $\bar{\rho} : \mathcal{N} \to \mathcal{M}$ is said to be a conjugate to $\rho : \mathcal{N} \to \mathcal{M}$ if there exists intertwiners $R \in (\text{id}_\mathcal{M}, \bar{\rho}\rho)$ and $\bar{R} \in (\text{id}_\mathcal{N}, \rho\bar{\rho})$ such that the equations

$$\begin{align*}
(1_\rho \otimes R^*) \cdot (\bar{R} \otimes 1_\rho) & \equiv \rho(R^*) \cdot \bar{R} = 1_{\rho} \\
(1_{\bar{\rho}} \otimes \bar{R}^*) \cdot (R \otimes 1_{\bar{\rho}}) & \equiv \bar{\rho}(\bar{R}^*) \cdot R = 1_{\bar{\rho}}
\end{align*}$$

(2.1)

$$\begin{align*}
(1_{\rho} \otimes R^*) \cdot (\bar{R} \otimes 1_{\rho}) & \equiv \bar{\rho}(R^*) \cdot \rho(R) = 1_{\rho} \\
(1_{\bar{\rho}} \otimes \bar{R}^*) \cdot (R \otimes 1_{\bar{\rho}}) & \equiv R^* \bar{\rho}(\bar{R}) = 1_{\bar{\rho}}
\end{align*}$$

(2.2)

hold, where the equivalence is given by taking the adjoint. The 2–morphisms $R, \bar{R}$ will graphically represent by

![Diagram](image)
and the above equations (2.1), (2.2) are sometimes called **zig-zag identities**, because in diagrams they are given by

\[
\begin{align*}
\rho & = \rho \\
\overline{\rho} & = \overline{\rho}
\end{align*}
\]

and

\[
\begin{align*}
\rho & = \rho \\
\overline{\rho} & = \overline{\rho}
\end{align*}
\]

If \( \rho \) is irreducible we ask the solution \( R, \overline{R} \) to be **normalized**, i.e. \( \| R \| = \| \overline{R} \| \). In the case that \( \rho \) is not irreducible we further ask that \( R, \overline{R} \) is a **standard** solution of the conjugate equation, i.e. \( R \) is of the form

\[
R = \sum_i (\overline{W}_i \otimes W_i) \cdot R_i,
\]

where \( R_i \in (\text{id}_M, \overline{\rho}_i \rho_i) \) is a normalized solution for an irreducible object \( \rho_i \preceq \rho \) and \( W_i \in (\rho_i, \rho) \) and \( \overline{W}_i \in (\overline{\rho}_i, \rho) \) are isometries expressing \( \rho \) and \( \overline{\rho} \) as direct sums of irreducibles. We define the dimension \( d_\rho \equiv d(\rho) \) of \( \rho \) to be \( d_\rho \in \mathbb{R}_+ \) with \( R^* R = d_\rho \cdot 1_M \) and note that \( d(\rho) = d(\overline{\rho}) \).

2.2.2. **Q-Systems.** The notion of a Q-system ([Lon94]) is similar to the notion of a Frobenius element in a monoidal category. Let \( M \) be a type III factor. We consider the \( C^* \)-tensor category \( \text{End}(M) \) of endomorphism of \( M \) with morphisms \( \text{Hom}_M(\rho, \sigma) = \{ T \in M : T \rho(x) = \sigma(x) T \text{ for all } x \in M \} \).

To make contact with the last section, we can see this as a \( 2-C^* \)-category with just one 0-cell \( M \).

**Definition 2.2.2.** A **Q-System** in \( M \) is a triple \( (\rho, T, S) \) with \( \rho \in \text{End}(M) \) and isometric intertwiners \( T \in \text{Hom}_M(\text{id}_M, \rho) \) and \( S \in \text{Hom}_M(\rho, \rho^2) \), such that \( T^* S = \rho(T^*) S = \lambda^{-\frac{1}{2}} \cdot 1 \) and \( S S = \rho(S) S \), for some \( \lambda \in \mathbb{R}_+ \).

We translate this definition into string diagrams and in this case we do not need shadings. We use the fact that \( T^* T = \lambda^2 \cdot 1 \).

They fulfill the following relations, which correspond to the law of a co-unit and co-associativity of a co-multiplication:

\[
\begin{align*}
\rho & = \rho \\
\rho & = \rho \\
\rho & = \rho \\
\rho & = \rho
\end{align*}
\]
We remark that for a Frobenius element in a general monoidal category is normally also asked the Frobenius law \( S^* \rho(S) = S S^* = \rho(S^*) S \):

\[
\begin{array}{c}
\rho \\
\rho \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\rho \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\rho
\end{array}
= \begin{array}{c}
\rho \\
\rho \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\rho \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\rho \\
\rho
\end{array}
= \begin{array}{c}
\rho \\
\rho \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\rho \\
\rho
\end{array}
\begin{array}{c}
\rho
\end{array}
\]

but it is here automatic due to the \(*\)-structure of the \(C^*\)-category as proved in [LR97].

**Lemma 2.2.3** ([Lon08b, Lemma 5.2.]). \( E(m) = S^* \rho(m) S \) defines a faithful normal conditional expectation of \( \mathcal{M} \) onto a subalgebra \( \mathcal{N} \subset \mathcal{M} \).

We note that from the definition of the Q-system follows that \( \rho \) is its own conjugate with \( R = \overline{R} = \sqrt{\lambda} \cdot ST \) and it is \( d(\rho) = \lambda \).

### 2.2.3. Q-systems associated with a subfactor.

Let \( \mathcal{N} \subset \mathcal{M} \) be a subfactor of finite index and \( \iota : \mathcal{N} \to \mathcal{M} \) the inclusion homomorphism and \( \overline{\iota} : \mathcal{M} \to \mathcal{N} \) the conjugated morphism with a canonical pair of isometric intertwiners \( \nu : \text{id}_M \to \gamma := \overline{\iota} \in \text{End}(\mathcal{M}) \) and \( w : \text{id}_N \to \theta := \iota \in \text{End}(\mathcal{N}) \) such that \( \iota(w)^* \nu = \lambda^{\frac{1}{2}} \text{id}_M \) and \( \overline{\iota}(\nu)^* w = \lambda^{\frac{1}{2}} \text{id}_N \). The endomorphism \( \gamma = \overline{\iota} \in \text{End}(\mathcal{M}) \) is called the **canonical endomorphism** and \( \theta = \iota \in \text{End}(\mathcal{N}) \) the dual canonical endomorphism. It can be shown that \( \gamma \) is of the form \( \gamma = \text{Ad} J_{\mathcal{N}, \Phi} \circ \text{Ad} J_{\mathcal{M}, \Phi} \) where \( \Phi \) is a vector cyclic and separating for both \( \mathcal{M} \) and \( \mathcal{N} \). Further we note that \( \theta \) is just the restriction of \( \gamma \) to \( \mathcal{N} \), more precisely \( \gamma = \theta \) by definition.

**Definition 2.2.4.** With a subfactor \( \mathcal{N} \subset \mathcal{M} \) with isometries \( \nu : \text{id}_M \to \gamma \) and \( w : \text{id}_N \to \theta \) we associate

1. **the canonical Q-system in** \( \mathcal{M} \), which is given by \( (\gamma, \nu, \iota(w)) \) and
2. **the dual canonical Q-system in** \( \mathcal{N} \), which is given by \( (\theta, w, \overline{\iota}(\nu)) \) =: \( (\theta, w, x) \).

In graphical notation:

\[
\gamma = \begin{array}{c}
\iota \\
\overline{\iota}
\end{array}, \quad \nu = \frac{1}{\sqrt{\lambda}} \quad : \begin{array}{c}
\sqrt{\Phi}
\end{array}, \quad \iota(w) = \frac{1}{\sqrt{\lambda}} \quad : \begin{array}{c}
\lambda^{\frac{1}{2}} \text{id}_M
\end{array}, \quad \theta = \begin{array}{c}
\overline{\iota} \\
\iota
\end{array}, \quad w = \frac{1}{\sqrt{\lambda}} \quad : \begin{array}{c}
\sqrt{\Phi}
\end{array}, \quad x = \frac{1}{\sqrt{\lambda}} \quad : \begin{array}{c}
\lambda^{\frac{1}{2}} \text{id}_N
\end{array}.
\]

We note that \( (\theta, w, \overline{\iota}(\nu)) \) is the canonical Q-system for \( \overline{\iota}(\mathcal{M}) \subset \mathcal{N} \) which is isomorphic to \( \mathcal{M} \subset \mathcal{M}_1 \), which is obtained from \( \mathcal{N} \subset \mathcal{M} \) by Jones basic construction. The application of the basic construction to \( \overline{\iota}(\mathcal{M}) \subset \mathcal{N} \) recovers (up to an isomorphism) the original subfactor \( \mathcal{N} \subset \mathcal{M} \) and in particular \( \mathcal{N} \subset \mathcal{M} \) is also determined by its dual Q-system \( (\theta, w, \overline{\iota}(\nu)) \).
Given a Q-system \((\theta, w, x)\) in \(\mathcal{N}\) a concrete realization of \(\mathcal{M}\) is given by a representation \(\pi\) of \(\mathcal{N}\) on a Hilbert space \(\mathcal{H}\) such that \((\gamma, v, i(w))\) is the Q-system in \(\mathcal{M} := \mathcal{N} \vee \{v\}\) and \(\gamma\) is the extension of \(\theta\) given by \(\gamma(v) := \pi(x)\).

### 2.2.4. Charged intertwiner.

Let \(\mathcal{N} \subset \mathcal{M}\) be an irreducible (i.e. \(\mathcal{N}' \cap \mathcal{M} = \mathbb{C} \cdot 1\)) inclusion of type III factors having finite index \(\lambda := [\mathcal{M} : \mathcal{N}]\). We denote by \(i : \mathcal{N} \to \mathcal{M}\) the inclusion map \(^1\) and \(\bar{i} : \mathcal{M} \to \mathcal{N}\) its conjugate. We note that \(i(\mathcal{N}') \cap \mathcal{M} = \text{Hom}(i, i)\) so \(\mathcal{N} \subset \mathcal{M}\) irreducible is equivalent with \(i\) being irreducible. Let \((\gamma, v, i(w))\) and \((\theta, w, \bar{i}(v))\) be their canonical and dual canonical Q-system, respectively.

We note that \(E(m) = w^*\bar{i}(m)w\) defines a normal faithful conditional expectation \(E : \mathcal{M} \to \mathcal{N}\) by noting that \(i(E(m) = i(w)^*\gamma(m)i(w)\) and using Lemma 2.2.3. We can express \(\mathcal{M}\) as \(i(\mathcal{N})\gamma\), namely:

**Proposition 2.2.5.** It is \(\mathcal{M} = i(\mathcal{N})\gamma\), i.e. each element \(m \in \mathcal{M}\) has a unique decomposition \(m = i(\gamma)v\) with \(v \in \mathcal{N}\). It is given by

\[
\nu = \lambda E(\nu^*\nu) = \lambda w^*\bar{i}(w^*)w = \lambda^2 w^*i(m).
\]

**Definition 2.2.6.** For \(\rho \in \text{End}(\mathcal{M})\) (or more generally \(\rho \in \text{Mor}(\mathcal{N}, \mathcal{M})\)) irreducible, i.e. \(\text{Hom}_\mathcal{M}(\rho, \rho) = \mathbb{C} \cdot 1\) and \(\sigma \in \text{End}(\mathcal{M})\) the linear space \(\text{Hom}(\rho, \sigma)\) is a Hilbert space with scalar product \((\cdot, \cdot) \equiv (\cdot, \cdot)_{\text{Hom}(\rho, \sigma)}\) given by

\[
(S, T)_{\text{Hom}(\rho, \sigma)} = 1 = S^*T \in \text{Hom}(\rho, \rho) \quad S, T \in \text{Hom}_\mathcal{M}(\rho, \sigma)
\]

called the **Hilbert space of isometries** \(\text{Hom}_\mathcal{M}(\rho, \sigma)\), also denoted by \((\rho, \sigma)\).

The **Hilbert space of charged intertwiners** for \(\rho_i\) is given by

\[
K_i := (i, \nu_i) = \{R \in \mathcal{M} : R t(x) = i(\nu_i) R \text{ for all } x \in \mathcal{N}\}
\]

where we here explicitly write the \(i\).

**Definition 2.2.7.** We denote by \((\rho, \sigma)\) the dimension of the linear space \(\text{Hom}(\rho, \sigma)\), i.e. \(\langle \rho, \sigma \rangle := \dim(\rho, \sigma) \equiv \dim \text{Hom}(\rho, \sigma)\).

**Proposition 2.2.8** (Frobenius reciprocity). For \(\rho : \mathcal{M} \to \mathcal{P}\), \(\sigma : \mathcal{N} \to \mathcal{P}\) and \(\beta : \mathcal{M} \to \mathcal{N}\) we have \((\rho, \sigma\beta) = (\bar{\sigma}\rho, \beta)\).

**Proposition 2.2.9** ([LR95]). There is an antilinear isomorphism between \((\rho_i, \theta)\) and \(K_i := (i, \nu_i)\) given by:

\[
T \mapsto X = i(T^*) \quad \text{ and } \quad X \mapsto [\mathcal{M} : \mathcal{N}] \cdot E(vX^*).
\]

We can represent the isomorphism and its inverse diagrammatically by the following maps:

\[\text{Diagram 1:} \quad \text{Diagram 2:} \quad \text{Diagram 3:}\]

\(^1\)One does not loose anything if one sets \(\mathcal{M} = \mathcal{N}\) or one assumes \(i\) to be the identity. But it is somehow convenient to write the \(i\) for some reasons. First is the graphical calculus for Q-systems as we will see, and second we can treat subfactors \(\rho(\mathcal{M}) \subset \mathcal{M}\) the same way then \(i(\mathcal{N}) \subset \mathcal{M}\), we still write \(\mathcal{N} \subset \mathcal{M}\) if no confusion arises.
2.3. Preliminaries on standard subspaces

By Frobenius reciprocity (Proposition 2.2.8) it is $\langle \rho_i, \theta \rangle \equiv \langle \rho_i, \bar{\iota} \rangle = \langle \iota, \iota \rho_i \rangle$. We show that the maps are inverse to each other, namely

$$\begin{align*}
X \iota &\mapsto \sqrt{\iota}, \\
X^* &\mapsto T
\end{align*}$$

and in particular the maps $T \mapsto X$ and $X \mapsto T$ are isomorphisms of finite dimensional vector spaces. \qed

Let $\{[\rho_i], i = 0, \ldots, N\}$ be the family of irreducible (and pairwise inequivalent) sectors in the decomposition of $[\theta]$, i.e.

$$[\theta] = \bigoplus_{i=0}^{N} N_i [\rho_i],$$

with $\rho_0 = \text{id}_{\mathcal{N}}$. We can write $\theta$ directly as

$$\theta(n) = \sum_{i=0}^{N} \sum_{k=1}^{N_i} w_{ik} \rho_i(n) w_{ik}^*,$$

where $\{w_{ik}\}_{k=1, \ldots, N_i}$ is an orthonormal basis of $(\rho_i, \theta)$ (with $w = w_{01}$). We have

$$v = \sum_{i=0}^{N} \sum_{k=1}^{N_i} \iota(w_{ik} w_{ik}^*) v = \sum_{i=0}^{N} \sum_{k=1}^{N_i} \iota(w_{ik} R_{ik}),$$

where $R_{ik} = \iota(w_{ik})^* v$ are the associated charged interwiners having the normalization

$$R_{ik}^* R_{j\ell} = \frac{d_i}{\lambda} \cdot \delta_{ij} \delta_{k\ell}.$$

**Proposition 2.2.10** ("Harmonic Analysis" for subfactors [LR95]). Each $m \in \mathcal{M}$ has a unique decomposition:

$$m = \sum_{i=0}^{N} \sum_{k=1}^{N_i} \iota(x_{ik}) R_{ik}$$

with $x_{ik} \in \mathbb{N}$.

The coefficients are given by $x_{ik} = \lambda E(m R_{ik}^*)$.

### 2.3. Preliminaries on standard subspaces

If we have $(\mathcal{M}, \Omega)$ a von Neumann algebra on $\mathcal{H}$ in standard form then the closure $H(\mathcal{M}, \Omega)$ of the real linear space $\{a \Omega : a = a^* \in \mathcal{M}\}$ contains all the information to define the modular conjugation and the modular operator. If one abstracts the properties of the real subspace $H(\mathcal{M}, \Omega)$ one gets notion of a standard subspace of a Hilbert space and one can define a (simplified) Tomita–Takesaki theory. This theory is—as we will see—particularly interesting for describing a prequantized theory, so a theory on the so-called one-particle Hilbert space.
2.3.1. Standard subspaces. We repeat some basic facts (for details see [Lon08b]) on standard subspaces. Let \( \mathcal{H} = (\mathcal{H}, (\cdot, \cdot)) \) be a Hilbert space and let \( H \subset \mathcal{H} \) be a real subspace. We denote by \( H' = \{ x \in \mathcal{H} : \text{Im}(x, H) = 0 \} \) the \textbf{symplectic complement}, which is closed. In particular it is \( H'' = \overline{H} \). A closed real subspace \( H \) is called:

- **cyclic** if \( H + iH \) is dense in \( \mathcal{H} \),
- **separating** if \( H \cap iH = \{ 0 \} \) and
- **standard** if \( H \) is cyclic and separating.

So a closed real subspace \( H \) is separating or cyclic if and only if its symplectic complement \( H' \) is cyclic or separating, respectively, and \( H \) is standard if and only if \( H' \) is standard.

We denote the set of all standard subspaces of \( \mathcal{H} \) by \( \text{Std}(\mathcal{H}) \). To a standard subspace we relate a pair \( (J_H, \Delta_H) \), where \( (\Delta_H^u)_{u \in \mathbb{R}} \) is a unitary one-parameter group called the \textbf{modular unitaries} or \textbf{modular group} and an antiunitary involution \( J_H \) called \textbf{modular conjugation}. Both are defined by polar decomposition of the densely defined, closed, antilinear involutive (i.e. \( S_H^2 \subset \text{id}_{\mathcal{H}} \)) operator \( S_H = J_H \Delta_H^{1/2} \) with domain \( H + iH \) defined by \( x + iy \mapsto x - iy \) for \( x, y \in H \).

**Proposition 2.3.1.** There is a bijective correspondence between

- standard subspace \( H \subset \mathcal{H} \) and
- densely defined, closed, antilinear involutions \( S \)
- operators \( (J, \Delta) \) on \( \mathcal{H} \), where \( J \) is an antiunitary involution \( J = J^* = J^{-1} \) and \( \Delta = S^*S \) is a positive, non-singular self-adjoint operator with \( J\Delta J = \Delta^{-1} \)

given by the map \( H \mapsto S_H \) as above, with inverse map associating with such an involution \( S \) the standard subspace \( H_S = \{ x \in \text{Dom}(S) : Sx = x \} = \ker(1 - S) \) and the other correspondence given by the polar decomposition. It holds \( S_{H'} = S_H^* \). For a unitary \( U \) we have \( UH = H \) if and only if \( [S_H, U] = 0 \).

A (simpler) real subspace version of the Tomita-Takesaki theorem gives:

**Proposition 2.3.2 (Modular theory).** Let \( S_H = J_H \Delta_H^{1/2} \) be the polar composition of \( S_H \) with \( J_H \) an antunitary and \( \Delta_H^{1/2} \) a positive operator. Then

\[ J_H H = H', \quad \Delta_H^u H = H \quad \text{for } u \in \mathbb{R}. \]

2.3.2. Semigroup associated with standard pairs.

**Definition 2.3.3.** Let \( H \) be a standard subspace of a Hilbert space \( \mathcal{H} \) and let us assume that there exists a one-parameter group \( T(t) = e^{itP} \) on \( \mathcal{H} \) such that:

- \( T(t)H \subset H \) for all \( t \geq 0 \),
- \( P > 0 \).

Then we call the pair \( (H, T) \) a \textbf{standard pair}. It is called \textbf{non-degenerated} if the kernel of \( P \) is \( \{ 0 \} \).

It holds a one-particle version of \textbf{Borchers theorem} which gives the relations between the one-parameter groups \( T(t) \) and \( \Delta_H^u \). By this relations one gets a representation of the \( ax + b \) group (also called \textbf{affine group}) and these relations are equivalent to the canonical commutation relations and because of the Stone–von Neumann theorem there exists just one unique irreducible standard pair, more precisely:

**Theorem 2.3.4 ([LW11 Theorem 2.2]).** Let \( (H, T) \) be a non-degenerate standard pair:

1. Then it holds for all \( t, s \in \mathbb{R} \):
   \[ \Delta_H^u T(t) \Delta_H^{-is} = T(e^{2\pi i t}), \quad J_H T(t)J_H = T(-t), \]
where $\Delta_H^0$ and $J_H$ are the modular unitaries and conjugation, respectively, associated with the standard space $H$, i.e. $J_H H = H'$ and $\Delta_H^0 H = H$.

(2) $(H, T)$ yields a unitary positive energy representation of the translation-dilation group of $\mathbb{R}$ also called the $ax + b$ group, by associating with $x \mapsto e^{-2\pi i x + t}$ the unitary element $T(t) = \Delta_H^0$.

(3) There is a unique irreducible standard pair and each standard pair is a multiple of this unique standard pair.

**Definition 2.3.5.** Let $(H, T)$ be a standard pair on $\mathcal{H}$. The semigroup of unitaries $V$ of $\mathcal{H}$ commuting with $T$ such that $VH \subset H$ is denoted by $\mathcal{E}(H, T) = \mathcal{E}(H)$. We call $V$ a Longo–Witten unitary of the standard pair $(H, T)$.

The elements of $\mathcal{E}(H)$ are characterized in [LW11]. We first state the case of the irreducible standard pair, where the semigroup $\mathcal{E}(H_0)$ can be identified with a semigroup of certain “symmetric inner functions”.

**Definition 2.3.6.** We denote by $\mathcal{S}$ the set of all **inner functions**, i.e. complex Borel functions $\phi : \mathbb{R} \to \mathbb{C}$ which are boundary values of a bounded analytic functions on $\mathbb{R} + i\mathbb{R}_+$ with $|\phi(p)| = 1$ for almost all $p \geq 0$, which are also **symmetric**, i.e. $\overline{\phi(p)} = \phi(-p)$ for almost all $p \in \mathbb{R}$.

**Theorem 2.3.7** ([LW11 Corollary 2.4]). Let $(H_0, T_0)$ be the unique irreducible standard pair then $V \in \mathcal{E}(H)$ if and only if $V = \phi(P)$ for some $\phi \in \mathcal{S}$.

In the reducible case the semigroup $\mathcal{E}(H)$ consists of matrices of boundary values of analytical functions and the condition $|f(p)| = 1$ is generalized to unitarity of the matrix.

**Remark 2.3.8.** Let $(H, T)$ be a non-zero, non-degenerated standard pair on a Hilbert space $\mathcal{H}$. Then it can be decomposed as a direct sum of the unique irreducible standard pair. Let

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i \quad H = \bigoplus_i H_i \quad T = \bigoplus_i T_i$$

be such a finite or infinite decomposition, where each $(H_i, T_i)$ is a standard pair in $\mathcal{H}_i$ and can be identified with the unique irreducible standard pair $(H_0, T_0)$ with generator $P_0$.

**Definition 2.3.9.** For $n \in \mathbb{N} \cup \{\infty\}$ we denote by $\mathcal{S}^{(n)}$ the set of matrices $(\phi_{hk})_{1 \leq h,k \leq n}$ where $\phi_{hk} : \mathbb{R} \to \mathbb{C}$ are complex Borel functions which are boundary values of a bounded analytic function on $\mathbb{R} + i\mathbb{R}_+$ such that $\phi_{hk}(p)$ is a unitary matrix for almost all $p$, which is symmetric, i.e. $\overline{\phi_{hk}(p)} = \phi_{hk}(-p)$.

**Theorem 2.3.10** ([LW11 Theorem 2.6]). Let $(H, T)$ be like in Remark 2.3.8. Then $V \in \mathcal{E}(H)$ if and only if it is a $n \times n$ matrix $(V_{hk})$ with entries in $\mathcal{B}(\mathcal{H})$ such that $V_{hk} = \phi_{hk}(P_0)$ for some $(\phi_{hk}) \in \mathcal{S}^{(n)}$.

We close this section by noting that we can see Theorem 2.3.7 as an abstract (real) version of the classical Beurling–Lax theorem. The Hardy space $H^2(\mathbb{S}^1)$ is given by

$$H^2(\mathbb{S}^1) := \left\{ f : \text{analytic on the unit disk } D, \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty \right\}.$$ 

Any function in $H^2(\mathbb{S}^1)$ has a $L^2$-boundary value and can be considered as an element of $L^2(\mathbb{S}^1)$. In this sense, $H^2(\mathbb{S}^1)$ is a subspace of $L^2(\mathbb{S}^1)$.

**Theorem 2.3.11** (Beurling–Lax theorem [Beu49, Lax59]). Let $K \subset H^2(\mathbb{S}^1)$ be invariant under the shift $S : f(z) \mapsto z \cdot f(z)$, i.e. $SK \subset K$ then

$$K = \theta H^2(\mathbb{S}^1) = \left\{ z \mapsto \theta(z)f(z) : f \in H^2(\mathbb{S}^1) \right\},$$

where $\theta$ is an inner function on $\mathbb{S}^1$, i.e. an analytic function on $\mathbb{D}$ such that $|\theta(z)| = 1$ for almost all $z \in \mathbb{S}^1$. 

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Proposition 2.3.12. Let \((H, T)\) be a non-degenerated standard pair with \(T(t) = e^{itP}\). A standard subspace \(K \subset H\) satisfies \(T(t)K \subset K\) for \(t \geq 0\) if and only if \(K = VH\) for some \(f \in \mathcal{E}(H, T)\). In particular, if \((H, T)\) is irreducible, \(K = \varphi(P)H\) with \(\varphi \in \mathbb{S}^1\) a inner symmetric function.

2.3.3. Araki double construction. We comment on the application of standard subspaces and a different formulation using complex subspaces used by some other authors in the construction of free fields.

The Hilbert space \(\mathcal{H}\) plays the rôle of the one-particle space while a standard subspace (or in general a real subspace) plays the rôle of the closure of real test functions localized in some given (space-time) region, which is embedded in the one-particle space. For example, in the massive case, this embedding is given by projection of the Fourier transformation on the upper mass shell. Because the one-particle space has a complex structure one has fixed the two point function and therefore the quasi free representation. This is particularly useful if one just work in a unique (vacuum) representation. Sometimes it is taken another philosophy, where one starts with \((\mathcal{K}, \Gamma)\) a complex Hilbert space \(\mathcal{K}\) with complex conjugation \(\Gamma\), i.e. an antiunitary involution \((\Gamma^2 = 1)\). A basis projection is a projection which fulfills \(\Gamma P \Gamma = 1 - P\) and it fixes the representation. The choice of a basis projection correspond to the choice of a complex structure on \(\mathcal{H}\).

If \(\mathcal{H}\) is a Hilbert space we denote by \(\overline{\mathcal{H}}\) the complex conjugated Hilbert space, which is as a real Hilbert space the same space as \(\mathcal{H}\), but has a new complex structure given by \(\mathcal{J} = -i\). We can write \(\mathcal{H} = \{x : x \in \mathcal{H}\}\) and we get a canonical antiunitary map \(x \mapsto \overline{x}\) from \(\mathcal{H} \to \overline{\mathcal{H}}\), so it is convenient to denote elements in \(\overline{\mathcal{H}}\) by \(\overline{x}\) so that in particular \(ix = -\overline{x}\) holds.

If we start with a Hilbert space \(\mathcal{H}\) as “one-particle space” we can pass to the double \(\mathcal{K} = \mathcal{H} \oplus \overline{\mathcal{H}}\) with \(\Gamma(x \oplus \overline{y}) = y \oplus \overline{x}\) and the projection \(P(x \oplus \overline{y}) = x \oplus 0\). Then \(P\) is obviously a basis projection. Conversely, if we start with \((\mathcal{K}, \Gamma)\) and a given basis projection \(P\), we can identify \(\text{Re } \mathcal{K} = \{x \in \mathcal{K} : \Gamma x = x\}\), “the completion of the space of real test functions” with \(P\mathcal{H}\) the “one-particle space” by the orthogonal transformation (isomorphism of real Hilbert spaces)

\[
\text{Re } \mathcal{K} \ni x \mapsto \sqrt{2}Px \in P\mathcal{K}, \quad P\mathcal{K} \ni y \mapsto \frac{1}{\sqrt{2}}y + \Gamma y \in \text{Re } \mathcal{K}
\] (2.3)

which in particular induces a complex structure \(\mathcal{J}\) on \(\text{Re } \mathcal{K}\). This complex structure is given by the pull-back of the multiplication by \(i\) on the one-particle space \(P\mathcal{H}\). It is easy to check that \(\mathcal{J} = i(2P-1)\), namely for \(y \in P\mathcal{K}\) it is \(i(2P-1)(y + \Gamma y) = i(2P-1)y + i\Gamma(1-2P)y = (iy + \Gamma iy)\).

So there is a one-to-one correspondence between \(\mathcal{H}\) and \((\mathcal{K}, \Gamma, P)\) by the above double construction. Closed real Hilbert subspaces of \(\mathcal{H}\) are in correspondence with closed subspaces \(q\) which are invariant under \(\Gamma\), i.e. \(\Gamma q = q\). For such a \(q\), we denote by \(\text{Re } q = \{x \in q : \Gamma x = q\}\). Then \(P\text{Re } q \subset P\mathcal{K}\) can be seen as a real subspace of \(\mathfrak{p} := P\mathcal{K}\) by the above identification. On the other hand if \(\mathfrak{h} \subset \mathcal{H}\) is a real subspace we define the real subspace \(\text{Re } \mathfrak{h} \subset \text{Re } \mathcal{K}\) to be the image under the inverse of (2.3) and \(q = \text{Re } q + i \text{Re } q\), which is obviously \(\Gamma\) invariant.

Proposition 2.3.13. There is a one-to-one correspondence between Hilbert spaces \(\mathcal{H}\) and triple \((\mathcal{K}, \Gamma, P)\), where \(\mathcal{K}\) is a Hilbert space, \(\Gamma\) is an antiunitary involution and \(P\) a basis projection. It is given by \(\mathcal{K} = \mathcal{H} \oplus \overline{\mathcal{H}}, \Gamma(x \oplus \overline{y}) = y \oplus \overline{x}\) and \(P(x \oplus y) = x \oplus 0\). The converse is given by \(\mathcal{K} = P\mathcal{H}\) which can identify with \(\text{Re } \mathcal{K}\) with complex structure by \(\mathcal{J} = i(2P-1)\).

For a corresponding \(\mathcal{H}\) and \((\mathcal{K}, \Gamma, P)\) there is one-to-one correspondence between real closed subspace of \(\mathcal{H}\) and complex closed and \(\Gamma\) invariant subspaces \(q\) of \(\mathcal{K}\). It holds further with \(\mathfrak{p} = P\mathcal{K}\):

1. \(\mathfrak{p} \cap q^{\perp} = \{0\}\) if and only if \(\mathfrak{h}\) is cyclic, i.e. \(\mathfrak{h} \cap i\mathfrak{h}\) is dense in \(\mathcal{H}\).
2. \(\mathfrak{p} \cap q = \{0\}\) if and only if \(\mathfrak{h}\) is separating, i.e. \(\mathfrak{h} \cap i\mathfrak{h} = \{0\}\).
2.4. SECOND QUANTIZATION

We want to introduce the Bosonic and Fermionic second quantization. Given a Hilbert space $\mathcal{H}$, the one-particle Hilbert space, and a real subspace $H \subset \mathcal{H}$ one can associate two von Neumann algebras: the Bosonic $\mathcal{R}(H)$ and the Fermionic $\mathcal{C}(H)$ which are acting on the symmetric (Bosonic) and antisymmetric (Fermionic) Fock space, respectively. Unitaries on the one-parameter space $\mathcal{H}$ can be promoted to second quantization unitaries on the respective Fock spaces and it turns out that if $H$ is standard then Fock space vacuum vector $\Omega$ is cyclic and separating, in other words $(\mathcal{R}(H), \Omega)$ and $(\mathcal{C}(H), \Omega)$ happen to be in standard form and the modular unitary groups are given by second quantization of the modular unitary group of the standard subspace $H$. Further it holds some abstract version Haag duality in the Bose case or Haag–Araki duality in the Fermi case, namely $R(H') = R(H)'$ and $\mathcal{C}(H^{\perp}) = \mathcal{C}(H)^{\perp}$ where $H^{\perp} = iH'$ is the real orthogonal complement and $\mathcal{M}^{\delta}$ is the graded commutant. Some generalization will be given in Section 10.3.

2.4.1. Bosonic. Let $\mathcal{H}$ be a Hilbert space and $\omega(\cdot, \cdot) = \text{Im}(\cdot, \cdot)$ the sesquilinear form. Then we can define unitaries $W(f)$ for $f \in \mathcal{H}$ fulfilling

$$W(f)W(g) = e^{-\text{i}\omega(f, g)}W(f + g) = e^{-2\text{i}\omega(f, g)}W(g)W(f)$$

and acting naturally on the Bosonic Fock space $e^{\mathcal{H}}$ over $\mathcal{H}$. This space is given by $e^{\mathcal{H}} = \sum_{n=0}^{\infty} P_n \mathcal{H}^\otimes n$, where $P_n$ is the projection $P_n(x_1 \otimes \cdots \otimes x_n) = 1/n! \sum_{\sigma} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ and the sum goes through all permutations. The set of coherent vectors $e^h := \bigoplus_{n=0}^{\infty} h^{\otimes n}/\sqrt{n!}$ with $h \in \mathcal{H}$ is total in $e^{\mathcal{H}}$ and it is $(e^f, e^g) = e^{\omega(f, h)}$. The vacuum is given by $\Omega = e^0$ and the action of $W(f)$ is given by $W(f)e^0 = e^{-\frac{1}{2}||f||^2}e^f$, in other words the vacuum representation $\omega(\cdot) = (\Omega, \cdot \Omega)$ is characterized by $\omega(W(f)) = e^{-\frac{1}{2}||f||^2}$.

Finally, for a real subspace $H \subset \mathcal{H}$ we define the von Neumann algebra

$$R(H) = \{W(f) : f \in H\}'' \subset \text{B}(e^{\mathcal{H}}).$$

Let $U$ be a unitary in $\text{B}(\mathcal{H})$ then $\Gamma(U) = \bigoplus_{n=1}^{\infty} U^{\otimes n}$ acts on coherent states by $\Gamma(U)e^h = e^{U^*h}$ and is therefore a unitary (cf. [Gui11]) on $e^{\mathcal{H}}$, the second quantization unitary of $U$. Second quantization unitaries implement Bogoliubov automorphism, namely $\Gamma(U)W(f)\Gamma(U)^* = W(Uf)$ and in particular we have covariance $\Gamma(U)R(H)\Gamma(U)^* = R(UH)$. The map $R$ has the following properties:

Proposition 2.4.1 ([Lon08a]).

(1) Let $H, K \subset \mathcal{H}$ be real linear subspaces. Then $R(K) = R(H)$ iff $K = \bar{H}$.
(2) Let $H$ be closed. $H$ is separating or cyclic iff $R(H)$ is separating or cyclic, respectively.
(3) Let $H$ be standard, then the modular unitaries $\Delta^H_{(R(H), \Omega)}$ and the modular conjugation $J_{(R(H), \Omega)}$ associated with $(R(H), \Omega)$ are given by

$$\Delta^H_{(R(H), \Omega)} = \Gamma(\Delta^H), \quad J_{(R(H), \Omega)} = \Gamma(J_H)$$

and in particular $R(H') = R(H)'$. 

Proof: First $q \cap p = \{0\}$ is equivalent with $P_q$ being dense in $p$. But this corresponds to $H + iH$ being dense $\mathcal{H}$.

Let us assume $x \in H \cap iH$, i.e. $x, ix \in H$. If we identify $x \in H$ with $x \in \text{Re} q$, then we get that also $Jx \in \text{Re} q$ and $ix \in q$. So we have $y := Jx + ix = i(2P - 1)x + ix = 2Px$ is in $p \cap q$.

Conversely, if $x \in p \cap q$, then $Px = x$ and $(1 - P)\Gamma x = \Gamma x$. By defining $y := x + \Gamma x \in \text{Re} q$, we check that $Jy = i(2P - 1)(x + \Gamma x) = ix - i\Gamma x = ix + \Gamma ix$, is real and because $ix \in q$ we get $Jx \in q$ and we have by correspondence found a $y \in H \cap iH$. 


2.4.2. Fermionic. Let $\mathcal{H}^1$ be a complex Hilbert space and $\mathcal{H} = \Lambda(\mathcal{H}^1)$ the antisymmetric or Fermionic Fock space is obtained by completing the exterior algebra with the inner product:

$$(e_1 \land \cdots \land e_m, f_1 \land \cdots \land f_n) = \delta_{mn} \det A_{ij}$$

where $A_{ij} = (e_i, f_j)$ for $1 \leq i, j \leq n$.

For $A \in B(\mathcal{H}^1)$ with $\|A\| \leq 1$ we define $\Lambda(A)$ to be $A^{\otimes k}$ on $\mathcal{H}^k := \Lambda^k(\mathcal{H}^1) \subset (\mathcal{H}^1)^{\otimes k}$. If $U$ is a unitary we call $\Lambda(U)$ second quantization unitary. The space is $\mathbb{Z}_2$ graded by $\Gamma := \Lambda(-1)$. We define $Z = \frac{1-i}{1+i}$ and note that $Z^2 = 1$. For $f \in \mathcal{H}^1$ let $a(f)$ be the bounded operator obtained by continuing the exterior multiplication $f \land \cdots$. The operators fulfill the complex Clifford relations $a(f)^* a(g) + a(g)a(f)^* = (f, g)$ and $\{a(f), a(g)\} = \{a(f)^*, a(g)^*\} = 0$ for all $f, g \in \mathcal{H}^1$. For a standard subspace $K \subset \mathcal{H}^1$ we define the von Neumann algebra

$$C(K) = \{c(f) : f \in K\}'' \subset B(\mathcal{H})$$

where $c(f) = a(f) + a(f)^*$, which fulfills the real Clifford relations $c(f)c(g) + c(g)c(f) = 2 \text{Re}(f, g)$. By $\Omega = 1 \in \Lambda^1(\mathcal{H}^1)$ we denote the vacuum which is cyclic and separating for $C(K)$ for every standard subspace $K \subset \mathcal{H}^1$. For a real subspace $K$ we define the real orthogonal complement to be $K^\perp = iK^\perp = \{x \in \mathcal{H} : \text{Re}(x, K) = 0\}$.

**Proposition 2.4.2** ([Foi83, Was98] see also Proposition 10.3.9).

1. Let $H, K \subset \mathcal{H}^1$ be real linear subspaces. Then $C(K) = C(H)$ iff $K = H$.
2. Let $H$ be closed. $H$ is separating or cyclic iff $C(H)$ is separating or cyclic, respectively.
3. Let $H$ be standard, then the modular unitaries $\Lambda^H := \Lambda(\mathcal{H}^{1,H})$ and the modular conjugation $J_{C(H)}$ associated with $(R(H), \Omega)$ are given by

$$\Lambda^H = \Lambda(\mathcal{H}^1), \quad J_{C(H)} = \Lambda^*(J_H)$$

and in particular it holds Haag-Araki duality, i.e. $C(K^\perp)$ equals $C(K)^\perp := ZC(K)^\perp Z^*$, the twisted commutant of $C(K)$.

For a unitary $U$ on $\mathcal{H}^1$ it holds $\Lambda(U)c(f)\Lambda(U^*) = c(Uf)$, which implies that $C$ is covariant with respect to the unitaries $U(\mathcal{H}^1)$, i.e. $\Lambda(U)C(\mathcal{H}^1)\Lambda(U^*) = C(U\mathcal{H}^1)$.

We note that for example in the case of the complex Fermion the one-particle space is obtained from a Hilbert space $\mathcal{H}^1$ (the space of test functions) and a projection $P$ by $\mathcal{H}^1 = P \mathcal{H}^1 \oplus P^\perp \mathcal{H}^1$ and one gets a new representation of the complex Clifford algebra on $\Lambda(P \mathcal{H}^1)$ by defining

$$a_P(f) := a(Pf) + a(P^\perp f)^*$$

where $a(f)$ is the creation operator. For a standard subspace $K \subset \mathcal{H}^1_P$ which is invariant under the multiplication of $i\mathcal{H}^1$ in $\mathcal{H}^1$, the von Neumann algebra $C(K)$ on $\Lambda(\mathcal{H}^1_P)$ coincides with the von Neumann algebra $\{a_P(f), a_P(f)^* : f \in K\}$, so in this case

$$R(H) = \{c(f) : f \in H\}'' = \{a_P(f), a_P(f)^* : f \in K\}''$$

holds. Indeed, the one inclusion follows from $c(f) = a_P(f) + a_P(f)^*$ and the other follows from Araki-Haag duality and $\{a_P(f), c(g)\} = (g, f)_{\mathcal{H}_P^1} = \text{Re}(g, f)_{\mathcal{H}_P^1} - i \text{Re}(g, i_{\mathcal{H}_P^1}f)_{\mathcal{H}_P^1} = 0$ for $f \in K$ and $g \in K^\perp$. We further note that the space $\Lambda(\mathcal{H}^1_P)$ is as a real Hilbert space the same as $\Lambda(\mathcal{H}^1)$ and can be identified canonically with $\Lambda(P\mathcal{H}^1) \otimes \Lambda(P^\perp \mathcal{H}^1)$.
2.4.3. Araki’s self-dual CAR algebra. We will also use Araki’s self-dual approach to the CAR algebra [Ara70]. Let $\mathcal{K}$ be a Hilbert space with complex conjugation $\Gamma$, i.e. an antiunitary involution ($\Gamma^2 = 1$). The self-dual CAR algebra $\text{CAR}(\mathcal{K}, \Gamma)$ is the unital $C^*$-algebra generated by $\psi(f)$ depending linearly on $f \in \mathcal{K}$ subject to the relations
\[
\psi(f)^* = \psi(\Gamma f),
\]
\[
\{\psi(f)^*, \psi(g)\} = \psi(f)^*\psi(g) + \psi(g)\psi(f)^* = (f, g) \cdot 1 \quad f, g \in \mathcal{K}.
\]
Any real isometry $V = \Gamma V \Gamma \in \mathcal{B}(\mathcal{K})$ is called a Bogoliubov operator and induces a Bogoliubov endomorphism $\rho_V$ specified by
\[
\rho_V(\psi(f)) = \psi(V f) \quad f \in L^2(\mathbb{S}^1);
\]
an example is the $\mathbb{Z}_2$ gauge automorphism $\rho_{-1}$.

A state $\omega$ of $\text{CAR}(\mathcal{K}, \Gamma)$ is called quasifree if it is determined by its two-point function $\omega(\psi(f)\psi(g))$ in the following way:
\[
\omega(\psi(f_1) \cdots \psi(f_n)) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{n(n-1)/2} \sum_{\sigma} \text{sgn}(\sigma) \prod_{j=1}^{n/2} \omega(\psi(f_{\sigma(j)}^* \psi(f_{\sigma(j+n/2)}))) & \text{if } n \text{ is even,} \end{cases}
\]
where the sum runs over all permutations of $\{1, \ldots, n\}$ satisfying:
\[
\sigma(1) < \cdots < \epsilon(n/2), \quad \sigma(j) < \epsilon(j + n/2) \quad j = 1, \ldots, n/2.
\]
For $(\mathcal{K}, \Gamma)$ let $K(\mathcal{K}, \Gamma)$ be the set of operator $S \in \mathcal{B}(\mathcal{K})$ with $S + \Gamma S \Gamma = 1$ and $0 \leq S = S^* \subset 1$. An operator $S \in K(\mathcal{K}, \Gamma)$ defines a quasi-free state $\varphi_S$ of $\text{CAR}(\mathcal{K}, \Gamma)$ by $\varphi_S(\psi(f)^* \psi(g)) = (f, S g)$ and all quasi-free states arise in this way. We remind that if $P \in K(\mathcal{K}, \Gamma)$ is a projection, then it is called a basis projection. If $P$ is a basis projection then the GNS representation $\pi_P$ of the quasi-free state associated with $P$ is represented on the Fermionic Fock space $\Lambda(P\mathcal{K})$ by
\[
\pi_P(\psi(f)) := a(Pf) + a(\Gamma Pf)^* ,
\]
where $a(f)$ is the creation operator.

We end by the remark, that using the Araki double construction of Subsection 2.3.3 we get an equivalent description of the Fermionic second quantization of Subsection 2.4.2. Namely, we get that $c(f) = \sqrt{2}\psi(f)$ for $f \in \text{Re}\mathcal{K}$ with $c(f)$ from Subsection 2.4.2 and the identification of $\text{Re}\mathcal{K}$ and $\mathcal{K}$ by $\Lambda$ equation (2.3). Let $q$ be a $\Gamma$-invariant complex subspace of $\mathcal{K}$, then we have equality of the von Neumann algebras
\[
C(H) = \pi_P(C\text{AR}(q, \Gamma))'' \subset \mathcal{B}(\Lambda(P\mathcal{K})) ,
\]
where $H$ is a real subspace of $\mathcal{H} = \mathcal{K}P$ corresponding $q$ under the double construction, namely the correspondence of $H$ with $(\mathcal{K}, \Gamma, P)$ in Proposition 2.3.13. It is now also clear that the Fock vacuum vector $\Omega$ is cyclic or separating for $\pi_P(C\text{AR}(q, \Gamma))^\prime\prime$ if $q$ is cyclic or separating, respectively.

2.4.3.1. Shale–Stinespring quantization criterion. Let $(\mathcal{K}, \Gamma)$ like before and $P \in K(\mathcal{K}, \Gamma)$ be a basis projection. A Bogoliubov operators $Z$ which commutes with $P$ is canonical implemented by $\Lambda(Z)$. If $Z$ is not commuting with $P$ it is not canonical implemented. There is a necessary sufficient condition for a unitary Bogoliubov operator $Z$ to be implementable.

Definition 2.4.3. We call
\[
\mathcal{O}(\mathcal{K}, \Gamma) = \{ U \in \mathcal{U}(\mathcal{K}) : U = \Gamma UT \}
\]
the orthogonal group of $(\mathcal{K}, \Gamma)$ and
\[
\mathcal{O}_P(\mathcal{K}, \Gamma) = \{ U \in \mathcal{O}(\mathcal{K}, \Gamma) : PU(1 - P) \text{ is Hilbert–Schmidt} \}.
\]
the restricted orthogonal group of \((K, \Gamma)\) associated with the basis projection \(P\).

**Remark 2.4.4.** If \(U \in \mathcal{O}(\mathcal{H}, \Gamma)\) and \([P, U]\) Hilbert–Schmidt, then \(U \in \mathcal{O}_P\).

**Proof.** If \([P, U]\) is Hilbert–Schmidt then \(P[P, U] = PU - PUP = PU(1 - P)\). \(\Box\)

**Theorem 2.4.5** ([Ara87, Theorem 6.3. (p.77)]). An Boguliubov unitary \(Z \in \mathcal{O}(K, \Gamma)\) is unitary implemented on the Fock space \(\mathcal{F}_P = \Lambda(PK)\), i.e. it exists a unitary \(Q_Z \in \mathcal{U}(\mathcal{F}_P)\) such that \(\rho_Z(x) = Q_Z x Q_Z^*\) for \(x \in \text{CAR}(\mathcal{H}, \Gamma)\), if and only if \(Z \in \mathcal{O}_P(\mathcal{H}, \Gamma)\).
CHAPTER 3

Preliminaries on algebraic quantum field theory

3.1. Möbius group and representations

The chiral parts of 2D chiral conformal (or Möbius covariant) quantum field theory can be seen as two quantum field theories on the light rays. They extends to a theory on the compactification of one light ray, which is the circle. This motivates to look into quantum field theory on the circle. The symmetry group which leaves the vacuum state invariant turns out to be the Möbius group. We give some basics about the Möbius group and its positive energy representation (discrete series).

We are concerned with the circle $\mathbb{S}^1$ as a spacetime and the Möbius group $\text{M"ob}$ as symmetry group of $\mathbb{S}^1$. The compactified real line $\overline{\mathbb{R}}$ can be identified with the circle $\mathbb{S}^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ by the Caley map

$$C(x) = \frac{x - i}{x + i} = e^{i2 \arctan x} \quad \iff \quad x = C^{-1}(z) = -i \frac{z - 1}{z + 1}, \quad x = C^{-1}(e^{i\theta}) = \tan \frac{\theta}{2}.$$ 

We speak about the circle picture and real line picture, respectively. The Möbius group $\text{M"ob}$ can naturally be identified with $\text{PSU}(1, 1)$ in the circle picture:

$$\text{PSU}(1, 1) = \text{SU}(1, 1)/\{+I, -I\}, \quad \text{SU}(1, 1) = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array} \right) \in \text{GL}(2, \mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1 \right\}$$

by the action

$$g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \quad z \mapsto gz = \frac{\alpha z + \beta}{\beta \bar{z} + \bar{\alpha}}.$$ 

In the real line picture $\text{M"ob}$ can be identified with $\text{PSL}(2, \mathbb{R})$

$$\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{+I, -I\}, \quad \text{SL}(2, \mathbb{R}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}(2, \mathbb{R}) : \det A = ad - bc = 1 \right\}$$

which acts naturally on the compactified real line $\overline{\mathbb{R}}$ with the action given by

$$g : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}, \quad x \mapsto gx = \frac{ax + b}{cx + d}.$$ 

The following subgroups of $\text{M"ob}$: rotations $\{ R(\theta) \}_{\theta \in \mathbb{R}/2\pi \mathbb{Z}}$, dilations $\{ \delta(s) \}_{s \in \mathbb{R}}$ and translations $\{ \tau(t) \}_{t \in \mathbb{R}}$ are given by

$$R(\theta) = \left( \begin{array}{cc} e^{i \frac{\theta}{2}} & 0 \\ 0 & e^{-i \frac{\theta}{2}} \end{array} \right), \quad \delta(s) = \left( \begin{array}{cc} \cosh \frac{s}{2} & -\sinh \frac{s}{2} \\ -\sinh \frac{s}{2} & \cosh \frac{s}{2} \end{array} \right), \quad \tau(t) = \left( \begin{array}{cc} 1 - \frac{t}{2} & -\frac{t}{2} \\ \frac{t}{2} & 1 + \frac{t}{2} \end{array} \right)$$

as subgroups of $\text{PSU}(1, 1)$ and by

$$R(\theta) = \left( \begin{array}{cc} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{array} \right), \quad \delta(s) = \left( \begin{array}{cc} e^{\frac{s}{2}} & 0 \\ 0 & e^{-\frac{s}{2}} \end{array} \right), \quad \tau(t) = \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right).$$
as subgroups of $\text{PSL}(2, \mathbb{R})$. The action of these subgroups in the circle and real line pictures are given by

\begin{align*}
R(\theta)z &= e^{i\theta}z, \\
\delta(s)z &= \frac{z \cosh \frac{s}{2} - \sinh \frac{s}{2}}{\cosh \frac{s}{2} - z \sinh \frac{s}{2}}, \\
\tau(t)z &= \frac{z - \frac{2i}{\tau}(z + 1)}{1 + \frac{2i}{\tau}(z + 1)}, \\
\end{align*}

respectively, where $z \in \mathbb{S}^1$ and $x \in \mathbb{R}$. For rotations the circle picture is easier, while for dilations and translations the real line picture is easier.

### 3.1.1. Positive energy representations of $\text{Mob}$

Let $U$ be a unitary representation of $\text{Mob}$ on $\mathcal{H}$. It is called a **positive energy representation** if the generator $L_0$ of the rotations $U(R(\theta)) = e^{i\theta L_0}$ has positive spectrum. The rotation subgroup $U(R(\theta))$ defines a grading on $\mathcal{H}$

\[ \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n = \{ x \in \mathcal{H} : U(R(\theta))x = e^{i n \theta}x \} \]

and there exists a number $m$ such that $\mathcal{H}_m \neq \{0\}$ and $\mathcal{H}_n = \{0\}$ for $n < m$ which is called the **lowest weight** of $U$.

**Theorem 3.1.1.** For each $m \in \mathbb{N}$ there exists a unique irreducible representation of $\text{Mob}$ with lowest weight $m$. Every positive energy of $\text{Mob}$ representation is a direct sum of irreducible representations.

**Example (Lowest weight 1).** Consider $C^\infty(\mathbb{S}^1, \mathbb{R})$, where we write the periodic function $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$ as a Fourier series

\[ f(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ik\theta}, \quad \hat{f}_k = \int_0^{2\pi} e^{-ik\theta} f(\theta) \frac{d\theta}{2\pi} = \overline{\hat{f}_{-k}}. \]

We introduce a semi-norm

\[ \|f\|^2 = \sum_{k=1}^{\infty} k \cdot |\hat{f}_k|^2 \]

and a complex structure, i.e. an isometry $\mathcal{J}$ with respect to $\| \cdot \|$ satisfying $\mathcal{J}^2 = -1$, by $\mathcal{J} : \hat{f}_k \mapsto -i \text{sign}(k) \hat{f}_k$. Finally, we get the Hilbert space

\[ \mathcal{H} = C^\infty(\mathbb{S}^1, \mathbb{R})/\mathbb{R} \| \cdot \| \]

by completion with respect to the norm $\| \cdot \|$, where $\mathbb{R}$ is identified with the constant functions. By abuse of notation we denote also the image of $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$ in $\mathcal{H}$ by $f$. The scalar product (linear in the second component) and the sesquilinear form $\omega(\cdot, \cdot) \equiv \text{Im}(\cdot, \cdot)$ are given by

\[ (f, g) = \sum_{k=1}^{\infty} k \hat{f}_k \hat{g}_{-k}, \quad \omega(f, g) = -\frac{1}{2} \sum_{k \in \mathbb{Z}} k \hat{f}_k \hat{g}_{-k} = \frac{1}{2} \int_0^{2\pi} f(\theta)g'(\theta) \frac{d\theta}{2\pi} = \frac{1}{4\pi} \int f \text{dg}, \]

respectively. The unitary action $U_1$ of $\text{Mob}$ on $\mathcal{H}$ is induced by the action on $C^\infty(\mathbb{S}^1, \mathbb{R})$ given by $(U_1(g)f) := f(g^{-1}(\theta))$. 

3.1.2. Double cover and universal cover of M"ob. We denote by $\text{M"ob}^{(2)} \cong \text{SU}(1, 1) \cong \text{SL}(2, \mathbb{R})$ the double cover of $\text{M"ob}$:

$$I \rightarrow \mathbb{Z} \rightarrow \text{M"ob}^{(2)} \rightarrow \text{M"ob} \rightarrow 1$$

which acts faithful on the double cover $\mathbb{S}^{1(2)} \cong \mathbb{R}/4\pi\mathbb{Z}$ of the circle, with the projection $\mathbb{R}/4\pi\mathbb{Z} \rightarrow \mathbb{S}^{1} : \theta \mapsto e^{i\theta}$. The action of the rotation is given by $R(\theta) \theta = \theta + \theta$.

We denote by $\overline{\text{M"ob}}$ the universal covering of $\text{M"ob}$:

$$I \rightarrow \mathbb{Z} \rightarrow \overline{\text{M"ob}} \rightarrow \text{M"ob} \rightarrow 1$$

which acts on the universal cover $\mathbb{R}$ of the circle with $\pi : \mathbb{R} \rightarrow \mathbb{S}^{1} : \theta \mapsto e^{i\theta}$ and the rotation act on $\mathbb{R}$ also by $R(\theta) \theta = \theta + \theta$.

Let $j : z \mapsto \overline{z}$ the reflection on the circle. It acts by $\rho_{j}(g) = jgj$ on and we can define $\text{M"ob}_{2} = \text{PSL}(2, \mathbb{R})_{\pm} = \text{PSL} \rtimes \mathbb{Z}_{2}$ which turns out to be isomorphic to $\text{PSL}(2, \mathbb{R}) \cong \text{SL}(2, \mathbb{R})_{\pm} / \{\pm I, -I\}$,

$$\text{SL}(2, \mathbb{R})_{\pm} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) : \det A = ad - bc = \pm 1 \right\}$$

and has two connected components $\text{PSL}(2, \mathbb{R})_{+} = \text{PSL}(2, \mathbb{R})$ and its coset $\text{PSL}(2, \mathbb{R})_{-}$ given by $\rho_{j}, \rho_{j}$ lifts to $\text{M"ob}^{(2)}$ and $\overline{\text{M"ob}}$ and we can define analogously $\text{M"ob}_{2}^{(2)} \cong \text{SL}(2, \mathbb{R})_{\pm}$ and $\text{M"ob}_{2} = \text{M"ob} \rtimes \mathbb{Z}_{2}$.

Following groups are involved:

\[
\begin{array}{cccc}
1 & 1 & \rightarrow & \\
\downarrow & \downarrow & & \\
\mathbb{Z} & \mathbb{Z} & & \\
\downarrow & \downarrow & & \\
1 & \text{M"ob} & \text{M"ob}_{2} & \mathbb{Z}_{2} & 1 \\
\downarrow_{\text{M"ob}} & \downarrow_{\text{M"ob}} & \downarrow_{\text{M"ob}} & \\
1 & \overline{\text{M"ob}} & \text{M"ob}_{2} & \mathbb{Z}_{2} & 1 \\
\downarrow & \downarrow & \downarrow & \\
1 & 1 & 1 & \\
\end{array}
\]

and we get a similar picture with $\overline{\text{M"ob}}, \text{M"ob}_{2}, \mathbb{Z}$ replaced by $\text{M"ob}^{(2)}, \text{M"ob}^{(2)}, \mathbb{Z}_{2}$. We denote again the subgroups of rotations, dilations and translations by $R(\theta), \delta(t)$ and $\tau(s)$, respectively. We remark that $\text{M"ob}$ has no representation as matrix group.

3.1.3. Dilation associated with an interval. Let $I_{0} = e^{i(0, \pi)}$, i.e. the upper semicircle which is identified with the positive half-line $\mathbb{R}_{+} \subset \mathbb{R}$ in the real line picture. If no confusion arises, we also write $I_{0} = \mathbb{R}_{+}$. The dilation $\delta(s)$ given in the real picture by $\delta(s)x = e^{s}x$ for $x \in \mathbb{R}$ leaves the interval $I_{0}$ invariant, i.e. $\delta(s)I_{0} = I_{0}$. For every interval $I \in \mathcal{I}$ we define a one-parameter group, the dilation of an interval $I$ with $\delta_{I}(s)I = I$ by

$$\delta_{I}(s) := g\delta(s)g^{-1}, \quad I = gI_{0}, s \in \mathbb{R}.$$  

This is well defined, because $g$ is unique up to right multiplication by an element which leaves $I_{0}$ invariant, i.e. up to right multiplication by an element in an Abelian subgroup $A \subset \text{M"ob}$. But also $\delta(s) \in A$ and $A$ is Abelian. It follows from the definition that $\delta_{I}(s)I = I.$
Similar we define the reflection \( \mathbb{M}ob_2 \ni r_0 : z \mapsto \overline{z} \) in the circle picture, where \( \overline{z} \) is the complex conjugate of \( z \) and which corresponds to \( x \mapsto -x \) in the real line picture.

For general \( I \in \mathbb{I} \) we choose \( g \in \mathbb{M}ob \) such that \( I = gI_0 \) and define the **reflection of an interval** \( I \) by \( r_I = gr_Ig^{-1} \).

An **(anti-) unitary representation** of \( \mathbb{M}ob_2 \) is a representation such that

\[
U(g) = \begin{cases} 
\text{unitary} & g \in \mathbb{M}ob \\
\text{antiunitary} & g \in \mathbb{M}ob_2 \setminus \mathbb{M}ob
\end{cases}
\]

and it is called positive energy if it restriction to \( \mathbb{M}ob \) is positive energy.

**Theorem 3.1.2** ([Lon08b]). Every unitary positive energy representation \( U \) of \( \mathbb{M}ob \) extends to an (anti-)unitary representation \( \tilde{U} \) of \( \mathbb{M}ob_2 \) on the same Hilbert space \( \mathcal{H} \) and every (anti-)unitary positive energy representation of \( \mathbb{M}ob_2 \) arises in this way. \( U_1 \) is equivalent with \( U_2 \) if and only if \( \tilde{U}_1 \) is equivalent with \( \tilde{U}_2 \) and \( U \) is irreducible if and only if \( \tilde{U} \) is irreducible and in this case the choice of \( J = U(r) \) is unique up to a phase \( z \in \mathbb{T} \).

---

**Figure 1.** Geometric flow of \( \delta_I \)
### 3.2. Conformal nets

In this section we give some background on the operator algebraic approach of conformal field theory. The notion of a conformal net is the Haag–Kastler formulation of one half (or chiral part) of a 2D chiral conformal field theory.

#### 3.2.1. Definition and first consequences.

In this subsection we introduce the notion of a **local Möbius covariant net** of von Neumann algebras which we will simply call **conformal net** and its first consequences.

**Definition 3.2.1.** A *local Möbius covariant net* (conformal net) $\mathcal{A}$ on $\mathbb{S}^1$ is a family $\{\mathcal{A}(I)\}_{I \in \mathcal{I}}$ of von Neumann algebras on a Hilbert space $\mathcal{H}$, with the following properties:

- **Isotony.** $I_1 \subset I_2$ implies $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.
- **Locality.** $I_1 \cap I_2 = \emptyset$ implies $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$.
- **Möbius covariance.** There is a unitary representation $\mathcal{U}$ of $\text{Möb}$ on $\mathcal{H}$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.
- **Positivity of energy.** $U$ is a positive energy representation, i.e. the generator $L_0$ (conformal Hamiltonian) of the rotation subgroup $U(\mathbb{R}) = e^{i\theta L_0}$ has positive spectrum.
- **Vacuum.** There is a (up to phase) unique rotation invariant unit vector $\Omega \in \mathcal{H}$ which is cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

We note that this are kind of the standard assumption and that these assumption have physical motivations. The set of axioms (Definition 3.2.1) is not claimed to be minimal or independent in some sense. But we will now see that some properties, which we could have assumed follow automatically from the axioms.

**Theorem 3.2.2** (Reeh–Schlieder theorem [FJ96]). Let $\mathcal{A}$ be a local Möbius covariant net on $\mathbb{S}^1$. The vacuum vector $\Omega$ is cyclic and separating for any local algebra $\mathcal{A}(I)$ with $I \in \mathcal{I}$.

We remind that we defined (in Section 3.1.3) for every $I \in \mathcal{I}$ a dilation $\delta_I(s) \in \text{Möb}$ associated with $I$ leaving $\mathcal{A}$ invariant, i.e. $\delta_I(s)I = I$ a and reflection $r_I \in \text{Möb}$ with $r_I I = I'$.

**Theorem 3.2.3** (Bisognano–Wichmann property [GF93, BGL93]). For $I \in \mathcal{I}$ let $(J_I, \Delta_I) := (J_{\mathcal{A}(I)\Omega}, \Delta_{\mathcal{A}(I)\Omega})$ be the modular conjugation and operator of $(\mathcal{A}(I), \Omega)$ then:

1. It holds the Bisognano–Wichmann property
   \[ \Delta_I^t = U(\delta_I(-2\pi t)), \quad t \in \mathbb{R} \]
2. $U$ extends to an (anti–) unitary representation $\tilde{U}$ of $\text{Möb}_2 \cong \text{PSL}_\pm(2, \mathbb{R})$ determined by
   \[ \tilde{U}(r_I) = J_I \]
   (independent of $I$) such that $\mathcal{A}$ is covariant w.r.t. $\text{Möb}_2$, i.e.
   \[ U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), g \in \text{Möb}_2, I \in \mathcal{I}. \]

This means that the action of the modular group $\Delta_I^t$ and of $J_I$ has a geometric meaning and implies a property which is stronger than locality, namely **Haag duality**.

**Corollary 3.2.4** (Haag duality). It holds $\mathcal{A}(I') = \mathcal{A}(I)$ for every $I \in \mathcal{I}$.

**Proof.** By locality $\mathcal{A}(I) \subset \mathcal{A}(I')'$ holds and by the Bisognano–Wichmann property $\Delta_{\mathcal{A}(I)\Omega} = \Delta_{\mathcal{A}(I')\Omega}^{-1} \equiv \Delta_{\mathcal{A}(I')\Omega}$, where we used that $\delta_I(s) = \delta_{I'}(-s)$ which can be easily checked. Then by Takesaki’s theorem (Theorem 2.1.11) there exist a conditional expectation $E$ from $\mathcal{A}(I')'$ to $\mathcal{A}(I)$ and by the Reeh–Schlieder property this has to be the identity. \qed
Proposition 3.2.5 (Irreducibility [Lon08a]). Let $\mathcal{A}$ fulfill all properties of a local Möbius covariant net on $\mathbb{S}^1$ without the uniqueness of $\Omega$. Then are equivalent:

1. $\Omega$ are the only $U(\text{Möb})$ invariant vectors.
2. The algebras $\mathcal{A}(I)$ with $I \in \mathcal{I}$ are factors (unless $\dim \mathcal{H} = 1$).
3. The net is irreducible, i.e. $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = \mathcal{B}(\mathcal{H})$.
4. It is $\bigcap_{I \in \mathcal{I}} \mathcal{A}(I) = \mathbb{C} \cdot 1$.

Proposition 3.2.6 (Type III$_1$ property [Dri75, Lon79]). Let $\mathcal{A}$ fulfill all properties of a local Möbius covariant net on $\mathbb{S}^1$, then $\mathcal{A}(I)$ is a type III$_1$ factor.

Proposition 3.2.7 (Additivity [FJ96]). A conformal net is additive, i.e. for $I \in \mathcal{I}$ and a possibly infinite family of intervals $\{I_i\}$ in $\mathcal{I}$ we have

$$ I \subset \bigcup_i I_i \implies \mathcal{A}(I) \subset \bigvee_i \mathcal{A}(I_i). $$

As in [KL04] we define:

Definition 3.2.8. Two conformal nets $\mathcal{A} \equiv \{\mathcal{A}(I)\}_{I \in \mathcal{I}}$, $\mathcal{H}_A, U_A, \Omega_A$ and $\mathcal{B} \equiv \{\mathcal{B}(I)\}_{I \in \mathcal{I}}$, $\mathcal{H}_B, U_B, \Omega_B$ are said to be isomorphic, denoted by $\mathcal{A} \cong \mathcal{B}$, if there is a unitary operator $U : \mathcal{H}_A \to \mathcal{H}_B$ such that $U^* \mathcal{B}(I) U = \mathcal{A}(I)$ for all $I \in \mathcal{I}$ and $U \Omega_A = U \Omega_B$.

3.2.2. Representations.

Definition 3.2.9. Let $\mathcal{A}$ be a conformal net. A representation $\pi = \{\pi_I\}_{I \in \mathcal{I}}$ of $\mathcal{A}$ is a family of representations $\pi_I$ of $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$ which fulfill:

$$ \pi_I |_{\mathcal{A}(h_0)} = \pi_{I_0}, \quad I_0 \subset I. $$

It is called covariant if

$$ \text{Ad} U_\pi(g) \circ \pi_I = \pi_{gI} \circ \text{Ad} U(g), $$

where $U_\pi$ is a unitary representation of the universal covering group $\text{Möb}$ of $\text{Möb}$ with positive energy. A representation $\pi$ of $\mathcal{A}$ on $\mathcal{H}_\pi$ is called non-degenerate if it is unital, i.e. $\pi_I(1) = 1_{\mathcal{H}_\pi}$.

Lemma 3.2.10 (Local equivalence [Müg10]). Let $\mathcal{A}$ be a conformal net with separable vacuum Hilbert space $\mathcal{H}$ and let $\pi$ be a non-degenerate separable representation.

Then, for every $I \in \mathcal{I}$ there is a unitary $V_I : \mathcal{H}_\pi \to \mathcal{H}$, such that

$$ V_I \pi(x) V_I^* = x \quad x \in \mathcal{A}(I). $$

We assume $\mathcal{H}_\pi$ to be separable and this implies that $\pi$ is locally normal, namely $\pi_I$ is normal for all $I \in \mathcal{I}$.

Definition 3.2.11. A representation $\rho$ of a conformal net $\mathcal{A}$ is called localized in some interval $I_0 \in \mathcal{I}$ if $\mathcal{H}_\rho = \mathcal{H}_A$ and $\rho_I = \text{id}_{\mathcal{A}(I_0)}$.

Proposition 3.2.12 (Localizability). Let $\mathcal{A}$ be a conformal net with separable vacuum Hilbert space $\mathcal{H}$ and let $\pi$ be a non-degenerate separable representation and let $I_0 \in \mathcal{I}$. Then there exist a to $\pi$ unitary equivalent representation $\tilde{\pi}$ localized in $I_0$.

Proposition 3.2.13. Let $\pi$ be localized in $I_0 \in \mathcal{I}$, then $\pi_I$ is an endomorphism of $\mathcal{A}(I)$ (i.e. $\pi_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$) for each $I \in \mathcal{I}$ with $I \supset I_0$. 

Let $\rho$ be a (covariant) representation localized in $I_0$. By a local cocycle [Lon03] localized in a proper interval $I \subset I_0$, we mean an assignment of a symmetric neighborhood $U$ of the identity of $\text{Möb}$ such that $I_0 \cup gI_0 \subset I$ for all $g \in U$ and a strongly continuous unitary valued map $g \in U \mapsto z_\rho(g) \in \mathcal{A}(I)$ such that with $a_g := \text{Ad} U(g)$:

$$z_\rho(gh) = z_\rho(g) \cdot a_g(z_\rho(h)),$$

$$\text{Ad} z_\rho(g)^* \circ \rho_I(a) = a_g \circ \rho_{g^{-1}I} \circ a_{g^{-1}}(a),$$

for some open interval $I \subset \mathcal{I}$ with $\tilde{I} \supset I$. By covariance and Haag duality exists a local cocycle given by

$$z_\rho(g) = U^*_\rho(g)U(g)^* \in \mathcal{A}(I), \quad g \in U.$$

Let us assume that $\mathcal{A}$ is strong additive net (defined in Subsection 3.2.6) on $\mathcal{H}$. Then the restriction of $\mathcal{A}$ to a net on $\mathbb{R}$ is Haag dual and we can apply the theory of [FRS89], but by localizability we can also simply restrict to all representations in a given interval $I \subset \mathcal{I}$ and we can use the theory of sectors and endomorphisms of Section 2.2. Let $\text{Rep}_I(\mathcal{A}) \subset \text{End}(\mathcal{A}(I))$ the category of all non-degenerate representations of $\mathcal{A}$ on $\mathcal{H}$ which are localized in $I$. It can be shown that this is a full subcategory of the category of all non-degenerated separable representations of $\mathcal{A}$. Further $\text{Rep}_J(\mathcal{A})$ is a full subcategory of $\text{Rep}_I(\mathcal{A})$ with $I \subset J$. We can compose representation by composing the localized endomorphisms. The statistical dimension for a representation $\pi$ is defined as the square root of the index of the inclusion

$$\pi_I(\mathcal{A}(I)) \subset \pi_J(\mathcal{A}(I')),\]$$

and does not depend on $I$. For $\rho \in \text{Rep}_I(\mathcal{A})$ it equals the dimension $d(\rho)$ of $\rho$ as in Section 2.2 and it just depend on the sector $[\rho_I]$. If $d(\rho) < \infty$ there exists a conjugated representation $\tilde{\rho}$.

Let $\rho$ be an irreducible covariant representation, then we denote by $h_\rho$ the lowest eigenvalue of the image of $L_0$ (the generator of rotations) and define the conformal spin to be $\omega_\rho = e^{2\pi h_\rho/\nu}$.

The braiding or statistics operator is defined as follows. Let $\rho_1, \rho_2 \in \text{Rep}_I(\mathcal{A})$ and let $I_1, I_2 \subset I$ disjoint intervals, with $I_1 < I_2$ ($I_1$ clockwise from $I_2$ inside $I$). Then we choose $\tilde{\rho}_1 \in [\rho_I]$ localized in $I_1$ and unitary intertwiners $U_i \in (\rho_i, \tilde{\rho}_i)$ for $i = 1, 2$ and define

$$\epsilon(\rho_1, \rho_2) := \rho_2(U_1^*)U_2^*U_1\rho_1(U_2)$$

and it follows from $\tilde{\rho}_1 \circ \tilde{\rho}_2 = \tilde{\rho}_2 \circ \tilde{\rho}_1$ the intertwining property $\epsilon(\rho_1, \rho_2) \in (\rho_1 \rho_2, \rho_2 \rho_1)$. It does not depend on the several choices an defines a braiding in the category $\text{Rep}_I(\mathcal{A})$. We note that if $\rho_1$ is localized in $I_1 \subset I$ we can choose $\rho_1 = \tilde{\rho}_1$ and $U_1 = 1$ the formula simplifies to

$$\epsilon(\rho_1, \rho_2) = U_2^*\rho_1(U_2).$$

One get the opposite braiding for $I_2 < I_1$ denoted by $\epsilon^-$ and we have $\epsilon^-(\rho_1, \rho_2) = \epsilon(\rho_2, \rho_1)^*$. We note that on the space $(\rho^n, \rho^l)$ we get a representation $\pi$ of the braid group on $n$ on strands defined by the generators $\{\sigma_i\}_{i=1 \leq i \leq n-1}$ with relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad i = 1, \ldots, n - 2$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i - j| > 1,$$

which is defined by $\pi(\sigma_i) = \rho^{n-1}(\epsilon(\rho, \rho))$. 

Proof: Let $J \subset \mathcal{I}$ with $J \subset I'$ such that there is a $K \in \mathcal{I}$ with $I \cup J \subset K$. By locality it holds for $x \in \mathcal{A}(I)$ and $y \in \mathcal{A}(J)$ that $\pi_K(x) \equiv \pi_I(x)$ commutes with $\pi_K(y) \equiv y$. By additivity (Proposition 3.2.7) $\pi_I(x)$ commutes with $\mathcal{A}(I')$ and by Haag duality (Corollary 3.2.4i) follows that $\pi_J(x) \in \mathcal{A}(I')$. \hfill $\Box$
3.2.3. Conformal subnets. Let \( \mathcal{A} \) be a conformal net and \( U \) its associated positive energy representation of \( \text{M} \). We call a family \( \{ \mathcal{B}(I) \}_{I \in \mathcal{I}} \) with \( \mathcal{B}(I) \subset \mathcal{A}(I) \) for all \( I \in \mathcal{I} \) a conformal subnet if \( \mathcal{B} \) is isotonous, i.e. \( I, J \in \mathcal{I} \) with \( I \subset J \) implies \( \mathcal{B}(I) \subset \mathcal{B}(J) \) and covariant, i.e. it is \( U(g) \mathcal{B}(I) U(g)^* = \mathcal{B}(J) \) for all \( I \in \mathcal{I} \) and \( g \in \text{M} \). The structure of conformal subnets is studied in [Lon03].

Let \( e \) be the projection on the closure \( \mathcal{H}_B \) of \( \bigvee_{I \in \mathcal{I}} \mathcal{B}(I) \Omega \). Then \( \mathcal{B} \) is itself a conformal net on \( \mathcal{H}_B := e \mathcal{H} \) with unitary representation \( U \upharpoonright \mathcal{H}_B \) also denoted by \( U \), namely \( \Omega \) is cyclic for \( \mathcal{H}_B \) by definition and all other properties are inherit by the ones of \( \mathcal{A} \). By the Reeh–Schlieder property \( \Omega \) is cyclic and separating for all \( \mathcal{B}(I) \) with \( I \in \mathcal{I} \) and in particular \( e \) is the Jones projection (see also [LR95]) of the inclusion \( \mathcal{B}(I) \subset \mathcal{A}(I) \).

**Lemma 3.2.14.** Let \( \mathcal{B} \) be a conformal subnet of \( \mathcal{A} \). If \( e = 1 \) then the conformal nets \( \mathcal{B} \) and \( \mathcal{A} \) are identical.

**Proof.** Let \( I \in \mathcal{I} \). Then it is \( \mathcal{B}(I) \subset \mathcal{A}(I) \) and by the Bisognano–Wichmann property the modular group of \( \mathcal{A}(I) \) with respect to the vector state of \( \Omega \) is given by \( \sigma_I = \text{Ad} U(\delta_1(-2\pi t)) \) and by covariance of \( \mathcal{B} \) it leaves \( \mathcal{B}(I) \) invariant. By Takesaki’s Theorem (Theorem 2.1.11) there exists a normal conditional expectation from \( \mathcal{A}(I) \) onto \( \mathcal{B}(I) \) which has to be the identity of \( \mathcal{A} \) by \( e = 1 \). \( \square \)

3.2.4. Simple current extension. A local extension of a conformal net is \( \mathcal{A} \) is a conformal net \( \mathcal{B} \) which contains \( \mathcal{A} \) as a subnet, more precisely there is a representation \( \pi \) containing the vacuum representation of \( \mathcal{A} \) on \( \mathcal{H}_B \) such that \( \pi(\mathcal{A}(I)) \subset \mathcal{B}(I) \) and \( \pi(\mathcal{A}(I)) \) is covariant with respect to the representation of \( \text{M} \) on \( \mathcal{H}_B \). We are interested in the case when the inclusion is irreducible.

**Lemma 3.2.15** (Simple current extension [KL06, Lemma 2.1]). Let \( \mathcal{A} \) be a conformal net. Suppose we have a finite system \( \{ \lambda_i \} \) of irreducible DHR sectors each with statistical dimension and conformal spin 1, i.e. \( d_{\lambda_i} = \omega_{\lambda_i} = 1 \). Then the crossed product of \( \mathcal{A} \) with the finite Abelian group \( G \) given by \( \{ \lambda_i \} \) produces a local extension \( \mathcal{B} \) of \( \mathcal{A} \).

The extended net \( \mathcal{B} \) is called the simple current extension of \( \mathcal{A} \) by \( G \).

3.2.5. Diffeomorphism covariance. Let us denote the group of orientation preserving diffeomorphisms of the circle \( S^1 \) by \( \text{Diff}_+(S^1) \). We note that \( \text{M} \subset \text{Diff}_+(S^1) \).

**Definition 3.2.16.** A conformal net is called diffeomorphism covariant, if there is a projective unitary representation of \( \text{Diff}_+(S^1) \) extending the representation \( U \) of \( \text{M} \) and which also denoted by \( U \), such that for all \( I \in \mathcal{I} \)

\[
U(g, A(I)) U(g)^* = A(gI) \quad g \in \text{Diff}_+(S^1),
\]

\[
U(g) x U(g)^* = x \quad x \in A(I), \quad g \in \text{Diff}(I),
\]

where \( \text{Diff}(I) \) are all \( g \in \text{Diff}_+(S^1) \) with \( g \uparrow l' = 1 \).

By Haag duality and \( I \in \mathcal{I} \) we have \( U(g) \in \mathcal{A}(I) \) for each \( g \in \text{Diff}(I) \). Then it is immediate that

\[
\text{Vir}_{A(I)} := \{ U(g) : g \in \text{Diff}(I) \}''
\]

defines a subnet of \( A \) and it is called the Virasoro net. The positive energy representation of \( \text{Diff}_+(S^1) \) restricts to \( \mathcal{H}_V = \bigvee_I \text{Vir}(I) \Omega \) to an irreducible positive energy representation with an \( \text{M} \) invariant vector \( \Omega \) (see [Car04, CW05]). These representations are completely classified by a positive number.
c, the central charge and \( \text{Vir} \cong \text{Vir}_c \) for some unique \( c \) so we can assign a central charge \( c \) to a diffeomorphism covariant conformal net. The central charge can take values in
\[
c \in \left\{ 1 - \frac{6}{m(m + 1)} : m = 2, 3, \ldots \right\} \cup [1, \infty).
\]

3.2.6. Completely rational conformal nets. A conformal net \( \mathcal{A} \) is said to be strongly additive if for \( I_1, I_2 \in \mathcal{I} \) adjacent intervals and \( I = (I_1 \cup I_2)' = I_1 \cup_{\mathcal{I}} I_2' \in \mathcal{I} \) if \( \mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I) \) holds. The net \( \mathcal{A} \) satisfies the split property if for \( J_0, I \in \mathcal{I} \) with \( J_0 \subset I \) the inclusion \( \mathcal{A}(J_0) \subset \mathcal{A}(I) \) is a split inclusion, namely there exists an intermediate type I factor \( M \) such that \( \mathcal{A}(J_0) \subset M \subset \mathcal{A}(I) \) or equivalently \( \mathcal{A}(I_0) \vee \mathcal{A}(I)' \) is canonically isomorphic to \( \mathcal{A}(I_0) \otimes \mathcal{A}(I)' \). The split property implies that \( \mathcal{H} \) is separable.

**Theorem 3.2.17** ([DLR01]). Let \( \mathcal{A} \) be a conformal net on \( \mathcal{H} \). If \( e^{-\beta L_0} \) is trace class (i.e. \( \text{tr}_\mathcal{H} e^{-\beta L_0} < \infty \) for all \( \beta > 0 \), where \( L_0 \) is the generator of the rotations), then \( \mathcal{A} \) satisfies the split property.

Let \( I_1, I_2 \in \mathcal{I} \) be two intervals with disjoint closure and \( I_3, I_4 \in \mathcal{I} \) the two components of \( (I_1 \cup I_2)' \), in other words the intervals \( I_1, \ldots, I_4 \) divide the circle into four parts. Then we denote by \( \mu_{\mathcal{A}} \) or \( \mu(\mathcal{A}) \) the Jones–Kosaki index (see Definition 2.2.1) of the inclusion.
\[
\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'
\]
which does not depend on the intervals \( I_i \). Finally the net \( \mathcal{A} \) is called completely rational if it is strongly additive, split and \( \mu(\mathcal{A}) < \infty \). In [KLM01] it is shown that the index of the inclusion (3.4) is the global index
\[
\mu(\mathcal{A}) = \sum_i d(\rho_i)^2
\]
associated with all sectors \( \{[\rho_i]\}_{i=0,\ldots,n} \), i.e. equivalence classes of irreducible separable representations with finite dimensions. The category of representations form a modular tensor category, where each sector is a direct sum of sectors with finite dimension.

3.2.7. Fermi nets. Because we will use free Fermionic nets for some constructions, we give a generalization of the notion of a local Möbius covariant net to a graded local Möbius covariant net or also called Fermi net.

**Definition 3.2.18.** A graded local Möbius covariant net or Fermi net \( \mathcal{F} \) on \( \mathbb{S}^1 \) is a family \( \{\mathcal{F}(I_i)\}_{I_i \in \mathcal{I}} \) of von Neumann algebras on a \( \mathbb{Z}_2 \)-graded Hilbert space \( \mathcal{H} \), i.e. there is a unitary operator \( \Gamma \) with \( \Gamma^2 = 1 \) with the following properties:

A. **Isotony.** \( I_1 \subset I_2 \) implies \( \mathcal{F}(I_1) \subset \mathcal{F}(I_2) \).
B. **Graded locality.** \( I_1 \cap I_2 = \emptyset \) implies \( [\mathcal{F}(I_1), \text{Ad} Z(\mathcal{F}(I_2))] = \{0\} \), where \( Z = e^{\frac{i\pi}{2}} \).
C. **Möbius covariance.** There is a unitary representation \( U \) of \( \text{Möb}(\mathbb{S}^1) \) on \( \mathcal{H} \) such that \( U(g)\mathcal{F}(I)U(g)^* = \mathcal{F}(gI) \).
D. **Positivity of energy.** \( U \) is a positive energy representation, i.e. the generator \( L_0 \) (conformal Hamiltonian) of the rotation subgroup \( U(\mathbb{R}\theta) \) has positive spectrum.
E. **Vacuum.** There is a (up to phase) unique rotation invariant and even (i.e. \( \Omega = \Omega \)) unit vector \( \omega \in \mathcal{H} \) which is cyclic for the von Neumann algebra \( \mathcal{F}(I) \).

If \( Z = 1 \) then this definition coincides with the one of a local Möbius covariant net (Definition 3.2.1). Among the consequences of the definition are ([CKL08]):

**Proposition 3.2.19.** Let \( \mathcal{F} \) be a Fermi net on \( \mathcal{H} \).
3.3. Poincaré covariant nets on Minkowski space

We will mainly consider nets in 2 spacetime dimensions. The $d$-dimensional ($d \geq 2$) Minkowski space $M_d$ can be identified with $\mathbb{R}^d$ with the inner product

$$x \cdot y = x^0y^0 - \sum_{i=1}^{d-1} x^i y^i,$$

where we write $x \in M_d$ as $x = (x^0, \ldots, x^{d-1})$. We write $x^2 = x \cdot x$. Two points $x, y \in M_d$ are called mutually

- **spacelike**: if $(x - y) \cdot (x - y) < 0$,
- **timelike**: if $(x - y) \cdot (x - y) > 0$,
- **lightlike**: if $(x - y) \cdot (x - y) = 0$.

For an open region $O \subset \mathbb{R}^d$ we denote by $O'$ the **causal complement**, which is the interior of the set of all points spacelike to $O$. For $d = 2$ we also often write $(t, x) \in M_2$. Let $\mathcal{L}$ be the **Lorentz group**, i.e. all matrices $\Lambda \in \text{GL}(d, \mathbb{R})$ with $\Lambda x \cdot \Lambda y = x \cdot y$. The **Poincaré group** $\mathcal{P} = \mathcal{L} \rtimes \mathbb{R}^d$ is given by the semidirect product with the translation group $\mathbb{R}^d$, namely, by the composition law:

$$(a, \Lambda) \circ (b, \Lambda') = (a + \Lambda b, \Lambda \cdot \Lambda'), \quad a, b \in \mathbb{R}^d, \Lambda, \Lambda' \in \mathcal{L}.$$

The group $\mathcal{P}$ is non connected, non compact and perfect Lie group. The components of Lorentz and Poincaré group are denoted by

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\downarrow, \quad \mathcal{P} = \mathcal{P}_+^{\uparrow} \cup \mathcal{P}_+^{\downarrow} \cup \mathcal{P}_-^{\uparrow} \cup \mathcal{P}_-^{\downarrow},$$

respectively. Here $\pm$ correspond to $\det \Lambda = \pm 1$ and $\uparrow, \downarrow$ correspond to time-orientation preserving and changing transformations, respectively. $\mathcal{L}_+^\uparrow$ and $\mathcal{P}_+^{\uparrow}$ are the components of the identity $I$. In $d = 2$ we also write $(a, \Lambda) \in \mathcal{P}_+^{\uparrow}$ with the action on $\mathbb{R}^2$

$$(a, \Lambda)x = \begin{pmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{pmatrix} x + a.$$

We denote by $V_+ = \{ (\omega, p) \in \mathbb{R}^2 : \omega > |p| \}$ the **forward light cone**, by $V_- = \{ (\omega, p) \in \mathbb{R}^2 : \omega \geq |p| \}$ its closure and by $W_2 = \{ (t, x) \in \mathbb{R}^2 : x > |t| \}$ **standard right wedge** in 2D Minkowski space.

Let $\mathcal{D}$ be the family of **double cones** in 2D Minkowski space. They are defined to be non-empty intersections of translations of the forward and backward light cone.

**Definition 3.3.1.** A **local Poincaré covariant net** $\mathcal{A}$ on $M_2$ is a family $\{\mathcal{A}(O)\}_{I \in \mathcal{D}}$ of von Neumann algebras on a Hilbert space $\mathcal{H}$, with the following properties:

A. **Isotony.** $O_1 \subset O_2$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

B. **Locality.** $O_1 \cap O_2 \neq \emptyset$ implies $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$.

C. **Poincaré covariance.** There is a unitary representation $U$ of $\mathcal{P}_+^{\uparrow}$ on $\mathcal{H}$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$.

D. **Positivity of energy.** $U$ is a positive energy representation, i.e. the joint spectrum of the generator of the translation lies in $V_+$. 

E. Vacuum. There is a (up to phase) translation invariant unit vector \( \Omega \in \mathcal{H} \) which is cyclic for the von Neumann algebra \( \bigvee_{O \in D} \mathcal{A}(O) \).

F. Additivity. If \( O = \bigcup_i O_i \) then \( \mathcal{A}(O) = \bigcup_i \mathcal{A}(O_i) \).

We can extend the net to wedges \( W = gW_R \) with \( g \in \mathcal{P}_+ \) by additivity. We may also assume the following properties:

- **Bisognano-Wichmann property.** The modular group associated with \( (\mathcal{A}(W_R), \Omega) \) is given by \( \Delta^{t}(\mathcal{A}(W_R), \Omega) = U((0, -2\pi t)) \) and the representation of \( \mathcal{P}_+^1 \) extends to a representation of \( \mathcal{P}_+^1 \) by \( U(-I) := J_{\mathcal{A}(W_R), \Omega} \).
- **Haag duality.** For \( O \) a wedge or double cone it holds \( \mathcal{A}(O') = \mathcal{A}(O)' \).

An example of a Poincaré covariant net is a chiral conformal net. Let \( \mathcal{A}_\pm \) be two conformal nets on \( \mathcal{H}_\pm \) with vacuum \( \Omega_\pm \) and \( U_\pm \) the unitary representation of \( \text{M"ob} \). We see them by restriction as nets on \( \mathbb{R} \). Then we get a net \( \mathcal{A} \) on \( M_2 \) by defining \( \mathcal{A}(O) := \mathcal{A}_+(I_1) \otimes \mathcal{A}_-(I_2) \), \( O = I_1 \times I_2 := \{ (t, x) \in M_2 : t - x \in I_1, t + x \in I_2 \} \) on \( \mathcal{H}_+ \otimes \mathcal{H}_- \). It is covariant with respect to \( \text{M"ob} \times \text{M"ob} \) by the action \( U_+ \otimes U_- \) and in particular Poincaré covariant, because \( \mathcal{P}_+^1 \subset \text{M"ob} \times \text{M"ob} \).

### 3.4. Longo–Rehren boundary conformal quantum field theory

Let \( I_1, I_2 \) be two intervals of the time axis such that \( I_2 > I_1 \) and let us define the **double cone** \( O = I_1 \times I_2 := \{ (t, x) \in \mathbb{R}^2 : t - x \in I_1, t + x \in I_2 \} \) like in Figure 2. We call such a double cone \( O = I_1 \times I_2 \) **proper** if it has a positive distance to the boundary, i.e. \( \overline{I_1} \) and \( \overline{I_2} \) have empty intersection; the set of **proper double cones** we denote by \( K_+ \).

Let \( \mathcal{A} \) be a conformal net. With \( \mathcal{A} \) are associated two nets on the Minkowski half-plane: the trivial boundary CFT \( \mathcal{A}_+ \) defined by \( \mathcal{A}_+(O) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \) and its dual \( \mathcal{A}_+^{\text{dual}}(O) = \mathcal{A}(L) \cap \mathcal{A}(K)' \)

\( O = I_1 \times I_2 \);

\( \overline{I_1 \cup K \cup I_2}'' = L \).

![Figure 2. Double cone \( O = I_1 \times I_2 \) in \( M_+ \)](image-url)
Definition 3.5.1. A related to a local Möbius covariant net of von Neumann algebras on a Hilbert space $H$ is a family of von Neumann algebras $\{\mathcal{B}(O)\}_{O \in \mathcal{K}}$ on $H$ which fulfills:

A. **Isotony.** $O_1 \subset O_2$ implies $\mathcal{B}(O_1) \subset \mathcal{B}(O_2)$.

B. **Locality.** If $O_1, O_2 \in \mathcal{K}_+$ are spacelike separated then $[\mathcal{B}(O_1), \mathcal{B}(O_2)] = \{0\}$.

C. **Time-translation covariance.** There is a unitary continuous one-parameter group $T(t) = e^{itP}$ on $\mathcal{H}$ with positive generator $P$, such that

$$T(t)\mathcal{B}(O)T(t)^* = \mathcal{B}(O'),$$

where $O' = O + (t, 0)$ is the shifted double cone.

D. **Vacuum.** There is a (up to phase) unique $T$ invariant vector $\Omega \in \mathcal{H}$ which is cyclic and separating for every $\mathcal{B}(O)$ with $O \in \mathcal{K}_+$.

Let $\mathcal{A}$ be a Möbius covariant local net of von Neumann algebras on the Hilbert space $\mathcal{H}$, which we want to regard (by restriction) as a net on $\mathbb{R}$. All unitaries $V$ on $\mathcal{H}$, which commutes with the one-parameter group of translations $T(t) = U(\tau(t))$, satisfy $V\Omega = \Omega$ and the equivalent conditions

1. $V,A(I_2)V^*$ commutes with $A(I_1)$ for all intervals $I_1, I_2$ of $\mathbb{R}$ such that $I_2 > I_1$, i.e. $I_2$ is contained in the future of $I_1$,
2. $V,A(a,\infty)V^* \subset A(a, \infty)$ for all $a \in \mathbb{R}$,
3. $V,A(0,\infty)V^* \subset A(0, \infty)$,

form a semigroup denoted by $\mathcal{E}(A)$. Elements $V \in \mathcal{E}(A)$ are called **Longo–Witten unitaries** and $\eta = \text{Ad} V$ is called **Longo–Witten endomorphism**. Translations $V := T(t) = U(\tau(t))$ with $t \geq 0$ are elements in $\mathcal{E}(V)$. Also internal symmetries $V$ of $\mathcal{A}$, namely $V,A(I)V^* = A(I)$ for all $I \in \mathcal{I}$ give trivial examples of elements in $\mathcal{E}(A)$. Besides these trivial examples it is in general not much known if there exist other elements, but if they exist they are of the form stated as follows.

Remark 3.5.2 (cf. [LW11]). Let $\mathcal{A}$ be a conformal net, then $\mathcal{E}(A) \subset \mathcal{E}(H, T)$ with the one-parameter group of translations $T(t) = U(\tau(t)) = e^{itP}$ and the standard subspace $\mathcal{H} = \mathcal{A}(0, \infty)\omega\Omega$. In particular $\mathcal{H}_0 := H \ominus \mathbb{R}\Omega \subset \mathcal{H}_0$ with $\mathcal{H}_0 := \mathcal{H} \ominus \mathbb{C}\Omega$ is a non-degenerated standard pair and by Theorem

Proposition 3.4.1. $\mathcal{A}_+$ and $\mathcal{A}_+^{\text{dual}}$ are local.
2.3.10 it is \( V \uparrow_{\mathcal{H}_0} = (\varphi_{hk}(P_0)) \) (by definition \( V\Omega = \Omega \)) with \((\varphi_{hk})\) a matrix in \( \mathcal{S}^{(\infty)} \) (Definition 2.3.9) and \( P_0 = P \uparrow_{\mathcal{H}_0} \), cf. [LW11, Corollary 2.8].

Let \( \mathcal{A} \) be a conformal net and \( V \in \mathcal{E}(\mathcal{A}) \), then we define

\[
\mathcal{A}_V(\mathcal{O}) := \mathcal{A}(I_1) \vee V\mathcal{A}(I_2)V^*, \quad \mathcal{O} = I_1 \times I_2, \; I_2 > I_1.
\]

The special case \( V = 1 \) is exactly the conformal boundary net \( \mathcal{A}_+ \) defined in [LR04].

**Proposition 3.5.3** (cf. [LW11, Proposition 3.3 and Corollary 3.4]). *If* \( V \in \mathcal{E}(\mathcal{A}) \), *then* \( \mathcal{A}_V \) is a boundary net. The map \( \mathcal{E}(\mathcal{A}) \ni V \mapsto \mathcal{A}_V \) is one-to-one modulo internal symmetries, i.e. \( \mathcal{A}_{V_1} = \mathcal{A}_{V_2} \) with \( V_1, V_2 \in \mathcal{E}(\mathcal{A}) \) *iff* \( V_1 = V_2V \) with \( V \) an internal symmetry.

The study of such boundary nets \( \mathcal{A}_V \) associated with a conformal net \( \mathcal{A} \) simplifies therefore to the study of \( \mathcal{E}(\mathcal{A}) \). So the question is the characterization and classification of the semigroup \( \mathcal{E}(\mathcal{A}) \) for a given conformal net \( \mathcal{A} \). In Chapter II we investigate the explicit construction of families of such elements for several local Möbius covariant nets.

### 3.6. Borchers triple and wedge-local nets

An important tool by the construction are Borchers triple for example used in [Lec08], which give always a wedge-local net. We note that the construction of wedge-local nets is just an intermediate step and the ultimate goal is to obtain strictly local nets.

**Definition 3.6.1.** A **Borchers triple** on a Hilbert space \( \mathcal{H} \) is a triple \((\mathcal{M}, T, \Omega)\) of a von Neumann algebra \( \mathcal{M} \subset \mathcal{B}(\mathcal{H}) \), a strongly continuous unitary representation \( T \) of \( \mathbb{R}^2 \) on \( \mathcal{H} \) and a unit vector \( \Omega \in \mathcal{H} \), such that

- \( \text{Ad} T(t, x)(\mathcal{M}) \subset \mathcal{M} \) for all \((t, x) \in \mathbb{R}^2\).
- The joint spectrum of the generator of the translations is contained in the forward lightcone \( \mathcal{V}_+ \).
- \( \Omega \) is a unique (up to a phase) invariant vector under \( T \) and is cyclic and separating for \( \mathcal{M} \).

The fundamental examples for Borchers triple are Poincaré covariant nets. We can define a Borchers triple \((\mathcal{M}, T, \Omega)\) on the same Hilbert space by \( \mathcal{M} = \mathcal{A}(W_R) \), \( T = U \uparrow \mathbb{R}^2 \) the restriction to the translations. The properties follow from the translation covariance and the properties of \( \mathcal{M} \). If \( \mathcal{A} \) fulfills Bisognano–Wichmann property and Haag duality we can recover it from the Borchers triple as we will see below.

**Definition 3.6.2.** We call a Borchers triple \((\mathcal{M}, T, \Omega)\) **strictly local** if \( \Omega \) is cyclic for all \( \mathcal{M} \cap \text{Ad} T(a)(\mathcal{M}') \) for any \( a \in W_R \).

So the Borchers triple of a Poincaré covariant net is obviously strictly local.

We recall that one interprets \( \mathcal{M} \) as the algebra assigned to the wedge \( W_R \). Let \( \mathcal{W} \) be the set of wedges, i.e. the set of all \( W = gW_R \) where \( g \) is a Poincaré transformation. Then we define the **wedge-local net** \( \mathcal{W} \ni W \mapsto \mathcal{M}(W) \) associated with the Borchers triple \((\mathcal{M}, T, \Omega)\) by \( \mathcal{M}(W_R + a) = T(a)\mathcal{M}(0)T(a)^* \) and \( \mathcal{M}(W_R' + a) = T(a)\mathcal{M}'(a)^* \). With the help of the modular objects one can define a representation of the Poincaré group extending the one of translations \( T \) [Bor92, III.1. Thm.]. There is a one-to-one correspondence between

- Borchers triple \((\mathcal{M}, T, \Omega)\) and
- wedge-local Poincaré covariant nets \( \mathcal{A} \) on \( \mathbb{R}^2 \) fulfilling the Bisognano–Wichmann property.
3. PRELIMINARIES ON ALGEBRAIC QUANTUM FIELD THEORY

If \((\mathcal{M}, T, \Omega)\) is strictly local we can define for a double cone \(O = a + W_R \cap b + W'_R\) the Poincaré covariant net \(\mathcal{A}(O) = \text{Ad} T(a)(\mathcal{M}) \cap \text{Ad} T(b)(\mathcal{M}')\). If the Borchers is not strictly local, it is still a local net on \(\mathcal{H}' = \bigvee_{O \in \mathcal{T}} \mathcal{A}(O)\Omega\), but in can in principle happen that we get a trivial net, i.e. \(\mathcal{A}(O) = C \cdot 1\) and \(\mathcal{H}' = C \cdot \Omega\).

3.7. Massless scattering theory for Borchers triple

We fix a Borchers triple \((\mathcal{M}, T, \Omega)\) on \(\mathcal{H}\). For \(x \in \mathcal{B}(\mathcal{H})\) we write \(x(a) = \text{Ad} T(a)(x)\) for \(a \in \mathbb{R}^2\) for the by “a translated” operator. We consider observables sent to lightlike directions with a parameter \(\tau \in \mathbb{R}\):

\[
x_{\pm}(h_\tau) := \int_{\mathbb{R}} h_\tau(t) x(t, \pm \tau) dt,
\]

where \(h_\tau(t) = |t|^{-\epsilon} \cdot h(\epsilon \cdot (t - \tau))\) for some fixed \(0 < \epsilon < 1\) and some fixed nonnegative symmetric smooth function \(h : \mathbb{R} \to \mathbb{R}\) with \(\int_{\mathbb{R}} h(t) dt = 1\). Then for \(x \in \mathcal{M}\) and \(y' \in \mathcal{M}'\) the strong limits

\[
\Phi_{in}^{out}(x) := s\text{-lim}_{\tau \to +\infty} x_+ (h_\tau)
\]

\[
\Phi_{in}^{in}(x) := s\text{-lim}_{\tau \to -\infty} x_-(h_\tau)
\]

\[
\Phi_{out}^{in}(x) := s\text{-lim}_{\tau \to -\infty} y'_+(h_\tau)
\]

\[
\Phi_{out}^{out}(x) := s\text{-lim}_{\tau \to +\infty} y'_-(h_\tau)
\]

exist. The \(\Phi_{\pm}^{in/out}\) are called asymptotic fields and their properties are summarized in [DT11, Tan12a]. For example it holds for \(y' \in \mathcal{M}'\):

\[
\Phi_{in}^{out}(y') := J_{\mathcal{M}} \Phi_{+}^{out}(J_{\mathcal{M}} y' J_{\mathcal{M}}) J_{\mathcal{M}}
\]

\[
\Phi_{out}^{in}(y') := J_{\mathcal{M}} \Phi_{-}^{in}(J_{\mathcal{M}} y' J_{\mathcal{M}}) J_{\mathcal{M}}
\]

where \(J_{\mathcal{M}} \equiv J_{(\mathcal{M}, \Omega)}\) is the modular conjugation of \(\mathcal{M}\) with respect to \(\Omega\).

We want to consider the (massless) scattering theory of waves. A wave is a single excitation with lightlike spectrum. In two space time dimensions the situation is rather simple, because the (spacelike) momentum of such a wave is either positive or negative, so there are just two kind of waves.

**Definition 3.7.1.** We define the space \(\mathcal{H}_+\) of single (massless) excitations or waves with positive and \(\mathcal{H}_-\) of negative momentum by

\[
\mathcal{H}_\pm = \{ \xi \in \mathcal{H} : T(t, \pm \tau) \xi = \xi \text{ for } t \in \mathbb{R} \},
\]

respectively. We denote by \(P_\pm\) the projector on \(\mathcal{H}_\pm\).

For \(\xi_\pm \in \mathcal{H}_\pm\) there are sequences of local operators \(\{x_n\}, \{y_n\} \in \mathcal{M}\) and \(\{x'_n\}, \{y'_n\} \in \mathcal{M}'\), such that

\[
s\lim P_+ x_n \Omega = s\lim P_+ x'_n \Omega = \xi_+
\]

\[
s\lim P_- y_n \Omega = s\lim P_- y'_n \Omega = \xi_-
\]

and with this collision states can be defined as in [DT11] by

\[
\xi_+^{in} \times \xi_-^{in} = s\lim_{n \to \infty} \Phi_{+}^{in}(x'_n) \Phi_{-}^{in}(y_n) \Omega = (P_+ x'_n \Omega \to \xi_+, P_- y_n \Omega \to \xi_-)
\]

\[
\xi_+^{out} \times \xi_-^{out} = s\lim_{n \to \infty} \Phi_{+}^{out}(x'_n) \Phi_{-}^{out}(y_n) \Omega = (P_+ x_n \Omega \to \xi_+, P_- y'_n \Omega \to \xi_-)
\]
which do not depend on the particular choices made. We denote by $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ the subspace generated by $\xi_{+} \times \xi_{-}$ and $\xi_{+} \times \xi_{-}$, respectively. The isometry defined by

$$S : \mathcal{H}_{\text{out}} \rightarrow \mathcal{H}_{\text{in}}, \xi_{+} \times \xi_{-} \mapsto \xi_{+} \times \xi_{-}$$

is called the **scattering operator** or the **S-matrix** of the Borchers triple. We call the Borchers triple **interacting** if $S$ is not equal to a multiple of the identity operator and **asymptotically complete** if it holds $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}$.

**Definition 3.7.2.** Let $A_{\pm}, U_{\pm}, \Omega_{\pm}$ be two conformal nets, the chiral 2D net is defined by $A(O) = A_{+}(I) \otimes A_{-}(J)$ where $O = I \times J$.

**Definition 3.7.3.** Let $A_{\pm}$ a chiral conformal net on $\mathcal{H}_{\pm} \otimes \mathcal{H}_{\pm}$. A unitary $S \in \mathcal{B}(\mathcal{H}_{+} \otimes \mathcal{H}_{-})$ is called a **scattering operator** for $A_{+} \otimes A_{-}$ if

1. $S$ commutes with $T_{+} \otimes T_{-}$.
2. It is $S(\xi \otimes \Omega_{-}) = \xi \otimes \Omega_{-}$ for $\xi \in \mathcal{H}_{+}$ and $S(\Omega_{+} \otimes \eta) = \Omega_{+} \otimes \eta$ for $\eta \in \mathcal{H}_{-}$.
3. $x \otimes 1$ commutes with $\text{Ad} S(x') \otimes 1$ for $x \in A_{+}(\mathbb{R}_{-})$ and $x' \in A_{+}(\mathbb{R}_{+})$.
4. $\text{Ad} S(1 \otimes y)$ commutes with $1 \otimes y'$ for $y \in A_{-}(\mathbb{R}_{+})$ and $y' \in A_{-}(\mathbb{R}_{-})$.

**Proposition 3.7.4.** Let $A_{\pm} \otimes A_{\pm}$ a chiral conformal net on $\mathcal{H}_{+} \otimes \mathcal{H}_{-}$ and $S$ a scattering operator for $A_{+} \otimes A_{-}$. Then $(\mathcal{M}_{S}, T, \Omega)$ with

$$\mathcal{M}_{S} = (A_{+}(\mathbb{R}_{-}) \otimes 1) \lor \text{Ad} S(1 \otimes A_{-}(\mathbb{R}_{+}))$$

$$T(t, x) = T_{0}(t - x)/\sqrt{2} \otimes T_{0}(t + x)/\sqrt{2}$$

$$\Omega = \Omega_{0} \otimes \Omega_{0}$$

is an asymptotically complete Borchers triple with S-matrix $S$.

**Proof.** We prove a more general case in Section 9.1. 

The Borchers triple $(\mathcal{M}_{S}, T, \Omega)$ does not need to be strictly local. But if $(\mathcal{M}_{S}, T, \Omega)$ is a strictly local and asymptotically complete Borchers triple then it is actually of the above mentioned form (see [Tan12a]).
Part II

Construction of Longo–Witten unitaries and models in BQFT
Longo–Witten unitaries for conformal nets associated with lattices and loop group models

In this chapter we give new examples of models in boundary quantum field theory, i.e. local time-translation covariant nets of von Neumann algebras. Namely, we compute elements of this semigroup $\mathcal{E}(A_L)$ coming from Hölder continuous symmetric inner functions for a family of (completely rational) conformal nets $\{A_L\}_L$ which can be obtained by starting with nets of real subspaces, passing to its second quantization nets and taking local extensions of the former. This family is precisely the family of conformal nets associated with lattices $\{L\}$, which as we show contains as a special case the level 1 loop group nets of simply connected, simply laced groups. Further examples come from the loop group net of $\text{Spin}(n)$ at level 2 using the orbifold construction. The results were obtained in [Bis11].

4.1. Conformal field theory – conformal nets

In this section we are interested in local Möbius covariant nets (conformal nets). These are nets on the circle (or the real line), which physically describe the chiral part of the algebra of observables of a 2D QFT, where the real line (circle) is then identified with (the compactification) of one of the lightrays.

4.1.1. Nets of standard subspaces. Before describing nets of von Neumann algebras we want to go a step back and give some details on nets of real subspaces of a Hilbert space $\mathcal{H}_0$, whose “second quantization” leads to nets of von Neumann algebras, the so called second quantization nets. In analogy to the “free field construction” from Wigner particles the Hilbert space $\mathcal{H}_0$ will be called the “one-particle space”. See for example [BGL02] for a general construction of free Bosons using this techniques on more general space-times\(^1\) and [Lon08b] for such nets on the circle.

**Definition 4.1.1.** A local Möbius covariant net of standard subspaces of $\mathcal{H}$ is a family of standard subspaces $H(I) \subset H$ indexed by $I \in \mathcal{I}$ such that the following properties hold:

A. **Isotony.** $I_1 \subset I_2$ implies $H(I_1) \subset H(I_2)$.

B. **Locality.** $I_1 \cap I_2 = \emptyset$ implies $H(I_1) \subset H(I_2)'$.

C. **Möbius covariance.** There is a positive energy representation of $\text{M"ob}$ on $H$ such that $U(g)H(I) = H(gI)$ for all $g \in \text{M"ob}$ and $I \in \mathcal{I}$.

D. **Irreducibility.** $U$ is non-degenerate, i.e. does not contain the trivial representation.

Given a positive energy representation $U$ of $\text{M"ob}$ on $\mathcal{H}$ we can construct a local Möbius covariant net of standard subspaces as follows: we define the unitary one-parameter group $\Delta^it = U(\delta(2\pi t))$, where $\delta(t) = e^{\iota \pi t}$ are the dilations and the antiunitary involution $J = U(r)$ (where we use that $U$ extends to a representation of $\text{M"ob}_\pm$, cf. Theorem 3.1.2) and define the densely defined, closed, antilinear involution $S = J\Delta^{1/2}$. We denote by $I_0$ the interval corresponding to the upper semicircle or equivalently $(0, \infty)$. Then we set $H(I_0) \equiv H(0, \infty) = \{x \in \mathcal{H} : Sx = x\}$ to be the standard subspace.

---

\(^1\)In our case the “space-time” is the circle and the “wedges” correspond to open non-empty nowhere dense intervals.
associated with \( S \) and for general \( \mathcal{I} \ni I = g l_0 \) we set \( H(I) = U(g)H(0, \infty) \), which does not depend on the choice of \( g \in \text{M"ob} \). All local Möbius covariant nets of standard subspaces are obtained in this way \( [\text{Lon08b}] \).

For later use we make the construction of a family indexed by \( n \in \mathbb{N} \) of local Möbius covariant nets of real subspaces—or the net coming from \( n \) copies of the lowest weight \( 1 \) positive energy representation (cf. \( [\text{Lon08b}] \)) of \( \text{M"ob} \)—more explicit. Therefore let \( F \) be a \( n \)-dimensional Euclidean space with scalar product \( \langle \cdot, \cdot \rangle \). Let us define \( \mathcal{H}_{0,F} = \mathcal{H}_0 \otimes \mathbb{R} F \cong \bigoplus_{i=1}^{n} \mathcal{H}_0 \) which is in particular isomorphic to the \( n \)-fold direct sum of the unique irreducible positive energy lowest weight representation of \( \text{M"ob} \) with lowest weight \( 1 \) denoted by \( (U_0, \mathcal{H}_0) \). We denote by \( U_{0,F} \) the unitary representation of the \( \text{M"ob} \) on \( \mathcal{H}_{0,F} \). It can explicitly be constructed as follows. Let \( LF = C^\infty(\mathbb{S}^1, F) \cong C^\infty(\mathbb{S}^1, \mathbb{R}) \otimes \mathbb{R} F \) the set of all smooth maps (loops) from the circle \( \mathbb{S}^1 \) in \( F \). Because \( f \in LF \) is periodic it can be written as a Fourier series

\[
 f(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ik\theta}, \quad \hat{f}_k = \int_0^{2\pi} e^{-ik\theta} f(\theta) \frac{d\theta}{2\pi}
\]

with Fourier coefficients \( \hat{f}_k = \overline{\hat{f}_{-k}} \) in the complexified space \( F_\mathbb{C} := F \otimes _\mathbb{R} \mathbb{C} \). We introduce a semi-norm

\[
 \|f\|^2 = \sum_{k=1}^{\infty} k \cdot \|\hat{f}_k\|^2_{F_\mathbb{C}}
\]

and a complex structure, i.e. an isometry \( J \) w.r.t. \( \| \cdot \| \) satisfying \( J^2 = -1 \), by

\[
 J : \hat{f}_k \mapsto -i \text{sign}(k) \hat{f}_k
\]

and finally we get the Hilbert space \( \mathcal{H}_{F,0} \) by completion with respect to the norm \( \| \cdot \| \)

\[
 \mathcal{H}_{0,F} = LF/\mathbb{F}\| \cdot \|
\]

where \( F \) is identified with the constant functions. The scalar product \( \langle \cdot, \cdot \rangle \) can be obtained by polarization and the unitary action of \( \text{M"ob} \) is induced by the action on \( LF \), namely

\[
 U(g) : f \mapsto g_* f : (g_* f)(\theta) = f(g^{-1}(\theta)).
\]

Let \( f \in LF \). If no confusion arises we denote also its image \( \left[ f \right] \in \mathcal{H}_{0,F} \) of the inclusion \( i_F : LF \to \mathcal{H}_{0,F} \) by \( f \). On \( LF \) the sesquilinear form coming from the scalar product is given explicitly by

\[
 \omega(f, g) := \text{Im}(f, g) = -\frac{i}{2} \sum_{k \in \mathbb{Z}} k \langle \hat{f}_k, \hat{g}_{-k} \rangle = \frac{1}{2} \int_0^{2\pi} \langle f(\theta), g'(\theta) \rangle \frac{d\theta}{2\pi} =: \frac{1}{2} \int \langle f, g' \rangle.
\]

For \( I \in \mathcal{I} \) denote by \( H_F(I) \) the closure subspace of functions with support in \( I \). The family \( \{ H_F(I) \}_{I \in \mathcal{I}} \) is a local Möbius covariant net of standard subspaces. Indeed because \( U \) acts geometrical, and in particular \( U(\delta(t)) \) is the modular group of the abstract construction and leaves \( H_F(0, \infty) \) invariant, one can show that the explicit construction equals the modular construction mentioned above (cf. \( [\text{Lon08b}] \)).

**Proposition 4.1.2.** Let \((F, \langle \cdot, \cdot \rangle)\) be an Euclidean space, then there is a local Möbius covariant net of standard subspace \( H_F \) on the Hilbert space \( \mathcal{H}_{0,F} \).

We remark that by the geometric modular action follows that the net is Haag dual, i.e. \( H_F(I') = H_F(I) \) and also the restriction to \( \mathbb{R} \) can be shown to be Haag dual, i.e. \( H_F((\mathbb{R} \setminus I)^\circ) = H_F(I) \) for \( I \in \mathbb{R} \).
4.1.2. Second quantization nets. By second quantization of a net of standard subspaces we get a net of von Neumann algebras. We refer to Section 2.4.1 for the second quantization map $R$.

**Proposition 4.1.3** (Second quantization nets [Lon08a]). Let $\{H(I)\}_{I \in \mathcal{I}}$ be a local Möbius covariant net of standard subspaces on $\mathcal{H}$. Then $\mathcal{A}(I) = R(H(I))$ defines a local Möbius covariant net (of von Neumann algebras) on $e^H$.

Let $(F, \langle \cdot, \cdot \rangle)$ be an Euclidean space and $I \ni I \mapsto H_F(I) \subset \mathcal{H}_{0,F}$ the net of standard subspaces from Proposition 4.1.2. Then we denote by $\mathcal{A}_F$ the local Möbius covariant net on $\mathcal{H}_F := e^{\mathcal{H}_{0,F}}$ called the **Abelian current net over** $F$ given by $\mathcal{A}_F(I) := R(H_F(I))$. If $U_0$ is the action of $\text{Möb}$ on $\mathcal{H}_{0,F}$ then the action on $\mathcal{H}_F$ is given by $U(g) := \Gamma(U_0(g))$. In the case $F = \mathbb{R}$ the net is also called the **U(1)-current net** and was treated in an operator algebraic setting first in [BMT88]. We remark that $\mathcal{A}_F$ is clearly isomorphic to the $n$-fold tensor product of the U(1)-current net.

**4.1.2.1. Representations.** Let $\ell \in C^\infty(S_1, F)$ with support in some $I_0 \in \mathcal{I}$. Then we define for $I \supset I_0$ with $I_0 \subset I$

$$\rho_{\ell_1}(W(f)) = e^{i\ell(f_{\ell_1})}W(f)$$

where we have chosen a representant $f$ of $[f] \in \mathcal{H}_{F,0}$ with $f \upharpoonright I = 0$. This defines a representation localized in $I_0$. This representation is covariant with local cocycle localized in $I \supset I_0$ and $\mathcal{U}$ a symmetric neighbourhood of the identity of $\text{Möb}$ such that $I_0 \cup \text{gh} \subset I$ for all $g \in \mathcal{U}$ given by $\varepsilon(g) = W(M-L)$ where $L$ is a primitive of $\ell$, i.e. $L'(\theta) = \ell(\theta)$ and $L_{\theta}(\theta) = g_{L(\theta)} = L(g^{-1}\theta)$. The following lemma will be useful showing the equivalence of two cocycles.

Two representations are equivalent if they have the same charge, which is given for $\rho_\ell$ by

$$q_\ell = \int_0^{2\pi} \ell(\theta) \frac{d\theta}{2\pi} = \int \ell \in F,$$

namely for $q_\ell = q_m$ it is $z_{q_\ell} = \rho_m z$ with unitary intertwiner $z = W(M-L)$ where $M \in \mathcal{H}_{F,0}$ is a primitive of $m - \ell$. In other words the sectors depend only on this charge $q \in F$ and we denote the sector by $[q]$ with obvious fusion rules $[q] \times [r] = [q + r]$. We note that because there are infinitely many sectors (with dimension 1) the index of the inclusion (3.4) is infinite and the nets cannot be completely rational.

Equivalently the conformal net can be regarded as coming from a projective positive energy representation of the group $\mathcal{L}F$.

**4.1.2.2. Abelian currents from central extensions.** Basically to fix notation, we recall some facts about projective representations. If $\pi$ is a projective representation of a group $G$ on a Hilbert space $\mathcal{H}$, then there is a 2-cocycle with $c : G \times G \longrightarrow \mathbb{T} \subset \mathbb{C}$ given by

$$\pi(g)\pi(h) = c(g, h)\pi(gh)$$

fulfilling the cocycle relation $c(h,k)\pi(g, hk) = c(g, h)\pi(gh, k)$, which follows from associativity. Two representations are equivalent if and only if there is a coboundary

$$b_f(g, h) = \frac{f(g)f(h)}{f(gh)}$$

for all $g, h \in G$.

If $G \cong \mathbb{Z}^n$ this is true if and only if $c_1 = c_2$ (see for example [Kac98, Lemma 5.5] cf. also [DHR69, Lemma A.1.2]) where $c(g, h) = c(g, h)c(h, g)^{-1}$ is the **commutator map** or **antisymmetric part** of a cocycle $c$. The following lemma will be useful showing the equivalence of two cocycles.
**Lemma 4.1.4.** Let \( G = G_1 \times G_2 \) be an Abelian group and \( c, c' \in \mathbb{Z}^2(G, \mathbb{T}) \) be two 2-cocycle and \( c_i, c'_i \in \mathbb{Z}^2(G_i, \mathbb{T}) \) their restrictions to \( G_i \times G_i \) for \( i = 1, 2 \). If \([c_i] = [c'_i] \in H^2(G_i, \mathbb{T})\) then \( \hat{c} = \hat{c'} \) implies \([c'] = [c] \in H^2(G, \mathbb{T})\).

**Proof.** The proof is basically [DHR69, Proof of Lemma A.1.2.]. Because \( c \mapsto \hat{c} \) is a homomorphism it is enough to show that for \( c \in \mathbb{Z}^2(G, \mathbb{T}) \): if \( \hat{c} = 1 \) and \( c(g_i, h_i) = b_i(g_i)b_i(h_i)/b_i(g, h_i) \) then \( c \in B^3(G, \mathbb{T}) \), i.e. \( c = \delta b \) with \( b \in \mathbb{Z}^4(G, \mathbb{T}) \). Indeed, setting

\[
b(g_1g_2) = \frac{b_1(g_1)b_2(g_2)}{c(g_1, g_2)} = \frac{b_1(g_1)b_2(g_2)}{c(g_2, g_1)}
\]

for \( g_i \in G_i \) we calculate using the cocycle relation:

\[
c(g_1g_2, h_1h_2) = \frac{c(g_1, g_3h_1h_2)c(g_2, h_1h_2)}{c(g_1, g_2)} = \frac{c_1(g_1, h_1)c_2(g_1, h_2)c(g_2, h_1)}{c(g_1, g_2)c(h_1, g_2h_2)c(h_2, h_1)} = \frac{b(g_1g_2)b(h_1h_2)}{b(g_1h_1g_2h_2)}.
\]

Equivalently to say that \( \pi \) is a projective representation is the existence of a true representation also denoted by \( \pi \) of the group \( \hat{G} = G \times \mathbb{T} \) with multiplicative law \((g_1, t_1)(g_2, t_2) = (g_1g_2, c(g_1, g_2)t_1t_2) \) given by \( \pi(g, t) = \tau(t) \). One calls \( \hat{G} \) a central extension of \( G \).

Let \( G \) be a Lie group and \( \pi \) a continuous projective unitary representation of \( LG \) on a Hilbert space \( \mathcal{H} \). We assume that there is an action of the rotation, i.e. \( T \) acts unitarily on \( \mathcal{H} \) by \( U \) such that \( U(\theta)\pi(f)U(\theta)^* = \pi(R_0f) \) where \( R_0f(\theta') = f(\theta' - \theta) \) for \( f \in LG \). In other words we assume \( \pi \) extends to a representation of \( LG \rtimes \mathbb{T} \). Then \( \pi \) is called **positive energy** (cf. [Seg81, PS86]5) if

\[
\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n, \quad \mathcal{H}_n = \{ x \in \mathcal{H} : U(\theta)x = e^{i\theta}x \}
\]

with \( \dim \mathcal{H}_n < \infty \) and \( \mathcal{H}_0 \neq \{0\} \). That means the generator \( L_0 \) of \( U(\theta) = e^{i\theta L_0} \) has positive spectrum.

Let \( \mathcal{L}F \) be the central extension of \( LF \) defined by the cocycle

\[
c_F(f, g) = e^{-i\lambda(f, g)} = e^{-i/2 \int (f, g')}.
\]

Then the conformal net \( \mathcal{A}_F \) constructed above can be regarded as the conformal net associated with a positive energy representation of \( \mathcal{L}F \) with cocycle \( c_F \) or equivalently a (true) positive energy representation of \( LF \). For \( I \in \mathcal{I} \) we denote by \( L_I \) all loops with support in \( I \).

**Proposition 4.1.5.** Let \( \mathcal{H}_0 = LF/F \) be the one-particle space associated with \((F, \{ \cdot, \cdot \})\). There is unitary positive energy representation \( \pi \) of \( LF \) on the Fock space \( \mathcal{H} \) given by: \( \pi_0 : LF \equiv LF \rtimes \mathcal{T} \) \( \equiv LF \rtimes \mathbb{T} \), \( \mathcal{H}_0 \equiv \mathcal{H} \) \( \equiv \mathcal{H}_0 \). By Proposition 2.4.1 and because \( L_I \) is dense in \( H_F(I) \) again by construction.

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5We use a different convention, which fits with the definition of positive energy for conformal nets.

6This can be obtained by multiplying a given representation of \( \mathbb{T} \) with a character of \( \mathbb{T} \).
4.1.3. Conformal nets associated with lattices. We want to consider local extensions of the net \( \mathcal{A}_F \) associated with Abelian currents with values in \( F \). The case of \( F = \mathbb{R} \) (one current) was treated in \cite{BMT88} and the extensions are given by a charge \( g = \sqrt{2N} \) with \( N \in \mathbb{N} \). The general case was elaborated in \cite{Sta95} with the result that the extensions are given by even integral lattice (for \( n = 1 \) the lattice is \( g\mathbb{Z} \)). The same lattice models were also examined in \cite{DX06}, where they are equivalently defined as a positive energy representation of the loop group of the torus associated with the lattice. This gives a connection to the representations of loop groups at level 1 \cite{Seg81,PS86} for simply laced Lie groups. The lattice models are well known in the framework of vertex operator algebras. For a treatment of lattice models in vertex operator algebras and its connection to Kac–Moody algebras we refer e.g. to \cite[Chapter 5.4]{Kac98}.

Let \( L \) be an \textbf{integral (positive) lattice}, i.e. a free \( \mathbb{Z} \)-module with positive-definite integral bilinear form \( \langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{Z} \). A lattice is called \textbf{even} if \( \langle \alpha, \alpha \rangle \in 2\mathbb{N} \) for all \( \alpha \in L \), and we note that an even lattice is necessarily integral. To a lattice \( L \) we relate an Euclidean space \( (F, \langle \cdot, \cdot \rangle) \) where \( F = L \otimes_{\mathbb{Z}} \mathbb{R} \) and the scalar product \( \langle \cdot, \cdot \rangle \) is continued to \( F \times F \rightarrow \mathbb{R} \) by linearity. The dimension \( n = \dim F \) is called the \textbf{rank} (assumed to be finite).

Equivalently, we can view an even lattice \( L \) as a free discrete subgroup of a finite dimensional Euclidean space \( (F, \langle \cdot, \cdot \rangle) \) which spans \( F \) and satisfies \( \langle \alpha, \alpha \rangle \in 2\mathbb{N} \) for all \( \alpha \in L \). Let \( L \) be even and \( L^* := \{ x \in V : \langle x, L \rangle \subset \mathbb{Z} \} \) be the dual lattice \cite{CS98}. It is a lattice, but not necessarily an integer and can canonically be identified with \( \text{Hom}(L, \mathbb{Z}) \) by the scalar product. It is \( L \subset L^* \) and it can be shown that the group \( L^* / L \) is finite. In the case \( L^* = L \) the lattice is called \textbf{self-dual} or \textbf{unimodular} and this can be the case only for rank \( n \in 8\mathbb{N} \).

With an even lattice \( L \) we associate a torus \( T = F / 2\pi L \), and we will represent elements by \( e^{it} \) with \( f \in F \) and \( e^{if} = 1 \) if and only if \( f \in L \), formally

\[
F / 2\pi L \leftrightarrow T, \ [t] \mapsto e^{it}.
\]

4.1.3.1. \textit{Loop group associated with a torus}. Let \( \mathbb{L} T = C^\infty(\mathbb{S}^1, T) \) the loop group associated with the torus \( T \). We write \( e^{if} \) for an element in \( \mathbb{L} T \) where we mean the function \( e^{i\theta} \mapsto e^{if(\theta)} \) and \( f : \mathbb{R} \rightarrow F \) is a smooth function such that the winding number

\[
\Delta_f := \frac{1}{2\pi} (f(\theta + 2\pi) - f(\theta))
\]

is constant and takes values in \( L \). In particular \( f_0 : \theta \mapsto f(\theta) - \Delta_f \cdot \theta \) is a periodic function and we can decompose

\[
f(\theta) = \Delta_f \cdot \theta + f_0 + \sum_{n \in \mathbb{Z}^*} f_n e^{inti}
\]

where we call \( f_0 \) the \textbf{zeroth-mode}.

We are interested in projective positive energy representations of \( \mathbb{L} T \) or equivalently representations of a central extension:

\[
1 \rightarrow \mathbb{T} \rightarrow \mathbb{L} T \rightarrow \mathbb{L} T \rightarrow 1
\]

which are given by a cocycle \( c : \mathbb{L} T \times \mathbb{L} T \rightarrow \mathbb{T} \) specified in the following.

It is well-known (see for example \cite{Kac98}) that there exists a bilinear form \( b : L \times L \rightarrow \mathbb{Z}_2 \) such that

\[
b(\alpha, \alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle \quad \text{for all} \ \alpha \in L,
\]
e.g. if \( \{ \alpha_1, \ldots, \alpha_n \} \) is a basis of \( L \) one can choose
\[
b(\alpha_i, \alpha_j) = \begin{cases} 
(\alpha_i, \alpha_j) \mod 2 & i < j \\
\frac{1}{2} (\alpha_i, \alpha_i) \mod 2 & i = j \\
0 & i > j.
\end{cases}
\]
Therefore a bimultiplicative map \( \varepsilon(\alpha, \beta) : L \times L \rightarrow \{+1, -1\} \cong \mathbb{Z}_2 \) exists, satisfying \( \varepsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2} \). Such a map is a 2-cocycle satisfying:
\[
\varepsilon(\alpha, \beta + \gamma)\varepsilon(\beta, \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha + \beta, \gamma)
\]
\[
\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}.
\]

Now we specify the central extension \( \mathcal{L} \mathcal{T} \) by choosing a 2-cocycle \( c : \mathcal{L} \mathcal{T} \times \mathcal{L} \mathcal{T} \rightarrow \mathbb{T} \) as in [Seg81]
\[
c(e^{i\theta}, e^{i\theta}) \equiv c(f, g) = \varepsilon(\Delta_f, \Delta_g)e^{i\frac{\Delta(f, g)}{2}}
\]
\[
2 : S(f, g) = \int_{-\pi}^{\pi} \langle f'(\theta), g(\theta) \rangle \frac{d\theta}{2\pi} + \langle \Delta_f, g(0) \rangle.
\]
We note that the central extension (up to equivalence) does not depend on the explicit choice of the 2-cocycle in its equivalence class. Further we write the relations in \( \mathcal{L} \mathcal{T} \) formally as \( e^{i\theta}\varepsilon^{i\pi} = c(e^{i\theta}, e^{i\pi})e^{i(f + g)} \). It is straightforward to verify the following relations.

**Lemma 4.1.6** (cf. [DX06]). Let \( e^{i\theta}, e^{i\pi} \in \mathcal{L} \mathcal{T} \), then we have the following relations in \( \mathcal{L} \mathcal{T} \).
\[
e^{i\theta}\varepsilon^{i\pi}(e^{i\theta})^{-1} = e^{i\pi(\Delta_f, \Delta_g)}e^{i\int f'\,d\theta}e^{i(\Delta_f, \pi_0) - i(\Delta_g, \pi_0)}e^{i\pi}.
\]

**Proof.** We observe that \( (e^{i\theta})^{-1} = c(e^{i\theta}, e^{-i\theta})^{-1}e^{-i\theta} = c(f, f)e^{-i\theta} \) and we get:
\[
e^{i\theta}e^{i\pi}(e^{i\theta})^{-1} = c(f, g)c(g, -f)c(f, -f)c(f, f)e^{i\pi} = c(f, g)c(g, f)e^{i\pi} = (-1)^{\Delta_f, \Delta_g}e^{i\pi S(f, g) - S(g, f)}e^{i\pi}.
\]

Using \( f(\theta) = \Delta_f \cdot \theta + f_0 + f_1(\theta) \) and \( g(\theta) = \Delta_g \cdot \theta + g_0 + g_1(\theta) \) we have
\[
S(f, g) = \frac{1}{4\pi} \int_0^{2\pi} \langle f_1'(\theta), g_1(\theta) \rangle d\theta + \pi \langle \Delta_f, \Delta_g \rangle + \frac{1}{2} \langle \Delta_f, g_0 \rangle + \frac{1}{2} \langle f_1(2\pi), \Delta_g \rangle + \frac{1}{2} \langle \Delta_f, g_1(0) \rangle
\]
and this gives
\[
S(f, g) - S(g, f) = \frac{1}{2\pi} \int_0^{2\pi} \langle f_1'(\theta), g(\theta) \rangle d\theta + \frac{1}{2} \langle \Delta_f, g_0 \rangle - \frac{1}{2} \langle \Delta_g, f_0 \rangle
\]
\[
+ \frac{1}{2} \langle \Delta_f, g(0) - g_1(0) \rangle - \frac{1}{2} \langle \Delta_g, f(0) - f_1(0) \rangle
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \langle f_1'(\theta), g(\theta) \rangle d\theta + \langle \Delta_f, g_0 \rangle - \langle \Delta_g, f_0 \rangle
\]
which inserted in (4.7) completes the proof.

The following proposition is proved in [DX06, Proposition 3.4] and shows that the central extension is local, i.e. loops supported in disjoint intervals commute.

**Proposition 4.1.7** (Locality cf. [DX06, Prop. 3.4]). If \( \text{supp } e^{i\theta} \cap \text{supp } e^{i\pi} = \emptyset \) then \( e^{i\theta}e^{i\pi} = e^{i\pi}e^{i\theta} \).
For \( I \in \mathcal{I} \) we denote by \( \mathcal{L}_I T = \{ e^I / \mathcal{L} T : \text{supp} e^I \subset I \} \) all loops with support in \( I \) and by \( \mathcal{L}_I T \) the preimage of \( \mathcal{L}_I T \) under the covering map. Using this locality and well-known results of positive energy representations of loop groups it is shown in [DX06] that there is a conformal net associated with \( \mathcal{L} T \), precisely:

**Proposition 4.1.8** (Local conformal net associated with \( \mathcal{L} T \) cf. [DX06]). There is a correspondence between the elements of \( L^* / L \) and positive energy representation of \( \mathcal{L} T \). Let \( \pi_{(L,0)} \) be the (vacuum) representation corresponding to \([0] \in L^* / L\), then

\[
I \mapsto \mathcal{A}_L(I) := \pi_{(L,0)}(\mathcal{L}_I T)^\vee
\]

is a completely rational conformal net with \( \mu \)-index \( \mu = |L^* / L| \) and has \( \mu \) sectors of statistical dimension 1 corresponding to the positive energy representations of \( \mathcal{L} T \).

**Proof.** For the first statement see [DX06, Lemma 3.5] and [PS86, Section 9.5]. \( \mathcal{A}_L T \) is a local net by Proposition 4.1.7 and [DX06, Proposition 3.1] shows that it is a strongly additive conformal net fulfilling the split property. [DX06, Proposition 3.15] shows the correspondence between sectors and elements of \( L / L^* \) and the \( \mu \) index is given in [DX06 Corollary 3.19]. \( \square \)

**Remark 4.1.9.** We note that the construction depends only on \( L \) and we denote the obtained net also by \( \mathcal{A}_L \), the conformal net associated with the lattice \( L \).

In the rest of the section we give the construction in a more explicit manner. In particular the Hilbert space \( \mathcal{H}_L \) of \( \mathcal{A}_L \) can naturally be identified with \( L \) copies of the Hilbert space \( \mathcal{H}_F \) of \( \mathcal{A}_F \) (more precisely \( L^2(L, \mathcal{H}_F) \)) where \( F = L \otimes \mathbb{Z} \mathbb{R} \). This enables us to show that \( \mathcal{A}_L(I) \) is a crossed product of \( \mathcal{A}_F(I) \) with \( L \).

The identity component \( (\mathcal{L} T)^0 \) of \( \mathcal{L} T \) can be identified with \( H_F \times T \), where \( H_F = LF / F \times T \) is the Heisenberg group with multiplication law \((f, c_1)(g, c_2) = (f + g, e^{-i/2 \omega(f,g)}c_1c_2)\). The representation \( \pi_0 \) of \( LF \) is a representation of \( H_F \) because the constant loops lie in the kernel of the representation and it turns out to be the unique irreducible representation (cf. proof of 9.5.10 [PS86]) with positive energy. Let \( \bar{W} = H_F \times F \) (the idea is to add an operator \( Q \) which measures the charge). All irreducible representations of positive energy of \( \bar{W} \) are classified by a charge \( \alpha \in F \) and are of the form \((\pi_{\alpha_*}(\mathcal{H}_F), \alpha)\) given by \( \pi_{\alpha_*}(f, v) = e^{-i2x(\alpha, v)}\pi_0(f) \). As a set it is \( \mathcal{L} T \equiv (\mathcal{L} T)^0 \times L \) and it is shown in [PS86] that all irreducible representations of \( \mathcal{L} T \) of positive energy are given by points \( \lambda \in L^* / L \) and are acting on the Hilbert space

\[
\mathcal{H}_{(L,0)} = \bigoplus_{\alpha \in L} (\mathcal{H}_F)_{\alpha}.
\]

The Hilbert space \( \mathcal{H}_{(L,0)} \) on which \( \mathcal{L} T \) acts is graded by the lattice \( L \), and we call \( \alpha \) the charge of the subspace \( (\mathcal{H}_F)_\alpha \) of \( \mathcal{H}_{(L,0)} \). We define for \( \alpha \in L \) charge shift operators \( \Gamma_\alpha \) by \((\Gamma_\alpha x)_\beta = x_{\beta - \alpha}\) and introduce the unbounded charge operator \( Q \) satisfying \((Qx)_\alpha = \alpha x \). The \( \Gamma_\alpha \) does not fulfill exactly the commutation relations suitable for the representation of \( \mathcal{L} T \). But the commutation relations between different \( \Gamma_\alpha \) can be changed by a so called Klein transformation. Let \( \eta : L \times L \longrightarrow T \) be a bimultiplicative map (2-cocycle) and \( \tilde{\Gamma}_\alpha = \eta(-Q, \alpha)\Gamma_\alpha \) then we have:

**Lemma 4.1.10.** \( \alpha \mapsto \tilde{\Gamma}_\alpha \) defines a representation of the central extension \( \tilde{L} \) of \( L \) by the cocycle \( \eta(\cdot, \cdot) \).

**Proof.** We note that \( \tilde{L} = L \times T \) with multiplication law

\[
(\alpha, c)(\beta, d) = (\alpha + \beta, \eta(\alpha, \beta)cd)
\]
and the representation is obtained by applying \((\alpha, c) \mapsto c \Gamma_\alpha\). Indeed we calculate
\[
\Gamma_\alpha \Gamma_\beta = \eta(-Q, \alpha) \eta(-Q, \beta) \eta(-Q, \alpha + \beta) \Gamma_{\alpha + \beta} = \eta(\alpha, \beta) \eta(-Q, \alpha + \beta) \Gamma_{\alpha + \beta} = \eta(\alpha, \beta) \Gamma_{\alpha + \beta}.
\]

We choose \(\eta(\alpha, \beta) = c(e^{i\alpha}, e^{i\beta})\) where \(t_\alpha(\theta) = \alpha \cdot \theta\) and get a representation of \(\{(e^{i\alpha}, c) : \alpha \in L\} \subset \mathcal{L}T\) by \((e^{i\alpha}, c) \mapsto c \cdot \tilde{\Gamma}_\alpha\).

**Proposition 4.1.11.** The vacuum representation of \(\mathcal{L}T\) acts by the above construction irreducible on
\[
\mathcal{H}_L := \mathcal{H}_{(L,0)} \equiv \bigoplus_{\alpha \in L} \mathcal{H}_F(\alpha)
\]
i.e. the local net \(\mathcal{A}_L\) acts on \(\mathcal{H}_L\).

**Proof.** Let \(e^{i\gamma}, e^{i\varphi} \in \mathcal{L}T\). We note first that for \(f = f_\Delta + f_0 + f_1\), with \(f_\Delta(\theta) = \Delta_f \cdot \theta\) and \(f_0\) zero-mode like before, we have \(e^{i\gamma} = ke^{if_0}e^{i\theta}e^{if_\Delta}\) with an irreducible phase \(k = e^{i\gamma/f(2\pi, \Delta_f)} \in \mathbb{T}\).

We claim that
\[
\pi'(e^{i\gamma}) = e^{-i(f_0, Q)} W(f_1) \Gamma_{\Delta_f}
\]
defines a projective representation of \(\mathcal{L}T\) with a to \((\cdot, \cdot, \cdot)\) equivalent cocycle \(c'(\cdot, \cdot, \cdot)\). Then there exists a coboundary \(b_\theta(\cdot, \cdot, \cdot)\) like in (4.5) with \(c(f, g) = b_\theta(f, g)c'(f, g)\) and \(\pi(f) = h(f)\pi'(f)\) is the wanted representation.

We can write \(\mathcal{L}T = \mathcal{L}T_0 \times L\) where \(\mathcal{L}T_0\) is the connected component of the identity and \(\alpha \in L\) is identified with the loop \(t_\alpha(\theta) = \alpha \cdot \theta\). The cocycles restricted to \(L\) are equivalent by Lemma 4.1.10. Further the Weyl relations give exactly the relations of the cocycle \((\cdot, \cdot, \cdot)\), namely
\[
\pi(e^{i(f_0 + f_1)})\pi(e^{i(g_0 + g_1)}) = e^{-i(f_0, Q)} W(f_1) e^{-i(g_0, Q)} W(g_1) = e^{-i(f_0 + g_0, Q)} e^{i/2 \int \langle f_0, g_1 \rangle} W(f_1) + g_1) = e^{i/2 \int \langle f_1, g_1 \rangle} \pi(e^{i(f_0 + g_0, f_1 + g_1)}) = e^{c(f_0 + f_1, g_0 + f_1)} \pi(e^{i(f_0 + g_0, f_1 + g_1)}),
\]
so the cocycles restricted to \(\mathcal{L}T_0\) are also equal. By Lemma 4.1.4 it is sufficient to check that the pairwise commutation relations of \(\pi'(e^{i\gamma})\) and \(\pi'(e^{i\varphi})\) equal the one of \(\mathcal{L}T\) given in Lemma 4.1.6, indeed
\[
\pi(e^{i\gamma})\pi(e^{i(g_0 + g_1)})\pi(e^{i\varphi}) = \Gamma_{\Delta_f} e^{-i(g_0, Q)} W(g_1) \Gamma_{\Delta_f} = e^{-i(g_0, Q - \Delta_f)} W(g_1) = e^{i(\Delta_f - g_0)} \pi(e^{i(g_0, Q + g_1)}).
\]

**Proposition 4.1.12.** The local algebras \(\mathcal{A}_L(\mathbb{I})\) are given by a crossed product of \(\mathcal{A}_F(\mathbb{I})\) with \(L\).

**Proof.** Let \(I\) be a proper interval and \(y \in \mathbb{S}_1 \setminus \mathbb{T}\). The local loop group \(\mathcal{L}I\) is generated by loops \(e^{i\gamma}\) with \(f(x) \in 2\pi\mathbb{Q}\) for \(x \not\in I\). We note that \(\mathcal{L}IT = (\mathcal{L}IT_0) \times L\) as a set, where \((\mathcal{L}IT_0)\) is the connected component of the identity consisting of loops \(e^{i\gamma}\) with \(\Delta_f = 0\) and \(L\) is identified with \(\{e^{i\alpha} : \alpha \in L\}\), where \(t_\alpha\) are functions like above with \(\Delta_{t_\alpha} = \alpha\). We choose a basis \(\{\alpha_i\}\) of \(L\) and some smooth "step function" \(M : \mathbb{R} \rightarrow \mathbb{R}\) with \(M(\theta + 2\pi) = M(\theta)\) and with \(\Delta_M = 1\), such that for \(x \not\in I\) it is
\( M(x) \in \mathbb{Z} \) and therefore \( m(x) := M'(x) = 0 \). The loop \( e^{iM_a} \) has winding number \( \alpha_i \) and implements an automorphism \( \beta_i \) of \( \pi((\mathcal{L}T)_0)^\prime \)

\[
\pi(e^{i\beta_i}) = e^{-i(\alpha_i g_0)} W([f]) =: \tilde{W}(f)
\]

\[
\beta_i := \text{Ad} \pi(e^{iM_a})
\]

\[
\beta_i(\tilde{W}(f)) = e^{i \int (f, \text{max})} \tilde{W}(f),
\]

which defines an automorphic action \( \beta \) of \( L \) on the algebra \( \pi((\mathcal{L}T)_0)^\prime \). We note that with the notation from above \( \mathcal{H}_F \equiv (\mathcal{H}_F)_0 = \pi((\mathcal{L}T)_0) \supseteq \mathcal{H}_{(L,0)} \) and denote by \( \pi_F : (\mathcal{L}T)_0 \to \mathcal{U}(\mathcal{H}_F)_0 \) the representation of \( (\mathcal{L}T)_0 \) on \( (\mathcal{H}_F)_0 \) obtained by restriction of \( \pi \). By construction we get \( \mathcal{A}_F(I) = \pi_F((\mathcal{L}T)_0)^\prime \). This is the vacuum representation and it is \( W([f]) = \tilde{W}(f) \). Finally we can see \( \beta_i \) as an automorphism of \( \mathcal{A}_F(I) \)

\[
\beta_i(W([f])) = e^{i \int (f - f(y), \text{max})} W([f])
\]

and it is clear that \( \pi((\mathcal{L}T)^\prime_0) = \mathcal{A}_F(I) \rtimes_{\beta} L \), where the action is free and faithful. \( \square \)

**Remark 4.1.13.** By construction we have that \( \mathcal{A}_{L \rtimes Q} \equiv \mathcal{A}_L \otimes \mathcal{A}_Q \).

The adjoint action of a (localized) loop \( e^{i\beta} \) with \( \Delta_f = \lambda \in L^* \) gives a localized endomorphism of \( \mathcal{A}_L \) which belongs to the sector \([\lambda] \in L^*/L \). The conformal spin is well known to be \( e^{i\pi(\lambda, \lambda)} \).

**Proposition 4.1.14.** Let \( L \subset Q \) be two even lattices of the same rank \( n \). Then the local conformal net \( \mathcal{A}_Q \) is the simple current extension (see Subsection 3.2.4) of \( \mathcal{A}_L \) by the subgroup \( Q/L \) by the group \( L^*/L \) of all sectors of \( \mathcal{A}_L \).

**Proof.** By construction it is \( \mathcal{A}_L \subset \mathcal{A}_Q \) and \( \mathcal{H}_L \subset \mathcal{H}_Q \). Let us denote by \( \mathcal{B} \) the net obtained by the simple current extension by \( Q/L \), which is the crossed product with automorphisms given by the adjoint action by loops \( e^{i\beta} \) with \( \Delta_f \in Q/L \). So clearly we can see \( \mathcal{B} \) as conformal subnet of \( \mathcal{A}_Q \) and they coincide because \( \mathcal{B}(I)\Omega = \mathcal{H}_Q \). \( \square \)

**Remark 4.1.15.** In [KL06] another construction of lattice models is give, which starts with a conformal net \( \mathcal{A} \), which is the simple current extension of the dimension 1 sector of Vir_{c=1/2} \otimes \text{Vir}_{c=1/2}. In [KL06] Remark 2.3 the authors conjecture that \( \mathcal{A} \) is a Buchholz–Mack–Todorov extension, namely the one with \( g = 2 \) which is in our language the conformal net \( \mathcal{A}_{2\mathbb{Z}} \), where \( 2\mathbb{Z} \) is the lattice with \( \langle \alpha, \beta \rangle = \alpha \beta \). We will show in Proposition 7.2.3 that this conjecture is true. They take even lattices \( L \supseteq (2\mathbb{Z})^n \) (coming from codes) and take the simple current extension by the group \( L/(2\mathbb{Z})^n \subset (1/2\mathbb{Z})^n/(2\mathbb{Z})^n \) of the net \( \mathcal{A}_{2\mathbb{Z}} \), which is under then isomorphic to \( \mathcal{A}_{2\mathbb{Z}} \), using Remark 4.1.13. By Proposition 4.1.14 the extended net \( \mathcal{A}_{2\mathbb{Z}} \) is isomorphic to our net \( \mathcal{A}_L \) and for this class of lattices the two constructions give isomorphic nets.

### 4.1.4. Loop group models of simply laced groups at level 1.

We show the relation of the lattice model associated with the root lattice \( L \) of simply laced group \( G \) to the level 1 representation of the loop group of \( LG \) [PS86, Seg81, Sta95].

Let \( G \) be a compact, connected, simply connected, simply laced Lie group with maximal torus \( T \). Simply laced means that there is an invariant inner product on its Lie algebra \( g \) for which all roots have the same length or equivalently the Weyl group of \( G \) acts transitively on the roots. By [GF93, Theorem 3.12] the vacuum positive energy representation \( \pi \) of \( LG \) at level \( k \) gives rise to a conformal net denoted by \( \mathcal{A}_{G_k} \), the loop group net of \( G \) at level \( k \), defined by \( \mathcal{A}_{G_k}(I) = \pi(L_k G)^\prime \).

Let \( \mathfrak{t} \) be the Lie algebra of \( T \) and let us identify

\[
t/2\pi L \leftrightarrow T \subset G, [t] \leftrightarrow e^t.
\]
The roots of $G$ are linear maps $\alpha : t \mapsto \mathbb{R}$. For each $\alpha$ we define a $h_\alpha \in t$ such that $\alpha(t) = \langle h_\alpha, t \rangle$ for $t \in t$ where $\langle \cdot, \cdot \rangle$ is the Cartan–Killing form which we can assume to be normalized such that $\langle h_\alpha, h_\alpha \rangle = 2$. This can always be realized, due to $G$ being simply laced. In the case $\text{SU}(N)$ and $\text{Spin}(2N)$ it is given explicitly by $\langle x, y \rangle = -\text{tr}(xy)$ and $\langle x, y \rangle = -1/2 \text{tr}(xy)$, respectively. It is well known that $h_\alpha \in L$ and that the set of the $h_\alpha$ with $\alpha$ a root coincide with the $x \in L$ such that $\langle x, x \rangle = 2$.

By abuse of notation we identify $\alpha$ with $h_\alpha \in L$, i.e. $\langle \alpha, \beta \rangle \equiv \langle h_\alpha, h_\beta \rangle$. We note the missing $i$ in the exp map due to the conventions $t^* = -t$ for $t \in t$, i.e. by identifying $F = -it$ we get the relation to the former notation.

Let $\pi$ be a positive energy representation of $LT$ with cocycle (4.6), where $T$ is the torus associated with $L$ but also a maximal torus of $G$ by the above discussion. We note that the cocycle of the level 1 representation of $LG$ restricted to $LT$ is (equivalent to) our cocycle (4.6) by [PS86, Proposition 4.8.3]. More remarkable is the following result by Segal [Seg81], stating that the representation of $LT$ extends to $LG$. This is mainly achieved by taking a limit of loops with winding number $\Delta_f = \alpha$, and building so called “vertex” or “blib” operators which turn out to generate—together with the generators of loops with trivial winding number—a representation of the polynomial algebra $L_{\text{alg}}$, which is then exponentiated.

**Proposition 4.1.16** ([Seg81, Proposition 4.4]). Let $G$ be a compact, connected, simply connected, simply laced Lie group with maximal torus $T$. If $\pi$ is a positive energy projective representation of $LT$ with cocycle above, then the action of $LT$ extends canonically to an action of $LG$.

Now we want to apply this result to show that certain loop group nets at level 1 are a special case of the conformal nets associated with lattices. The analog of the following result is well known in the theory of vertex operator algebras under the name **Frenkel–Kac–Segal** construction.

**Proposition 4.1.17** (Algebraic version of the Frenkel–Kac–Segal construction). Let $G$ be a compact, simple, connected, simply connected and simply laced Lie group and $L$ its root lattice as above. Then the conformal net $A_L$ is equivalent to the loop group net $A_{G,1}$ at level 1 associated with $LG$. In particular $A_{G,1}$ is completely rational and has $\mu$-index $\mu = \vert L^*/L \vert$.

The case $G = \text{SU}(N)$ is stated in [Xu09, 3.1.1] and the general case in [Sta95, p. 37/38]. In principle we could try to use the result of Segal to directly prove the proposition, but locality of the constructed exponentiated currents is not clear and has to be checked. Therefore we give a more indirect proof using an operator algebraic argument.

**Proof.** Let $\pi$ be the vacuum positive energy representation at level 1 of $LG$ and by Proposition 4.1.16 it can be assumed to act on the Hilbert space $\mathcal{H}_L$. We see $\pi$ as a representation of the central extension $\mathcal{L}G$. It is $\mathcal{L}T \subset \mathcal{L}G$ and in particular for every $I \in \mathcal{I}$ also $\mathcal{L}_I T \subset \mathcal{L}_I G$. This implies that $A_L(I) \equiv \pi(\mathcal{L}_I T)^\prime$ is a conformal subnet of $A_{G,1}(I) = \pi(\mathcal{L}_I G)^\prime$. Because $\mathcal{A}_{\overline{I}}(\overline{I}) \overline{\Omega} = \mathcal{H}_L$ by Lemma 3.2.14 the two nets $A_L$ and $A_{G,1}$ have to coincide. \hfill $\square$

**Example.** The simple, simply laced groups correspond to the Dynkin diagrams of type $A$, $D$ and $E$ (see Figure 1), namely $\text{SU}(n + 1)$ for $A_n$ with $n \geq 1$, $\text{Spin}(2n)$ for $D_n$ with $n \geq 4$ and in the exceptional case the compact, simply connected forms of $E_6, E_7, E_8$. The level 1 loop group nets of these groups are therefore given by lattice models of their root lattice $L$, which is characterized by the basis $\{ \alpha_1, \ldots, \alpha_n \}$ with $n$ the rank of $L$ and $\alpha_i$ represents the $i$-th vertex of the Dynkin diagram. The
inner product is specified by the Cartan matrix $(C_{ij})$ via

$$\langle \alpha_i, \alpha_j \rangle = C_{ij} = \begin{cases} 2 & i = j \\ -1 & i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

4.1.5. Local nets of standard subspaces on Minkowski half-plane. As an intermediate step we built up local time-translation covariant nets of standard subspaces related with the local Möbius covariant nets of standard subspaces $H_F$ from Proposition 4.1.2 using the semigroup $\mathcal{E}(H_F(0, \infty))$.

**Definition 4.1.18.** By a local, time-translation covariant net of standard subspaces on $M_+$ on a Hilbert space $\mathcal{H}$ we mean a family $\{K(O)\}_{O \in K_+}$ of standard subspaces of a Hilbert space $\mathcal{H}$ which fulfills:

A. **Isotony.** $O_1 \subset O_2$ implies $K(O_1) \subset K(O_2)$.

B. **Locality.** If $O_1, O_2 \in K_+$ are space-like separated then $K(O_1) \subset K(O_2)'$.

C. **Time-translation covariance.** There is a strongly continuous one-parameter group $U(t) = e^{itP}$ on $\mathcal{H}$ with positive generator $P$, such that

$$U(t)K(O) = K(O), \quad O \in K_+$$

where $O_t = O + (t, 0)$ is the in time-direction shifted double cone.

**Remark 4.1.19.** Let $F$ be an Euclidean space and $F_\mathbb{C} = F \otimes \mathbb{C}$ its complexification with canonical complex conjugation $x \mapsto \bar{x}$. We denote by $S_F$ the space of all complex Borel functions $\varphi : \mathbb{R} \to \mathcal{B}(F_\mathbb{C})$ which are boundary values of a bounded analytic function $\mathbb{R} + i\mathbb{R}_+ \to \mathcal{B}(H)$, i.e. for $x, y \in F_\mathbb{C}$ the function $p \mapsto (x, \varphi(p)y)$ is an analytic Borel function $\mathbb{R} + i\mathbb{R}_+ \to \mathbb{C}$ such that $\varphi$ is symmetric and inner, i.e. that $\varphi(p) = \overline{\varphi(-p)}$ and $\varphi(p) \in \mathcal{U}(F_\mathbb{C})$ for almost all $p > 0$, respectively.

We note that with $n = \dim F$ the $S_F$ space is naturally isomorphic to $S^{1,n}$ defined in Section 2.3.

**Theorem 4.1.20.** We take the standard subspace $H_F := H_F(0, \infty)$ and $T(t) = U(\tau(t))$ the one-parameter group of translation. Let $\mathcal{H}_{0,F} = \mathcal{H}_0 \otimes \mathbb{R} F \cong \bigoplus_{i=1}^n \mathcal{H}_0$ from Proposition 4.1.2, which decompose into $n$ copies of the irreducible standard pair $(H_0, T_0)$, i.e. $H_F = \mathcal{H}_0 \otimes \mathbb{R} F \cong \bigoplus_{i=1}^n H_0$. Then $\mathcal{E}(H_F, T)$ can be identified with $S_F$ by Theorem 2.3.10.

**Theorem 4.1.21.** Let $H$ be a local Möbius covariant net of standard subspaces, then for each $V \in \mathcal{E}(H(0, \infty), T)$, with $T(t) = U(\tau(t))$ the one-parameter group of translations, there is local, time-translation covariant net of standard subspaces on $M_+$ given by

$$K_V : O \equiv I_1 \times I_2 \mapsto K_V(O) := H(I_1) + VH(I_2).$$

Proof. Isotony is obvious. Locality is shown like in [LW11] and follows from $V \in \mathcal{E}(H(0, \infty))$. But then we have also standardness, namely $K(O)$ is cyclic because $H(I_1)$ is already cyclic and separating because $K(O)'$ contains $H(I_1')$ where $I_1'$ is the left component of the two piece complement of $I_1$. Time-translation covariance holds because $V$ commutes with $T$. \qed
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Corollary 4.1.22. Let \((F, \langle \cdot, \cdot \rangle)\) be a real \(n\)-dimensional Euclidean space and \(H_F\) the net of standard subspaces from Proposition 4.1.2. Then for each \(V \in \mathcal{E}(H_F(0, \infty), T)\), i.e. each element in \(S_F\) as described in Remark 4.1.20, there is a local, time-translation covariant net of standard subspaces on \(M_+\).

4.2. Boundary quantum field theory – nets on Minkowski half-plane

4.2.1. Second quantization boundary nets. Let \((F, \langle \cdot, \cdot \rangle)\) be a \(n\)-dimensional Euclidean space. For the net \(A_F\) of Abelian currents constructed in Section 4.1.2 we know all \(V = \Gamma(V_0) \in \mathcal{E}(A_F)\) which are second quantization unitaries by the following theorem.

Theorem 4.2.1 (cf. [LW11, Theorem 3.6]). \(V = \Gamma(V_0) \in \mathcal{E}(A_F)\) if and only if \(V_0 = \varphi(P_0)\) with \(\varphi \in S_F\).

Remark 4.2.2. These models are exactly the second quantization of the models constructed in Section 4.1.5.

The next task is to find which \(V \in \mathcal{E}(A_F)\) extend to \(\tilde{V} \in \mathcal{E}(A_L)\) for an even integral lattice \(L \subset F\).

4.2.2. Semigroup for subnets. If we have a conformal net \(B\) with conformal subnet \(A\) and \(V \in \mathcal{E}(B)\) the question arises when \(V\) restricts to an element in \(\mathcal{E}(A)\).

Lemma 4.2.3. Let \(\Omega \in \mathcal{H}\) be a cyclic and separating vector for the von Neumann factor \(B \subset B(\mathcal{H})\) and separating for the subfactor \(A \subset B\) and we assume there is a conditional expectation \(E_A : B \to A\) which leaves the state \(\phi_\Omega = \langle \Omega, \cdot \Omega \rangle\) invariant. Let \(V \in \mathcal{U}(\mathcal{H})\) with \(V\Omega = \Omega\) and \(VBV^* \subset B\). Then the following is equivalent:

1. \(V\) commutes with the Jones projection \(e_A\).
2. \(E_A\) and \(Ad\) commute, i.e. \(E_A(VbV^*) = VE_A(b)V^*\) for all \(b \in B\).

Proof. By definition we have \(E_A(b)e_A = e_Abe_A\) for all \(b \in B\). Let us assume \([e_A, V] = 0\), then

\[
VE_A(b)V^*\Omega = VE_A(b)\Omega \\
= VE_A(b)e_A\Omega \\
= Ve_Abe_AV^*\Omega \\
= e_AVbV^*e_A\Omega \\
= E_A(VbV^*)e_A\Omega \\
= E_A(VbV^*)\Omega
\]

and by the separating property \([E_A, Ad\] = 0 follows. On the other hand, let us assume now \([E_A, Ad\] = 0. Then

\[
e_AVb\Omega = e_AVbV^*e_A\Omega \\
= E_A(Ad_V(b))e_A\Omega \\
= Ad_V(E_A(b))e_A\Omega \\
= VE_A(b)\Omega \\
= VE_A(b)e_A\Omega \\
= Ve_Abe_A\Omega \\
= Ve_Ab\Omega
\]

and cyclicity implies that \(e_A V = Ve_A\).
Proposition 4.2.4. Let $\mathcal{B}$ be a conformal net on $\mathcal{H}$ with vacuum $\Omega$; let $\mathcal{A}$ be conformal subnet of $\mathcal{B}$ and let $e$ be the projection on $\overline{\mathcal{A}(t)\Omega}$ for some $t \in \mathcal{I}$. Further let $V \in \mathcal{E}(\mathcal{B})$ and $\eta = \text{Ad} V$, then the following statements are equivalent:

1. $V \upharpoonright_{\mathcal{H}} \in \mathcal{E}(\mathcal{A}),$ regarding $\mathcal{A}$ as a conformal net on $e\mathcal{H}$.
2. For every $a \in \mathbb{R}$ it is $\eta(E_{a}(b)) = E_{a}(\eta(b))$ for all $b \in \mathcal{B}(a, \infty)$, where $E_{a}$ is the conditional expectation $\mathcal{B}(a, \infty) \to \mathcal{A}(a, \infty)$.
3. It is $\eta(E_{0}(b)) = E_{0}(\eta(b))$ for all $b \in \mathcal{B}(a, \infty)$, where $E_{0}$ is the conditional expectation $\mathcal{B}(0, \infty) \to \mathcal{A}(0, \infty)$.
4. $V$ commutes with the projection $e$.

Proof. The projection $e$ does not depend on $I$ and is the Jones projection of the inclusion $\mathcal{A}(I) \subset \mathcal{B}(I)$ for any $I \subset \mathcal{I}$. Let $V \in \mathcal{E}(\mathcal{B})$ such that $V \upharpoonright_{\mathcal{H}} \in \mathcal{E}(\mathcal{A})$. We show that (4) is true, namely for $a \in \mathcal{A}(0, \infty)$ using $V A(0, \infty) V^{*} \subset A(0, \infty)$ we compute

$$e V a \Omega = e V a V^{*} \Omega = V a V^{*} \Omega = V a \Omega = V e a \Omega,$$

thus by continuity $[V, e] \upharpoonright e \mathcal{H} = 0$. Let us write $\mathcal{H} = e \mathcal{H} \oplus e^{\perp} \mathcal{H}$ and let $V_{1} = e V e = V e$ and $V_{2} = e^{\perp} V e^{\perp}$; we write

$$V = \begin{pmatrix} V_{1} & X \\ 0 & V_{2} \end{pmatrix}.$$

Because $V$ and $V_{1}$ are unitaries on $\mathcal{H}$ and $e \mathcal{H}$, respectively, it follows that $X = 0$. We claim that also $[V, e] \upharpoonright e^{\perp} \mathcal{H} = 0$. By Lemma 4.2.3 $\eta = \text{Ad} V$ commutes with the conditional expectation $E : B(0, \infty) \to A(0, \infty)$, i.e. $E(V a V^{*}) = V E(a) V^{*}$ for $a \in \mathcal{A}(0, \infty)$. We claim that $\text{Ad} V$ is an endomorphism of $A(0, \infty)$, namely

$$V A(0, \infty) V^{*} = V E(B(0, \infty)) V^{*} \quad \subset E(B(0, \infty)) = A(0, \infty).$$

Since $V$ and $e$ commute $V \upharpoonright e \mathcal{H} = e V e$ is a unitary on $e \mathcal{H}$ and commutes with $T(t) \upharpoonright e \mathcal{H} = e T(t) e$, i.e. $V \upharpoonright e \mathcal{H} \in \mathcal{E}(\mathcal{A})$. 

4.2.3. Extensions for the crossed product with free Abelian groups. Let $\mathcal{M}$ be a type III factor. $\text{End}(\mathcal{M})$ is a tensor-$C^{*}$-category with objects $\rho \in \text{End}(\mathcal{M})$ normal endomorphisms of $\mathcal{M}$ and arrows $\text{Hom}_{\mathcal{M}}(\rho, \eta) = \{ t \in \mathcal{M} : \rho(x) = \eta(x) t \}$ for all $x \in \mathcal{M}$. For any $\rho \in \text{End}(\mathcal{M})$ we have $\text{Hom}(\rho, \rho) \ni \text{id}_{\rho} : = 1$. The tensor product is defined by the composition $\eta \otimes \rho := \eta \rho$ and for $f \in \text{Hom}_{\mathcal{M}}(\rho, \rho')$ and $g \in \text{Hom}_{\mathcal{M}}(\eta, \eta')$ it is $f \otimes g := f \rho(g) = \rho'(g) f \in \text{Hom}_{\mathcal{M}}(\rho \eta, \rho' \eta')$.

Let $L$ be a free Abelian group of rank $n$ with generators ($\mathbb{Z}$-basis) $\{ \alpha_{1}, \ldots, \alpha_{n} \}$ and let $\beta$ be a faithful action on a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H}_{0})$ with cyclic and separating vector $\Omega$. The action of $L$
is characterized by the action of the automorphisms $\beta_t := \beta_{\eta_t}$. Let $H = \oplus_{a \in L} H_a \supset H_0$. We assume $\beta_t$ to be implemented by unitaries $U_t$ mapping $H \to H_{a+\alpha}$. We note that

$$L \ni g = \sum_{i=1}^n g_i a_i \mapsto U_g = U_{g_1} \cdots U_{g_n} \quad g_i \in \mathbb{Z}$$

defines a projective representation of $L$ on $H$.

Let $N = M \supseteq L \subset \mathcal{B}(H)$ be the von Neumann algebra generated by $M$ and $\{U_i\}$ on $H$. We are interested in extension of endomorphisms of $M$ to endomorphisms of $N$. The following is in principal a generalization of [LW11, Proposition 3.8 and 3.9].

**Lemma 4.2.5.** Let $M$ as above and $N_0 \subset M$ the algebra finitely generated by $M$ and $\{U_a\}_{a \in L}$. Further let $R : L \to L$ be an automorphism of $L$ and for $i = 1, \ldots, n$ let $\tilde{\beta}_i := \beta_{R(a_i)}$ be automorphisms of $M$ having $\tilde{U}_i = U_{R(a_i)}$ as implementing unitaries. If there exist unitaries $\tilde{\zeta}_i \in \text{Hom}_\Lambda(N, \eta, \eta \circ \beta_i)$ satisfying

$$\tilde{z}_i \tilde{\beta}_i(z_j) = \tilde{z}_j \tilde{\beta}_j(z_i)$$

then $\eta$ extends to an endomorphism $\tilde{\eta}_0$ of $N_0$ characterized by $\tilde{\eta}_0(U_i) = \tilde{z}_i U_i$.

**Proof.** Each $g \in L$ can uniquely be written as $g = \sum_i g_i a_i$ with $g_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. Further we denote $U_g := U_{g_1} \cdots U_{g_n}$ which defines a projective representation of $L$. For finite non-zero $a_g \in M$ we define:

$$\tilde{\eta}_0 : \sum_a a_g U_g \mapsto \sum_a a_g (z_1 U_1) \eta_1 \cdots (z_n U_n) \eta_0 := \sum_a a_g \tilde{\zeta}_g U_g$$

which is well-defined because the action is faithful. It is easy to check that $\tilde{\eta}_0$ is an endomorphism if $\tilde{\zeta}_g \in M$ is a “cycology” (similar like in [Kaw01]) satisfying

$$\tilde{v}_g \tilde{\beta}_g(\eta(x)) = \eta(\beta_g(x)) \tilde{v}_g \quad x \in M$$

$$v_{g+h} = v_g \tilde{\beta}_h(v_h)$$

with $\tilde{\beta}_g = \text{Ad} \tilde{U}_g$. Indeed, using the tensor category calculus we write for arrows $t : \sigma \eta \to \eta \sigma'$ and $s : \rho \sigma \eta \to \eta \rho'$

$$s \circ t := (s \otimes \text{id}_{\sigma'})((\text{id}_{\rho} \otimes t) : \rho \sigma \eta \xrightarrow{\text{id}_{\rho} \otimes \text{id}_t} \rho \eta \sigma' \xrightarrow{s \otimes \text{id}_{\sigma'}} \eta \rho' \sigma')$$

for example $z_i \triangleright z_j := (z_i \otimes \text{id}_{\tilde{\beta}_j})(\text{id}_{\tilde{\beta}_j} \otimes z_j) \equiv z_i \tilde{\beta}_j(z_j)$. The condition $z_i \tilde{\beta}_j(z_j) = z_j \tilde{\beta}_i(z_i)$ reads $z_i \triangleright z_j = z_j \triangleright z_i$. Let us write $z_i^- = \tilde{\beta}_i^{-1}(z_i^\ast)$ in particular $z_i \triangleright z_i^- = z_i^- \triangleright z_i = 1$ and we have also

$$z_j \triangleright z_i^- = z_j \tilde{\beta}_j(\tilde{\beta}_i^{-1}(z_i^\ast)) = \tilde{\beta}_i^{-1}(\tilde{\beta}_i(z_j) \tilde{\beta}_j(z_i^\ast)) = \tilde{\beta}_i^{-1}(z_i \tilde{\beta}_j(z_i^\ast)) = \tilde{\beta}_i^{-1}(\tilde{\beta}_j(z_i^\ast)) = \tilde{\beta}_i^{-1}(z_i^- z_j) = z_i^- \triangleright z_j.$$

With this notation it is

$$v_g = z_1^{g \eta_1} \cdots z_n^{g \eta_n} \equiv \underbrace{z_1^{\pm} \cdots z_n^{\pm}}_{\pm \mathbb{R}_1\text{-times}} \cdots \underbrace{z_1^{\pm} \cdots z_n^{\pm}}_{\pm \mathbb{R}_n\text{-times}} \in \text{Hom}_\Lambda(\tilde{\beta}_g \eta, \eta \tilde{\beta}_g)$$
which does not depend on the order of the $z_i$ and $z_i^-$, so in particular

\[ v_g \beta_h(v_h) \equiv (v_g \otimes \text{id}_{\theta_h})(\text{id}_{\beta_g} \otimes v_h) = v_g \circ v_h = v_{g+h}. \]

\[ \Box \]

**Proposition 4.2.6.** Let $\eta$ be a $\phi_\Omega$-preserving endomorphism of $\mathcal{M}$. Under the hypothesis of Lemma 4.2.5 the endomorphism $\tilde{\eta}$ extends to a $\phi_\Omega$-preserving endomorphism $\tilde{\eta}$ of $\tilde{\mathcal{M}}$ characterized by $\tilde{\eta}(U_i) = z_i \tilde{U}_i$.

**Proof.** $\tilde{\eta}_0$ preserves the conditional expectation $\sum a_n U_n \mapsto a_0$ so it preserves the state $\phi_\Omega$ and $\Omega$ is cyclic for $\tilde{\eta}_0(\tilde{\mathcal{M}}_0)$, because the space $\tilde{\eta}_0(\tilde{\mathcal{M}}_0)\Omega$ contains $\mathcal{H}_0$ and is $U_i$ invariant. Finally, there exists a unitary $\tilde{V}$ with $\tilde{V} \chi \Omega = \tilde{\eta}_0(x)\Omega$ and $\tilde{\eta} = \text{Ad} \tilde{V}$ is the extension. $\Box$

Let us in the case $\eta \in \text{Aut}(\mathcal{M})$ and $v_x \in \mathbb{T}$ speak of an *internal symmetry*. In the special case $z_i = 1$ for $i = 1, \ldots, n$ it is $\hat{\beta}_i \eta = \eta \hat{\beta}_i$ and $\eta$ extends to a symmetry $\tilde{\eta}$ related to the automorphism $R$ of $L$; in the case $\eta = \text{id}_\mathcal{M}$ we speak of a *toral symmetry*. On the other hand let us in the case $R = \text{id}_L$ talk about charge preserving endomorphisms. A charge preserving internal symmetry is toral.

**Remark 4.2.7.** Let $\hat{\tau} : U_i \mapsto z_i U_i$ a charge preserving transformation which extends $\tau$ and $\hat{\sigma} : U_i \mapsto c_i \tilde{U}_i$ inner then $\tilde{\tau} \tilde{\sigma} : U_i \mapsto c_i z_i \tilde{U}_i$ defines an extension of $\tau \sigma$.

Given $\tilde{\eta}$ an extension of $\eta$ with $R$ and $\tilde{\sigma}$ an inner transformation with $\tilde{\sigma} : \tilde{U}_i \mapsto c_i U_i^*$ where $c_i \in \mathbb{T}$ extending some $\sigma \in \text{Aut}(\mathcal{M})$ having $R^{-1} : L \mapsto L$ as automorphism of $L$. Then $\tilde{\eta} \tilde{\sigma}$ is charge preserving.

**Remark 4.2.8.** In the case when $\eta$ has a charge preserving extension $\tilde{\eta} : U_i \mapsto z_i U_i$, let us look into the full monoidal subcategory $C$ generated by $\beta_i$ and $\eta$. Then $v_{\hat{\beta}_i} \in \text{Hom}_C(\beta_i, \eta \beta_i)$ is similar (the number of endomorphisms is not finite) two the half-braiding with respect to $\eta$ defined in [Izu00]. The condition $z_i \beta_i(z_j) = z_j \beta_j(z_i)$ reflects the fusion-braid equation.

We also have a converse of Proposition 4.2.6. namely that extensions of this form are given by $z_i$ like in Lemma 4.2.5.

**Proposition 4.2.9.** If $\tilde{\eta}$ is an endomorphism of $\tilde{\mathcal{M}}$ and restricts to an endomorphism of $\mathcal{M}$ and $\eta_0(e_\alpha) = e_{R(\alpha)}$ such that $U_i U_j = z_i z_j \tilde{U}_i \tilde{U}_j$ then there exists $z_i \in \text{Hom}_\mathcal{M}(\beta_i, \eta \beta_i)$ with $z_i \beta_i(z_j) = z_j \beta_j(z_i)$.

**Proof.** If $\tilde{\eta}$ restricts to an endomorphism of $\mathcal{M}$ this means it commutes with the Jones projection $e_0$ by Proposition 4.2.4 and $z_i := \tilde{\eta}(U_i) \tilde{U}_i^* \in \tilde{\mathcal{M}}$. But also $z_i \in \mathcal{M}$ because it commutes with $e_0$. Finally

\[ z_i \beta_i(\eta(x)) = \tilde{\eta}(U_i) \eta(x) \tilde{U}_i^* \]

\[ = \eta(\beta_i(x)) \tilde{\eta}(U_i) \tilde{U}_i^* \]

\[ = \eta(\beta_i(x)) z_i \tilde{\eta}(U_i) \tilde{U}_i^* \]

which completes the proof. $\Box$
Proposition 4.2.10. Let $A$ be conformal net and $A_{\text{ext}}$ a local extension by $\{\beta_i\}_{i=1}^{n}$ automorphisms of $A$ localized in $(0, \infty)$, such that $A_{\text{ext}}(0, \infty) = A(0, \infty) \otimes _\beta L$. Further let $V \in \mathcal{E}(A)$, $\eta = \Ad V$. Then there exists an extension $\tilde{V}$ of $V$, with $\tilde{V} \in \mathcal{E}(A_{\text{ext}})$ associated to an automorphism $R : \alpha_i \rightarrow \tilde{\alpha}_i$ of $L$ if and only if

1. there are $z_i \in \text{Hom}_{A(0,\infty)}(\hat{\beta}_i, \eta \beta_i)$ for $i = 1, \ldots, n$ such that $z_i \hat{\beta}_i(z_i) = z_i \tilde{\beta}_i(z_i)$,

2. and there are unitary one-parameter groups $u_i(t)$ with $\Ad u_i(t) \beta_i(\tau_i(x)) = \tau_i(\beta_i(x))$ for all $x \in A(0, \infty)$ with $u_i(t) \beta_i(u_i(t)) = u_i(t) \beta_i(u_i(t))$ which extends $\tau_i = \Ad T(t)$ from $A$ to $A_{\text{ext}}$ by $\tilde{\tau}_i(U_i) = u_i(t)U_i$ satisfying

\[ z_i \tilde{u}_i(t)^* = \eta(u_i(t)^*) \tau_i(z_i). \]

Proof. The first part follows directly by Proposition 4.2.6 and the converse by Proposition 4.2.9.

We note that $\tau_i$ is extended to $A_{\text{ext}}$ via a cocycle $u_i(t)$ namely $\tilde{\tau}_i(U_i) = u_i(t)U_i$. That $\tilde{\tau}_i$ commutes with $\tilde{\eta}$ means equality of

\[ \tilde{\tau}_i(\tilde{\eta}(U_i)) = \tilde{\tau}_i(z_i \tilde{U}_i) = \tau_i(z_i) \tilde{u}_i(t) \tilde{U}_i \]
\[ \tilde{\eta}(\tilde{\tau}_i(U_i)) = \tilde{\eta}(u_i(t)U_i) = \eta(u_i(t)) z_i \tilde{U}_i \]
which is equivalent with

\[ \eta(u_i(t)^*) \tau_i(z_i) = z_i \tilde{u}_i(t)^* . \]

□

4.2.4. Boundary nets associated with lattices. We investigate in the semigroup elements for the conformal nets associated with lattices and give corresponding boundary nets. Here is more convenient to use the real line picture by identifying $x = \tan \theta/2$. For $f \in L^2(\mathbb{R}, F)$, we denote its Fourier transform by $\hat{f} \in L^2(\mathbb{R}, F_0)$, namely:

\[ \hat{f}(p) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ipx} f(x) dx \]
\[ f(x) = \int_{\mathbb{R}} e^{ipx} \hat{f}(p) dp . \]

We note that in $\mathcal{H}_{0,F}$ the complex structure is given by $\overline{\mathcal{J}f}(p) = -i \text{sign}(p) \hat{f}(p)$ and the action of the translation by $T(t) = e^{itp}$ by

\[ T(t)f(p) = e^{-i \text{sign}(p)t|p|} \hat{f}(p) = e^{-ip} \hat{f}(p) . \]

and the sesquilinear form by

\[ \omega(f, g) := \text{Im}(f, g) = \frac{1}{2} \int_{\mathbb{R}} (f'(x), g(x)) \frac{dx}{2\pi} =: \frac{1}{2} \int \langle f', g \rangle . \]

The norm of $\mathcal{H}_{0,F}$ is $\|f\|_{\mathcal{H}_{0,F}} = \text{const.} \int_0^{\infty} \|\hat{f}(p)\|_{F_0} dp$ and we note that $f \in L^2(\mathbb{R}, F)$ is in $\mathcal{H}_{0,F}$ if the norm $\|f\|_{\mathcal{H}_{0,F}}$ is finite.

Let $L$ be an even lattice. We write it as a sum of irreducible components $L = L_1 \oplus \cdots \oplus L_k$ with $\langle L_i, L_j \rangle = 0$. We call a linear, isometric, isomorphic map $L \cong L$ an automorphism of $L$ and denote the set of automorphisms of $L$ by $\text{Aut} L$.

Definition 4.2.11. Let $R : L \rightarrow L$ be an automorphism of $L = L_1 \oplus \cdots \oplus L_k$ and $F = L \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $S_{L,R}$ the space of elements $\varphi \in S_F$, such that $\varphi(p)$ maps $\mathbb{C} \alpha_i$ to $\mathbb{C} \varphi \alpha_i$ for all $i = 1, \ldots, n$ and for almost all $p$. 

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Lemma 4.2.12. With this notation, there is a bijection between \( S_{L,R} \) and \( S^{\times k} \). It is given by \( S^{\times k} \ni (\varphi_1, \ldots, \varphi_k) \mapsto \varphi \) with
\[
\varphi(p)\alpha_i := \varphi_j(p)R\alpha_i, \quad \alpha_i \in L_j.
\]

Proof. We write \( \varphi(p)\alpha_i = c_i(p)R\alpha_i \) with \( c_i \in S \). That \( \varphi(p) \in \mathcal{U}(F_C) \) is equivalent with,
\( c_i(p)c_j(p)\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i, \alpha_j \rangle \) for all \( i, j \). This means \( c_i(p) = c_j(p) \) on each component.

Let us abbreviate \( \tilde{\alpha}_i = R\alpha_i \). We call \( \varphi \in S \) H{"o}lder continuous at 0, if
\[
p \mapsto \frac{|\varphi(p) - 1|^2}{|p|}
\]
is locally integrable at 0 and denote the subset of H{"o}lder continuous functions by \( S^{\text{H{"o}l}} \). In an obvious way we denote \( S^\text{Hö} \subset S^\text{Hö} \).

Lemma 4.2.13. Let \( L \) be an even lattice, \( R \in \text{Aut}(L) \) and \( F = L \otimes \mathbb{Z} \mathbb{C} \). Let \( \eta = \text{Ad} V \) with \( V \in \mathcal{E}(\mathcal{A}_F) \) related to \( \varphi \in S^\text{Hö} \subset S_F \) like in Theorem 4.2.1. Then there exist unitaries \( z_i \in \mathcal{A}_F(0, \infty) \), such that
\[
(1) \ z_i \in \text{Hom}_{\mathcal{A}_F(0, \infty)}(\beta_i, \eta_i, \eta_i), \\
(2) \ z_i\tilde{\beta}_i(z_j) = z_j\tilde{\beta}_j(z_i), \\
(3) \ z_i\tilde{u}_i(t)^* = \eta(u_i(t)^*) \tau_i(z_i).
\]

Proof. The automorphisms localized in \( (0, \infty) \) can be chosen to be
\[
\beta_i(W(f)) = e^{i\int_{\{p: m\alpha \leq 0\}} W(f)}
\]
with \( m : \mathbb{R} \longrightarrow \mathbb{R} \) a Schwartz function with support in \( (0, \infty) \) and \( \int_{\mathbb{R}} m(x) = 1 \). Let \( R \in \text{Aut}(L) \) and \( \varphi \in S^\text{Hö} \) with corresponding \( (\varphi_1, \ldots, \varphi_k) \in S^{\text{Hö} \times k} \) given by Lemma 4.2.12 and let us formally define \( m_i := \varphi_j(p)m\tilde{\alpha}_j \) for \( \alpha_i \in L_j \), more precisely
\[
m_i(x) = \int e^{ipx} \varphi_j(p)\hat{m}(p)\tilde{\alpha}_j dp \quad \alpha_i \in L_j.
\]
Then \( m\tilde{\alpha}_i - m_i \) has zero integral, because \( \hat{m}(0)\tilde{\alpha}_i = \hat{m}_i(0) \) and it is in \( H_F(0, \infty) \) because \( \varphi_j \in S \) is analytic in the upper strip using the Paley-Wiener theorem. Further its principal \( M_i - M\tilde{\alpha}_i \) has support in \( (0, \infty) \) and is in \( H_0,F \) because the norm
\[
\int_0^\infty \| M_i(p) - M(p)\tilde{\alpha}_i \|^2 \frac{dp}{\|p\|^2} < \infty
\]
is finite due to the H{"o}lder continuity. In particular, we get \( M_i - M\tilde{\alpha}_i \in H_F(0, \infty) \).

We claim that \( z_i := W(M_i - M\tilde{\alpha}_i) \in \mathcal{A}_F(0, \infty) \) defines unitaries with the wanted properties. Namely, to check (1) let us calculate
\[
\text{Ad} z_i(\beta_i(W(f)))) = \text{Ad} z_i(\hat{\beta}_i(W(V_0 f)))
\]
\[
= e^{i \int \langle m\tilde{\alpha}, V_0 f \rangle} \text{Ad} z_i(W(V_0 f))
\]
\[
= e^{i \int \langle m\tilde{\alpha}, V_0 f \rangle} e^{i \int \langle (M_i - M\tilde{\alpha}_i) V_0 f \rangle} W(V_0 f)
\]
\[
= e^{i \int \langle M_i, V_0 f \rangle} W(V_0 f)
\]
\[
= e^{i \int \langle m\tilde{\alpha}, f \rangle} W(V_0 f)
\]
\[
= e^{i \int \langle m\tilde{\alpha}, f \rangle} \eta(W(f))
\]
\[
= \eta(\beta_i(W(f))).
\]
To verify (2) we compute
\[
z\tilde{\beta}_i(z_j) = W(M_i - M\tilde{a}_i)\tilde{\beta}_j(W(M_j - M\alpha_j))
= e^{i\beta} \int \langle m\tilde{a}_i, M_j - M\alpha_j \rangle W(M_i - M\tilde{a}_i)W(M_j - M\alpha_j)
= e^{i\beta} \int \langle (M'_i + m\tilde{a}_i, M_j - M\alpha_j) \rangle W(M_i + M_j - M(\tilde{a}_i + \alpha_j))W(M_i + M_j - M(\tilde{a}_i + \alpha_j))
\]
which is symmetric under \(i \leftrightarrow j\) realizing that
\[
\langle M'_i + m\tilde{a}_i, M_j - M\tilde{a}_i \rangle = \langle M'_i, M_j \rangle - \langle M'_i, M\tilde{a}_i \rangle + \langle m\tilde{a}_i, M\alpha_j \rangle
\]
and noting that \(\langle M'_i, M\tilde{a}_j \rangle = \langle M_j, M\tilde{a}_i \rangle\). This is true, because if \(\langle \tilde{a}_i, \tilde{a}_j \rangle \neq 0\), then \(a_i\) and \(a_j\) are connected and sit in the same component, e.g. \(L_k\) and \(M_i\) and \(M_j\) are obtained both by multiplication with the same function \(\varphi_k\).

To show (3) we recall that \(u_i(t) = W((M_t - M)a_i)\) and \(\tilde{u}_i(t) = W((M_t - M)\tilde{a}_i)\)
\[
z\tilde{u}_i(t) = W(M_i - M\tilde{a}_i)W(M\tilde{a}_i - M\tilde{a}_i)
= e^{i/2} \int \langle (M'_i, M\tilde{a}_i) - \langle M'_i, M\tilde{a}_i \rangle + \langle m\tilde{a}_i, M\alpha_j \rangle \rangle W(M_i - M\tilde{a}_i)
= \eta(W(M\alpha_i - M\tilde{a}_i)) \tau_i(W(M_i - M\tilde{a}_i))
= \eta(u_i(t)) \tau_i(z_i),
\]
where \(M_t\) as before, \(M_i(x) := M(x - t)\) and \(M_{\tilde{a}}(x) := M_i(x - t)\).

Remark 4.2.14. In particular, the theorem shows that \(V \in \mathcal{E}(A_L)\) corresponding to \(S_{\text{Hol}}^{1,1}\) extends to \(\hat{V} \in \mathcal{E}(A_L)\) by Proposition 4.2.10. For the case of boundary nets we can choose \(R = \text{id}_L\) because the obtained \(\hat{V}\) just differ by internal symmetries.

Putting this together we have proven:

Proposition 4.2.15. Let \(L = L_1 + \cdots + L_k\) be an even integral lattice with \(k\) components and \(\varphi \in S_{\text{Hol}}^{1,1}\) corresponding to \((\varphi_1, \ldots, \varphi_k) \in S_{\text{Hol}}^{1,1}\), then there is a local, time-translation covariant net on Minkowski half-plane associated with the conformal net \(A_L\) and \(\varphi\).

Corollary 4.2.16. Let \(G\) be a compact, simple, connected, simply laced Lie group and \(\varphi \in S_{\text{Hol}}^{1,1}\), then there is a local, time-translation covariant net on Minkowski half-plane associated with the conformal net \(A_L\) (associated with the level 1 representation of \(L\)) and \(\varphi\). Further if \(G\) is just semisimple, i.e. it is a product of \(k\) simple groups of type A, D and E, then we obtain a such net for every \((\varphi_1, \ldots, \varphi_k) \in S_{\text{Hol}}^{1,1}\).

4.2.5. Further examples coming from the orbifold construction. In this section we want to give further examples of boundary nets coming from the loop group net of \(G = \text{Spin}(2n)\) at level 2 using the orbifold construction.
Definition 4.2.17. Let $A$ be a conformal net on $\mathcal{H}$. Let $V : G \to \mathcal{U}(\mathcal{H})$ be a faithful unitary representation of a finite group $G$ on $\mathcal{H}$. It is said that $G$ acts properly on the conformal net $A$ if the following conditions are satisfied:

1. for each $I \in \mathcal{I}$ and each $g \in G$, $\alpha_g(a) := V(g)aV(g)^* \in A(I)$ for all $a \in A(I)$,
2. for each $g \in G$ it is $V(g)\Omega = \Omega$.

Definition 4.2.18. Let $\mathcal{A}$ be a conformal net on $\mathcal{H}$ and let $V : G \to \mathcal{U}(\mathcal{H})$ be a proper action on $\mathcal{H}$. Let $\mathcal{H}_0 = \{x \in \mathcal{H} : V(g)x = x$ for all $g \in G\}$ and $P_0$ the projection on $\mathcal{H}_0$. Then $\mathcal{B}(I) = \{a \in A(I) : \text{Ad } V(g)a = a\}$ is a conformal subnet and we denote by $\mathcal{A}^\natural(I) = \mathcal{B}(I)P_0$ the conformal net on $\mathcal{H}_0$, called the orbifold net.

We use following result from [Xu00] to obtain loop group net of $L\text{Spin}(m)$ at level 2. By identifying $\mathbb{R}^{2m} \ni (x, y) \mapsto x + iy \in \mathbb{C}^m$ where $x, y$ are “column” vectors with $m$ real entries we have the natural inclusion $L\text{SU}(m)_1 \times L\text{SU}(1) \subset L\text{Spin}(2m)_1$ where $U(1)$ acts on $\mathbb{C}^m$ as scalars. A further natural inclusion is given by $L\text{Spin}(m)_2 \subset L\text{SU}(m)_1 \subset L\text{Spin}(2m)_1$. Let $K := (I_m, -I_m) \in \text{SO}(2m)$ which lifts to $\text{Spin}(2m)$. Then it is $KAK = A$ for $A \in \text{SU}(m)$ and $KAK = -A$ for $A \in \text{Spin}(2m)$. $K$ defines a proper action of $\mathbb{Z}_2$ on $A(\text{SU}(m), 1)$.

Proposition 4.2.19 (Lemma 5.1 [Xu00]). The loop group net $A_{(\text{Spin}(m), 2)}$ of $\text{Spin}(m)$ at level 2 is isomorphic to the $\mathbb{Z}_2$ orbifold net $A_{(\text{SU}(m)_1)}^{\mathbb{Z}_2}$ of the level 1 loop group net $A_{\text{SU}(m), 1}$ associated with $L\text{SU}(n)$, i.e. $A(\text{Spin}(m), 2) \cong A_{(\text{SU}(m)_1)}^{\mathbb{Z}_2} \cong A_{\text{Spin}(m), 1}^{\mathbb{Z}_2}$.

Proposition 4.2.20. Let $\varphi \in S(\mathcal{H})$, then there is a local, time translation covariant net on Minkowski half-plane associated with the loop group net $A_{\text{Spin}(m), 2}$ of $\text{Spin}(m)$ at level 2.

Proof. Let $L$ be the $A_{n-1}$ lattice, $F = L \otimes \mathbb{R}$ the associated Euclidean space and $\eta = \text{Ad } V$ the endomorphism $A_\varphi$ associated with the function $\varphi(p) \cdot 1_{n-1}$. We choose the special cocycle

$$z_i = e^{i \int (m_{\varphi_i} - M_{\alpha_i}) W(M_i - M_{\alpha_i})} = \beta_i^{1/2}(W(M_i - M_{\alpha_i}))$$

similar like before which differs from the $z_i$ just by a phase and denote $\tilde{\eta} = \text{Ad } \tilde{V}$ the endomorphism of $A_\varphi \cong A_{\text{SU}(n), 1}$ coming from the cocycle $z_i$. Let $\tau : W(f) \mapsto W(-f)$, $U_{\alpha} \mapsto c_{\alpha}U_{\alpha}^*$. This gives a proper action of $\mathbb{Z}_2$. Finally $\eta$ and $\tau$ commute

$$\eta(\tau(U_i)) = \eta(c_{\alpha_i}U_i^*)$$

$$= \beta_i^{1/2}(W(M_i - M_{\alpha_i}))c_{\alpha_i}U_i^*$$

$$= \tau(z_i c_{\alpha_i})U_i$$

$$= \tau(z_i)U_i$$

$$= \tau(\eta(U_i))$$

and $\tilde{\eta}$ restricts to an endomorphism $\tilde{\eta}^r = \text{Ad } \tilde{V}_1$ of $A_{\text{SU}(n), 1}^{\mathbb{Z}_2} = A_{\text{Spin}(n), 2}$, because $\tilde{\eta}$ commutes with $\tau$ and therefore with the Jones projection on the fixpoint. In particular, we have constructed $\tilde{V}_1 \in \mathcal{E}(A_{\text{Spin}(n), 2})$. \qed
CHAPTER 5

Longo–Witten unitaries from Boson–Fermion correspondence

A new family for the $U(1)$–current net was obtained by using Boson–Fermion correspondence. This result was obtained in [BT11] for construction of massless models considered in Chapter 9. We give here some generalization of the results in [BT11] for products and Fermionic extensions.

5.1. Boson–Fermion correspondence

Here we will use the notion of a Fermi net (Section 3.2.7). Let $\mathcal{F}$ be a Fermi net. The representation $U$ of $\text{M"ob}^2$ restricts to a projective unitary representation of $\text{M"ob}$ [CKL08]. We denote by $R(\theta) = U(R(\theta))$ the rotation subgroup, where $R(\theta)$ is the $4\pi$-periodic lift of the rotation from $\text{M"ob}$ to $\text{M"ob}^2$. Further we denote the subgroup of translations by $T(i) = U(\tau(i))$, where $\tau$ is the lift of the translation from $\text{M"ob}$ to $\text{M"ob}^2$.

A Longo–Witten endomorphism of a Fermi net $\mathcal{F}$ is an endomorphism of the algebra $\mathcal{F}(\mathbb{R}_+)$ implemented by a unitary $V$ which commutes with the translation $T(i)$.

Note that a Longo–Witten endomorphism is uniquely implemented up to scalar. Indeed, since it commutes with translation, $\text{Ad}_V$ is an endomorphism of $\mathcal{F}(\mathbb{R}_+)$ for any $t \in \mathbb{R}$. If there is another unitary $W$ which satisfies $\text{Ad}_W(x) = \text{Ad}_V(x)$ for any $x \in \mathcal{F}(\mathbb{R}_+ + t)$, $t \in \mathbb{R}$, then by the irreducibility $W^*V$ must be scalar. We can fix this scalar by asking $V\Omega = \Omega$ and we note that this coincides with the definition before, and we write also $V \in \mathcal{E}(\mathcal{F})$.

5.1.1. Subnets and the character argument. Let $\mathcal{F}$ be a Fermi (or conformal) net on $\mathcal{H}_F$. Another assignment $\mathcal{A}$ of von Neumann algebras $\{\mathcal{A}(I)\}_{I \in \mathcal{I}}$ on $\mathcal{H}_F$ is called like in the local case (cf. Subsection 3.2.3) a subnet of $\mathcal{F}$ if it satisfies isotony, M"obius covariance with respect to the same $U$ for $\mathcal{F}$ and it holds that $\mathcal{A}(I) \subset \mathcal{F}(I)$ for every interval $I \in \mathcal{I}$. We simply write $\mathcal{A} \subset \mathcal{F}$. In this case, let us denote $\mathcal{H}_\mathcal{A} = \bigvee_{I \in \mathcal{I}} \mathcal{A}(I)\mathcal{O}$.

Then it is immediate to see that $\mathcal{A}(I)$ and $U$ restrict to $\mathcal{H}_\mathcal{A}$, and by this restriction $\mathcal{A} \upharpoonright_{\mathcal{H}_\mathcal{A}}$ becomes a Fermi net.

For a Fermi net $\mathcal{F}$ on $\mathcal{S}$, a gauge automorphism $\alpha$ is a family of automorphisms $\{\alpha_I\}$ of local algebras which satisfies the consistency condition

$$\alpha_{I_2}|_{\mathcal{A}(I_1)} = \alpha_{I_1}$$

for $I_1 \subset I_2$.

If a gauge automorphism $\alpha$ preserves the vacuum state ($\Omega$, $\cdot$ $\Omega$), it is said to be an inner symmetry. An inner symmetry $\alpha$ can be unitarily implemented by the formula $V_{\alpha,x}\Omega = \alpha(x)\Omega$, where $x$ is an element of some local algebra $\mathcal{F}(I)$. We say that a compact group $G$ acts on the net $\mathcal{F}$ if there are automorphisms $\{\alpha_g\}_{g \in G}$ which satisfy the composition law when restricted to local algebras. The fixed point subnet with respect to this action of $G$ is the subnet defined by $\mathcal{F}^G(I) := \mathcal{F}(I)^G$.

Let $\mathcal{F}$ be a Fermi net and $\mathcal{A}$ be a subnet. Recall that, for a M"obius covariant Fermi net, the Bisognano-Wichmann property is automatic. As a consequence, for each interval there is a conditional expectation $E_I : \mathcal{F}(I) \to \mathcal{A}(I)$ which preserves the vacuum state ($\Omega$, $\cdot$ $\Omega$) and is implemented by the projection $P_{\mathcal{A}}$ onto $\mathcal{H}_\mathcal{A}$ (by Theorem 2.1.11).

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Consider the case where $A = F^G$ is the fixed point subnet with respect to an action $\alpha$ of a compact group $G$ by inner symmetry. Then we have a unitary representation $V_\alpha$ of $G$ on $H_F$. If we write the set of invariant vectors with respect to $V_\alpha$ by $H^G_F$, it holds that $H^G_F = H_A$. Indeed, the inclusion $H_A \subset H^G_F$ is obvious. On the other hand, for $x \in F(I)$, we have
\[
\left(\int_G \alpha(x) dg\right) \Omega = \int_G (V_\alpha(g) x \Omega) dg,
\]
which implies that any vector in $H^G_F$ can be approximated from $H_A$ by the Reeh–Schlieder property.

For the later use, we put here a simple observation.

**Proposition 5.1.1.** In the situation above, if a Longo–Witten endomorphism is implemented by $W$ and $W$ commutes with $V_\alpha$, then $Ad W$ restricts to a Longo–Witten endomorphism of the fixed point subnet $A$.

Proof. The unitary $W$ commutes with the projection $P_{A_0}$, hence also with the conditional expectation $E$ onto $A$, cf. Lemma 4.2.3.

Let $F$ be Fermi (or local) net on $H_F$. The Hilbert space $H_F$ is graded by the action of the rotation subgroup $R(\theta) = e^{i\theta L_0}$:
\[
H_F = \mathbb{C} \Omega \oplus \bigoplus_{r \in \frac{1}{2} \mathbb{N}} H_r = \bigoplus_{r \in \frac{1}{2} \mathbb{N}_0} H_r
\]
with $H_r = \{ \xi \in H_F : R(\theta) \xi = e^{i\theta} \xi \}$ and the sum only going over $\mathbb{N}_0$ for a local net. The **conformal character** of the net $F$ is given as a formal power series of $t = e^{-\beta}$:
\[
\text{tr}_{H_F}(e^{-\beta L_0}) = \sum_{r \in \frac{1}{2} \mathbb{N}_0} \dim H_r \cdot t^r.
\]
Let us assume that there is an action of $G = U(1)$ by inner symmetry. We denote by $V(\theta)$ the implementing unitary. Then $V$ and $U$ commute and $H_F$ is graded also by the gauge action $V(\theta) = e^{i\theta Q}$:
\[
H_F = \mathbb{C} \Omega_0 \oplus \bigoplus_{r \in \frac{1}{2} \mathbb{N}_0} H_r = \bigoplus_{q \in \mathbb{Z}} H_{r,q}, \quad \text{with} \quad H_{r,q} := \bigoplus_{r \in \frac{1}{2} \mathbb{N}_0} H_{r,q}
\]
and the character is given as a formal power series in $t = e^{-\beta}$ and $z = e^{-E}$:
\[
\text{tr}_{H_F}(e^{-\beta L_0 - EQ}) = \sum_{r \in \frac{1}{2} \mathbb{N}_0, q \in \mathbb{Z}} \dim H_{r,q} \cdot t^r z^q.
\]
Recall that it holds that $H^G_F = H_A$. The operator $Q$ acts by $0$ on $H^G_F$, hence we can obtain the conformal character of $A$ just by taking the coefficient of $z^0$ in $\text{tr}_{H_F}(e^{-\beta L_0 - EQ})$.

Later in this section we need to compare the size of two subnets. Let $A \subset B \subset F$ be an inclusion of three Fermi nets. If the conformal characters of $A$ and $B$ coincide, then this means that the subspaces $H_A$ and $H_B$ coincide, since we have already an inclusion $H_A \subset H_B$ and the coefficients of the conformal character are the dimensions of eigenspaces of $L_0$. This in turn implies that two subnets $A$ and $B$ are the same by the given argument in the proof of Lemma 3.2.14.
5.1.2. The U(1)-current net $\mathcal{A}_\mathbb{R}$ (revisited). Let $U_1$ be the irreducible unitary positive-energy representation of Möbius with lowest weight 1 on a Hilbert space denoted by $\mathcal{H}^1_{\text{Möb}}$, which can be identified with the one-particle space of the U(1)-current net and remember that there is an embedding $C^\infty(\mathbb{S}^1, \mathbb{R})/\mathbb{R} \to \mathcal{H}^1_{\text{Möb}}$.

For $I \in \mathcal{I}$ we denote by $H(I) \equiv H_R(I)$ the local Möbius covariant net of standard subspaces on $\mathcal{H}^1_{\text{Möb}}$. Like in Section 4.1.2 we obtain the U(1)-current net $\mathcal{A}_\mathbb{R}$ on $\mathcal{H}_{\mathcal{A}_\mathbb{R}} := e^{\mathcal{H}^1_{\text{Möb}}}$ with $\Omega_0 = \mathbb{R}$ by defining $\mathcal{A}_\mathbb{R}(I) := R(H(I))$ which is covariant with respect to $U(g) := e^{U(g)}$. For $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$ we consider a self-adjoint operator $J(f)$ given by the generator of the unitary one-parameter group $W(t \cdot f) = e^{itJ(f)}$ with $t \in \mathbb{R}$. This defines the usual current (field operator) smeared with the real test function $f$, which fulfills $J(f)\Omega_0 = f \in \mathcal{H}^1_{\text{Möb}}$

$$
[J(f), J(g)] = 2i\omega(f, g) = \sum_k k\hat{f}_k\hat{g}_{-k} = \frac{i}{2\pi} \int f dg.
$$

It can be extended to complex test functions via $J(f + ig) = J(f) + iJ(g)$, and one obtains the usual operator valued ($z$-picture) distribution $J(z)$ with the relations

$$
J(z) = \sum_{n \in \mathbb{Z}} \hat{f}_n J_n = \int_{\mathbb{S}^1} f(z)J(z) \frac{dz}{2\pi}, \quad J(z) = \sum_n J_n z^{-n-1}
$$

$$
[J_m, J_n] = m\delta_{m+n,0},
$$

where the modes $J_n = J(e^n)$ with $e^n(\theta) = e^{i\theta}$ satisfy $J_n\Omega_0 = 0$ for $n \geq 0$.

The space $\mathcal{H}_{\mathcal{A}_\mathbb{R}}$ is spanned by vectors of the form $\xi = J_{-n_1} \cdots J_{-n_k} \Omega_0$ with $0 < n_1 \leq \cdots \leq n_k$ with "energy" $N = \sum n_m$, i.e. $R(\theta)\xi = e^{iN\theta}\xi$. Therefore it is graded with respect to the rotations

$$
\mathcal{H}_{\mathcal{A}_\mathbb{R}} = \mathbb{C}\Omega_0 \oplus \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{\mathcal{A}_\mathbb{R}, n}
$$

$$
\mathcal{H}_{\mathcal{A}_\mathbb{R}, n} = \bigoplus_{k=1}^n \bigoplus_{0 < n_1 \leq \cdots \leq n_k \atop n_1 + \cdots + n_k = n} \mathbb{C} J_{-n_1} \cdots J_{-n_k} \Omega_0
$$

and $\dim \mathcal{H}_{\mathcal{A}_\mathbb{R}, n}$ is the number of partitions of $n$ elements, whose generating function $p(t)$ is the inverse of Euler’s function $\phi(t) = \prod_{k=1}^\infty (1 - t^k)$ and therefore the conformal character of the U(1)-current net is given by ($t = e^{-\beta}$): 

$$
\operatorname{tr}_{\mathcal{H}_{\mathcal{A}_\mathbb{R}}} (e^{-\beta L_0}) = \sum_{n=0} \dim \mathcal{H}_{\mathcal{A}_\mathbb{R}, n} \cdot r^n = \prod_{n \in \mathbb{N}} (1 - t^n)^{-1}
$$

(a conformal character is defined as a formal power series, but it is often convergent for $|t| < 1$ and here we used the formula $(1 - z)^{-1} = 1 + z + z^2 \cdots$). It will be convenient to use the real parametrization $x \in \mathbb{R} \equiv \mathbb{S}^1 \setminus \{-1\}$ of the cut circle and use the conventions

$$
f(s) = \int_{\mathbb{R}} e^{-ipx} \hat{f}(p) dp.
$$

By writing $f(s) = f_0(\theta(s))$ for $f_0 \in C^\infty(\mathbb{S}^1, \mathbb{R})$ where $\theta(s) = 2 \arctan(s)$, the space $\mathcal{H}_{\mathcal{A}_\mathbb{R}}$ above can be identified with the space $L^2(\mathbb{R}_+, dp)$ in which the space $S(\mathbb{R}, \mathbb{R})$ embeds by restriction of the Fourier transformation to $\mathbb{R}_+$. In other words $\mathcal{H}_{\mathcal{A}_\mathbb{R}}^1$ can be seen as the closure of the space $S(\mathbb{R}, \mathbb{R})$ with complex structure $J \hat{f}(p) = i \text{sign}(p) \hat{f}(p)$ and the scalar product and sesquilinear form given by:

$$
\langle f, g \rangle = \int_{\mathbb{R}_+} \hat{f}(-p) \hat{g}(p) dp, \quad \omega(f, g) = \frac{-i}{2} \int_{\mathbb{R}} \hat{f}(-p) \hat{g}(p) dp + \frac{1}{4\pi} \int_{\mathbb{R}} f(x)g'(x) dx.
$$
5. LONGO–WITTEN UNITARIES FROM BOSON–FERMION CORRESPONDENCE

Using the above identification we denote for $f \in S(\mathbb{R}, \mathbb{R})$ by $J(f)$ the smeared current with $J(f)\mathcal{Q}_0 = f \in \mathcal{H}^1_{\text{As}}$. In this parametrization commutation relations read:

$$[J(f), J(g)] = \frac{i}{2\pi} \int_\mathbb{R} f(x)g'(x)dx = \int_\mathbb{R} \hat{f}(-p)\hat{g}(p)pd\rho.$$

5.1.3. The free complex Fermion net $\text{Fer}_C$. We construct the net of the free complex Fermion on the circle, which can be seen as the chiral part of the net of the free massless Dirac (or complex) Fermion on two dimensional Minkowski space. The notations of this section are basically in accordance with [Was98], but we use a different convention of positive-energy, which leads to the conjugated complex structure. For giving a simple description of the one-particle space, we consider first the Hilbert space $L^2(S^1)$ and the Hardy space $H^2(S^1)$ (see Subsection 2.3.2). It holds that

$$H^2(S^1) = \{ f \in L^2(S^1) : \hat{f}_n = 0 \text{ for } n < 0 \},$$

where $\hat{f}_n$ is the $n$-th Fourier component of $f$. We denote the orthogonal projection onto $H^2(S^1)$ by $P$.

We remember that the group

$$\text{Möb}^{(2)} \equiv \text{SU}(1, 1) = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ \beta^* & \alpha^* \end{array} \right) \in \text{GL}(2, \mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1 \right\}$$

acts on the circle $S^1$ by $g \cdot z = \frac{az + \beta}{\bar{\alpha}z + \bar{\beta}}$. Further there is a unitary action of $\text{SU}(1, 1)$ on $L^2(S^1)$ by

$$(U(g)f)(z) := (V_g f)(z) = \frac{1}{-\beta z + \alpha} f(g^{-1} \cdot z).$$

One sees that the projection $P$ commutes with $V_g$, since $V_g f$ is still an analytic function for $|\alpha| > |\beta|$. Then one defines a new Hilbert space, the \textbf{one-particle space of the free complex Fermi net} by

$$\mathcal{H}^1_{\text{Fer}_C} = \overline{\text{PL}^2(S^1)} \oplus (1 - P)L^2(S^1)$$

namely, $\mathcal{H}^1_{\text{Fer}_C}$ is identical with $L^2(S^1)$ as a real linear space and the multiplication by $i$ is given by $-i(2P - 1)$, or in other words, by $-i$ on $\text{PL}^2(S^1)$ and $i$ on $(1 - P)L^2(S^1)$. Because $P$ and $U(g)$ commute, the action of $\text{SU}(1, 1)$ remains unitary on $\mathcal{H}^1_{\text{Fer}_C}$.

Then for $I \in \mathcal{I}$ one takes real Hilbert subspaces $K(I) := L^2(I)$ of $\mathcal{H}^1_{\text{Fer}_C}$. This subspaces turn out to be standard [Was98, Theorem (p, 497)]. If $I_1$ and $I_2$ are disjoint intervals, $K(I_1)$ is real orthogonal to $K(I_2)$, in other words $K(I_1) \subset K(I_2)^\perp$, where $K^\perp = \{ \xi \in \mathcal{H} : \text{Re}(\xi, K) = 0 \}$. It turns out that $I \mapsto K(I)$ is a twisted-local Möbius covariant net of standard subspaces.

The Fermionic second quantization is explained in Section 2.4.2. With this we define the net $\text{Fer}_C(I) := C(K(I)) = \{ ap(f), ap(f)^* : f \in L^2(I)^\prime \}$ (where here $ap(f) := a(Pf) + a(P^\perp f)^*$) on $\mathcal{H}_{\text{Fer}_C} = \Lambda(\mathcal{H}^1_{\text{Fer}_C}) \cong \Lambda(\text{PL}^2(S^1)) \otimes \Lambda(P^\perp L^2(S^1))$ which is isometric by definition and fulfills twisted duality, namely by Haag-Araki duality $\text{Fer}_C(I^\prime) = C(K(I)^\perp) = C(K(I))^{\perp} = \text{Fer}_C(I)^\perp$. In addition, the net $\text{Fer}_C$ is Möbius covariant. Indeed, we can take the representation $\Lambda(U(\cdot))$ by promoting the one-particle representation $U$ to the second quantization unitary. It is easy to see that the covariance of this net $\text{Fer}_C$ follows from the covariance of the net of standard spaces $K$. The representation $\Lambda(U)$ has positive energy since so does the representation $U$, and leaves invariant the vacuum vector $\Omega_0$ of the Fock space. Summing up, the net $\text{Fer}_C$ is a Fermi net (cf. [Was98]). This net is referred to as the \textbf{free complex Fermi net} on $S^1$. The scalar multiplication by a constant phase $e^{-i\theta}$ in the original structure of the one-particle space is still a unitary operator in the new structure. Its promotion by the second quantization $V(\theta)$ implements an action of $U(1)$ on $\text{Fer}_C$ by inner symmetry. This will be referred to as the $U(1)$-gauge action.
For $r \in \frac{1}{2} + \mathbb{Z}$ let $\psi_r = a_r(e_{-r-\frac{1}{2}})$ and $\bar{\psi}_r = a_r(e_{r-\frac{1}{2}})^*$ where $e_r \in L^2(S^1)$ with $e_r(\theta) = e^{2i\theta}$. The $\psi_r, \bar{\psi}_r$ are the modes of the free complex Fermion, namely

\[
\begin{align*}
\{\psi_n, \psi_m\} &= \{\bar{\psi}_m, \bar{\psi}_n\} = 0 \\
\{\bar{\psi}_n, \psi_m\} &= \delta_{m+n,0} \\
\psi_n^* &= \bar{\psi}_{-n}
\end{align*}
\]

and it holds that $\psi_r\Omega_0 = \bar{\psi}_r\Omega_0 = 0$ for $r \in \frac{1}{2} + \mathbb{N}_0$. Each of $\psi_r$ or $\bar{\psi}_r$ has norm 1 following from the commutation relation. We can introduce the usual fields smeared with test functions $f, g \in L^2(S^1)$ and it holds that

\[
\Psi(f) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} f_r \Psi_r = \int_{S^1} f(z) z^{-\frac{1}{2}} \Psi(z) \frac{dz}{2\pi i}, \quad \Psi(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \Psi_r z^{-r-\frac{1}{2}},
\]

\[
\bar{\psi}(f) = \bar{\psi}(f)^* = a_p(e_{-r-\frac{1}{2}}) \quad \text{for} \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \text{SU}(1,1).
\]

We note that vectors of the form

\[
\xi = \psi_{-r_1} \cdots \psi_{-r_k} \bar{\psi}_{-s_1} \cdots \bar{\psi}_{-s_s} \Omega_0,
\]

with $0 < r_1 < \cdots < r_k$ and $0 < s_1 < \cdots < s_s$ form a basis of $H_{\text{Fer}_c} = \Lambda(H_{\text{Fer}_c}^1)$ and that (5.8) is an eigenvector for the rotations, $R(\theta) \xi = e^{i\theta} \xi$ with $N = \sum_{j=1}^k r_j + \sum_{j=1}^s s_j$ and of the gauge action $V(\theta) \xi = e^{i\theta} \xi$. Each vector of this basis the $r$-th energy level can either be empty, be occupied by $\psi_r$, or $\bar{\psi}_r$ or occupied by both. The contribution of this level to the character $\text{tr}_{H_{\text{Fer}_c}}(e^{-\mu L_0 - E \Omega})$ is then $1$, $z^{r'}$, $z^{-1} r'$ or $t^{r'}$, respectively, where $t = e^{-\mu}$ and $z = e^{-E}$. By summing over all possibilities one gets that the character of $\text{Fer}_c$ is given by (cf. [Kac98 Reh]):

\[
\text{tr}_{H_{\text{Fer}_c}}(e^{-\mu L_0 - E \Omega}) = \text{tr}_{H_{\text{Fer}_c}}(t^{L_0} z^{Q}) = \prod_{r \in \mathbb{N}_0 + \frac{1}{2}} \left(1 + z^{r'} + z^{-1} t^{r'} + t^{2r'}\right)
\]

\[
= \prod_{r \in \mathbb{N}_0 + \frac{1}{2}} \left(1 + z^{r'}(1 + z^{-1} t^{r'})\right)
\]

\[
= p(t) \sum_{q \in \mathbb{Z}} z^{q} t^{q^2},
\]

where the last equality follows directly from the Jacobi triple product formula (see [Apo76, Theorem 14.6])

\[
\prod_{r \in \mathbb{N}} \left(1 + zw^{2r-1}\right) \left(1 + z^{-1} w^{2r-1}\right) (1 - w^{2r}) = \sum_{q \in \mathbb{Z}} z^q w^q
\]

by setting $2r - 1 = 2n$ and $t = w^2$. In particular, for the local net $\text{Fer}_c^{U(1)}$ the character is given by $\text{tr}_{H_{\text{Fer}_c}}(e^{-\mu L_0}) = p(t)$, since it is the fixed point with respect to the $U(1)$-gauge action and the conformal character is the coefficient of $z^0$. 

5.1.4. The double construction and real Fermion. Let us define $\mathcal{H}_1$ by taking the real Hilbert space obtained by the closure of the antiperiodic functions

$$\left\{ f : \mathbb{R} \to \mathbb{R} : f(\theta + 2\pi) = -f(\theta) \right\} \subset C^\infty(\mathbb{R}, \mathbb{R})$$

under the norm

$$\|f\|^2 = 2 \sum_{r \in \mathbb{N}_0 + \frac{1}{2}} |\hat{f}_r|^2$$

where the function is expanded as:

$$f(\theta) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \hat{f}_r e^{ir\theta}, \quad \hat{f}_{-r} = \overline{\hat{f}_r}.$$  

An orthogonal basis of this space is given by

$$\{ e_r : \theta \mapsto \cos(r\theta), \sigma_r : \theta \mapsto \sin(r\theta) : r \in \mathbb{N}_0 + \frac{1}{2} \}.$$

This space gets a complex Hilbert space denoted $\mathcal{H}_{\frac{1}{2}}$ with the complex structure $\mathcal{J} : \hat{f}_k \mapsto -i \text{sign}(k)\hat{f}_k$, which acts on the basis by

$$\mathcal{J} c_r = s_r, \quad \mathcal{J} s_r = -c_r, \quad r \in \mathbb{N}_0 + \frac{1}{2}$$

and it has an unitary action of the double cover of the Möbius group $\hat{\text{M"ob}} (2) \cong SU(1,1)$ by

$$(U(g) f)(z) = \frac{1}{|\alpha - \beta z|} f(g^{-1} z), \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1). \quad (5.9)$$

There is an alternative way to describe the space $\mathcal{H}_{\frac{1}{2}}$ starting with a complex structure and basis projection as described in Section 2.3.3 Therefore we start with $(L^2(\mathbb{S}^1), \Gamma)$ with the complex conjugation $\Gamma f = \overline{f}$ and the action of $\hat{\text{M"ob}}_2$ given by the same formula $^1$ (5.9). We define the basis projection

$$P_{\text{NS}} = \sum_{r \in \mathbb{N}_0 + \frac{1}{2}} \langle e_{-r} | e_{-r} \rangle$$

and get an isorphism between the Hilbert space $PL^2(\mathbb{S}^1)$ and $\mathcal{H}_{\frac{1}{2}}$ by

$$e_{-r} \mapsto 1/\sqrt{2}(c_r - \mathcal{J}s_r) = \sqrt{2}c_r, \quad (r \in \mathbb{N}_0 + \frac{1}{2})$$

where $e_r(e^{i\theta}) = e^{i\theta}$ with $\theta \in (-\pi, \pi)$. This exactly the correspondence between $(L^2(\mathbb{S}^1), \Gamma, P_{\text{NS}})$ and $\mathcal{H}_{\frac{1}{2}}$ given by Proposition 2.3.13. A net of standard subspaces can e.g. be defined by $K(I) = \text{Re} L^2(I) \subset \text{Re} L^2(\mathbb{S}^1) \cong PL^2(\mathbb{S}^1)$ or by the closure of anti-periodic real functions with support in $I$, or abstractly using modular localization. This net fulfills twisted Haag duality, i.e. $K(I') = iK(I)$.

**Proposition 5.1.2** (Double construction). We can identify the one particle space $\mathcal{H}^1 = L^2(\mathbb{S}^1)_{1-P}$ of $\text{Fer}_C$ with $\mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{1}{2}}$ as representations of $\hat{\text{M"ob}} (2)$. It can be choosen such that the standard subspace $K(I)$ is identified $K(I) \oplus K(I)$. The $U(1)$ action on $\mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{1}{2}}$

$$V(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

is given by the multiplication by $e^{i(2\theta + \beta)} = e^{i(1-2p)}$ on $\mathcal{H}^1$.

$^1$We should see this as a priori as a projective representation up to a factor $\pm 1$ which we can make into a true representation.
It is convenient to write formally \( \mathcal{H}_1 \oplus i\mathcal{H}_2 \), which reflects that the sum of two real Fermions is a complex Fermion. Then the identification is up to a character of the rotations, which makes the antiperiodic function periodic, given by the obvious identification.

Proof. We can identify as real Hilbert spaces

\[
M : \mathcal{H}_R \oplus i\mathcal{H}_R \to L^2(S^1)
\]

\[
f \oplus ig \mapsto e_{-\frac{1}{2}}(f + ig)
\]

namely \( \|c_r \oplus is_r\| = \|c_r\|^2 + \|s_r\|^2 = 1 = \|e_{r+\frac{1}{2}}\| = \|M(c_r \oplus is_r)\| \) and as complex spaces \( M(\mathcal{H}_R \oplus i\mathcal{H}_R) = \mathcal{H}_1 \), namely for example for \( r > 0 \)

\[
M(J \oplus \bar{J})(c_r \oplus \pm is_r) = M(s_r \oplus \mp ic_r)
\]

\[
= e_{-\frac{1}{2}}(s_r \mp ic_r)
\]

\[
= \mp ie_{r-\frac{1}{2}}
\]

\[
= J e_{\pm r-\frac{1}{2}}
\]

\[
= \bar{J} M(c_r \oplus \pm is_r).
\]

It is \( MV_g \oplus V_g = V_g M \), namely

\[
(M(V_g \oplus V_g)(f \oplus ih))(z) = z^{-\frac{1}{2}} \frac{1}{|\alpha - \beta z|} (f + ih)(g^{-1}.z)
\]

\[
= \frac{1}{(\alpha - \beta z)^2} \left( \frac{\alpha z - \beta}{-\beta + \alpha z} \right)^{\frac{1}{2}} (f + ih) \left( \frac{\alpha z - \beta}{-\beta + \alpha z} \right)
\]

\[
= \frac{1}{\alpha - \beta z} \frac{1}{(\alpha - \beta z)^2} M(f \oplus ih)(g^{-1}.z)
\]

\[
= V_g M(f \oplus ih)(z).
\]

It is easy to check that \( MV(\theta) = e^{i\theta} M \) with \( \theta \) the imaginary unit from \( L^2(S^1) \). Finally if \( f \oplus ig \) has support on \( I \) if and only if \( M(f \oplus ig) \) has support in \( I \). \( \square \)

We can define the net of a real Fermion simply by \( \text{Fer}_R(I) := C(K^1_2(I)) \) on \( \Lambda(\mathcal{H}_2) \). Then it holds: \( \text{Fer}_C \cong \text{Fer}_R \otimes \text{Fer}_R \), where \( \otimes \) is the twisted tensor product.

### 5.1.5. U(1)-current net as a subnet of \( \text{Fer}_C \)

In this section we use the well-known fact that the Wick product \( \tilde{\psi} \psi \) of the complex Fermion \( \psi \) equals the \( U(1) \)-current and give an analogue of the Boson–Fermion correspondence (see e.g. [Kac98, 5.2]) in the operator algebraic setting. Let us denote by \( \mathcal{D}_0 \) the subspace of \( \Lambda(\mathcal{H}_p^1) \) of vectors with finite energy:

\[
\mathcal{D}_0 := \text{span} \left\{ \psi_{-r_1} \cdots \psi_{-r_l} \tilde{\psi}_{-s_1} \cdots \tilde{\psi}_{-s_j} : k, l \in \mathbb{N}_0, r_i, s_j \in \mathbb{N} + \frac{1}{2} \right\}.
\]
Then we define the unbounded operators on the domain $\mathcal{D}_0$:

$$J_n = \sum_{r+s=n} \bar{\psi}_r \psi_s : = \sum_{r<0} \bar{\psi}_r \psi_{n-r} - \sum_{r>0} \psi_{n-r} \bar{\psi}_r$$

$$= \sum_r (\bar{\psi}_r \psi_{n-r} - (\Omega_0, \bar{\psi}_r \psi_{n-r} \Omega_0))$$

with $r,s \in \frac{1}{2} + \mathbb{Z}$. Note that any vector in $\mathcal{D}_0$ is annihilated by $\psi_r$ for sufficiently large $r$, thus the action of $J_n$ on such a vector can be defined and remains in $\mathcal{D}_0$. In particular, we have $J_n \Omega_0 = 0$ for $n \in \mathbb{N}_0$.

**Lemma 5.1.3.** On $\mathcal{D}_0$ it holds that

1. $[J_n, \psi_k] = -\psi_{n+k}$ and $[J_n, \bar{\psi}_k] = \bar{\psi}_{n+k}$,
2. $[J_m, J_n] = m \delta_{n+m,0}$.

**Proof.** Using $[ab, c] = a\{b, c\} - \{a, c\} b$, one obtains $[\bar{\psi}_r \psi_n, \psi_k] = -\delta_{r+k,0} \psi_n$ and $[\psi_n \bar{\psi}_r, \psi_k] = \delta_{r+k,0} \psi_n$ from which directly follows $[J_n, \psi_k] = \sum_{r<0} [\bar{\psi}_r \psi_{n-r}, \psi_k] - \sum_{r>0} [\psi_{n-r} \bar{\psi}_r, \psi_k] = -\psi_{n+k}$. Analogously one shows $[J_n, \bar{\psi}_k] = \bar{\psi}_{n+k}$.

From the Jacobi identity, it follows immediately that $[J_n, J_m]$ commutes with all $\psi_k$ and $\bar{\psi}_k$ and hence $[J_n, J_m]$ is a multiple of the identity, therefore $[J_n, J_m] = (\Omega_0, [J_n, J_m] \Omega_0) \cdot 1$. It is

$$[J_n, J_p] = \sum_{r<0} [J_n, \bar{\psi}_r \psi_{p-r}] - \sum_{r>0} [J_n, \psi_{p-r} \bar{\psi}_r]$$

$$= -\sum_{r<0} (\bar{\psi}_r \psi_{p-r+n} - \bar{\psi}_{r+n} \psi_{p-r}) - \sum_{r>0} (\psi_{p-r} \bar{\psi}_{r+n} - \psi_{p-r+n} \bar{\psi}_r)$$

and in the case $p \neq -n$ we get $(\Omega_0, [J_n, J_p] \Omega_0) = 0$, and otherwise

$$(\Omega_0, [J_n, J_{-n}] \Omega_0) = \begin{cases} \sum_{r<0} (\Omega_0, [\bar{\psi}_{r+n} \psi_{-r-n} \Omega_0]) = \sum_{n \geq 0} \frac{1}{n+\frac{1}{2}} (\Omega_0, \{\bar{\psi}_r, \psi_{-r}\} \Omega_0) & n > 0 \\ -\sum_{r>0} (\Omega_0, [\psi_{r+n} \bar{\psi}_{-r-n} \Omega_0]) = -\sum_{n \geq 0} \frac{1}{n+\frac{1}{2}} (\Omega_0, \{\psi_r, \bar{\psi}_{-r}\} \Omega_0) & n < 0 \\ = n, \end{cases}$$

which completes the proof. \qed

Let $L_0$ be the generator of the rotation: $R(\theta) = e^{i\theta L_0}$. From its action (see the end of Section 5.1.3) one verifies that $\mathcal{D}_0$ is a core for $L_0$.

**Lemma 5.1.4** (Linear energy bounds). It holds that $[L_0, J_n] = -n J_n$ on $\mathcal{D}_0$. For a trigonometric polynomial $f = \sum_n \bar{f}_n e_n$ where the sum is finite and $\xi \in \mathcal{D}_0$, we have

$$\|J f(\xi)\| \leq c_f \| (L_0 + 1) \xi \|$$

$$\| [L_0, J f(\xi)] \| \leq c_{\delta, f} \| (L_0 + 1) \xi \|,$$

where $c_f$ depends only on $f$.

**Proof.** For the commutation relation, it is enough to choose an energy eigenvector $\xi \in \mathcal{D}_0$, i.e. $L_0 \xi = N \xi$. It is $J_n L_0 \xi = N J_n \xi$ and

$$L_0 J_n \xi = L_0 \left( \sum_{r<0} \bar{\psi}_r \psi_{n-r} \xi - \sum_{r>0} \psi_{n-r} \bar{\psi}_r \xi \right) = (N - n) J_n \xi,$$

and the first statement follows.
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We have seen that \( \psi_r \) and \( \tilde{\psi} \) have norm 1 (in Section 5.1.3). First we claim that \( \|J_n \xi\| \leq \|(2(L_0 + 1) + |n|)\xi\| \). Let \( \xi \) be an eigenvector of \( L_0 \), i.e. \( L_0 \xi = N \xi \). From the defining sum of \( J_n \), one sees that only \( 2N + |n| + 2 \) terms contribute to \( J_n \xi \). Hence we have \( \|J_n \xi\| \leq (2N + |n| + 2)\|\xi\| = \|(2(L_0 + 1) + |n|\xi)\| \). If the inequality holds for eigenvectors, then for \( \{\xi_r\} \) with different eigenvalues, we have \( \xi_r \perp \xi_s \) and \( J_n \xi_r \perp J_n \xi_s \), and hence

\[
\left\| J_n \sum_r \xi_r \right\|^2 = \sum_r \|J_n \xi_r\|^2 \\
\leq \sum_r \|(2(L_0 + 1) + |n|)\xi_r\|^2 \\
= \left\| (2(L_0 + 1) + |n|) \sum_r \xi_r \right\|^2
\]

and the general case follows.

For a smeared field, we have

\[
\|J(f)_\xi\| = \left\| \sum_n \hat{f}_n J_n \xi \right\| \leq \sum_n \|\hat{c}_{\hat{f}}(1)\| \|\hat{f}_n\| \|\xi\| + \hat{c}_{\hat{0}} \|\xi\| \leq (2\hat{c}_f + \hat{c}_{\hat{0}})\|\xi\| \leq (L_0 + 1)\|\xi\|.
\]

where \( \hat{c}_f = \sum_n |\hat{f}_n| \). By defining \( \hat{c}_f = 2\hat{c}_f + \hat{c}_{\hat{0}} \), we obtain the first inequality of the statement. The rest follows by noting that \( \|L_0, J(f)\| = \|\hat{f} \hat{0} f\| \).

For a smooth function \( f = \sum_{n \in \mathbb{Z}} \hat{f}_n e_n \in \mathcal{C}^\infty(\mathbb{S}^1) \), its Fourier coefficients \( \hat{f}_n \) are strongly decreasing and, in particular, it is summable: \( \sum_n |\hat{f}_n| = \hat{c}_f < \infty \). Hence we can naturally extend the definition of the smeared current to smooth functions using the above estimate by

\[
J(f) = \sum_{n \in \mathbb{Z}} f_n J_n = \sum_{r,s} f_{r+s} : \psi_r \tilde{\psi}_s :,
\]

and the same inequality as in Lemma 5.1.4 holds. The operator is closable since we have \( J(f) \subset J(\tilde{f}) \) and we still denote the closure by \( J(f) \). We note that from the above definition it follows that \( J(f) \) is obtained by a limit \( \sum_n : \psi(h_n) \tilde{\psi}(k_n) : \) with suitable functions such that \( \sum_n h_n(\theta) k_n(\theta) \rightarrow 2\pi f(\theta) \delta(\theta - \hat{0}) \). This implies covariance of the "field", i.e. \( U(g)J(f)U(g)^* = J(f \circ g^{-1}) \).

Recall that \( \|\psi_r\| = 1 \), hence the smeared field is still bounded: \( \|\psi(g)\| \leq \tilde{c}_g \). We claim that, for \( f, g \in \mathcal{C}^\infty(\mathbb{S}^1) \) and \( \xi \in \mathcal{D}_0 \): the vector \( \psi(g)\xi \) is in the domain of \( J(f) \). Indeed, for a trigonometric polynomial \( g \), we have the estimate

\[
\|J(f)\psi(g)\xi\| \leq c_f \|(L_0 + 1)\psi(g)\xi\| \\
\leq c_f (\hat{c}_g \|\xi\| + \|(L_0, \psi(g))\xi + \psi(g)L_0\xi\|) \\
\leq c_f (\hat{c}_g (\|\xi\| + \|(L_0, \xi)\| + \hat{c}_{\hat{0}} \|\xi\|))
\]

Then if we have a sequence of trigonometric polynomials \( g_n \) converging to a smooth function \( g \in \mathcal{C}^\infty(\mathbb{S}^1) \), the sequence \( \{J(f)\psi(g_n)\xi\} \) is also converging.

**Lemma 5.1.5.** For \( \xi, \eta \in \mathcal{D}_0 \), it holds that

\[
J(f), \psi(g)\xi = -\psi(f \cdot g)\xi \\
J(f), \tilde{\psi}(g)\xi = \tilde{\psi}(f \cdot g)\xi \\
(J(\tilde{\psi})\xi, J(g)\eta) = (J(\tilde{\psi})\xi, J(f)\eta) + 2i\omega(f, g)(\xi, \eta).
\]
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Proof: For trigonometric polynomials \( f, g \), the statements can be proved easily from Lemma 5.1.3. The general case is shown by approximating first \( f \) by polynomials, then \( g \), according to the convergence considered above (as for the third statement, obviously the order of limits does not matter).

We need the following well-known result [DF77, Theorem 3.1]:

**Theorem 5.1.6** (The commutator theorem). Let \( H \) be a positive self-adjoint operator and \( A, B \) symmetric operators defined on a core \( \mathcal{D}_0 \) for \( (H + 1)^2 \). Assume that there is a constant \( C \) such that

\[
\|A\xi\| \leq C\|(H + 1)\xi\|, \quad \|B\xi\| \leq C\|(H + 1)\xi\|, \\
\|[H, A]\xi\| \leq C\|(H + 1)\xi\|, \quad \|[H, B]\xi\| \leq C\|(H + 1)\xi\|, \\
(A\xi, B\eta) = (B\xi, A\eta) \quad \text{for any } \xi, \eta \in \mathcal{D}_0.
\]

Then \( A \) and \( B \) are essentially self-adjoint on any core of \( H \) and any bounded functional calculus of \( A \) and \( B \) commute.

**Remark 5.1.7.** In the original literature [DF77], this Theorem is proved under the assumption of certain operator inequalities. In fact, what is really used in the proof of commutativity of bounded functions is the norm estimates \( \|A(H + 1)^{-1}\| < C \), \( \|[H, A](H + 1)^{-1}\| \) \( < C \) etc. and they follow from the assumptions here. The essential self-adjointness of \( A \) and \( B \) can be proved by [RS75, Theorem X.37]. An analogous application of this theorem with norm estimates can be found in [BSM90].

By the commutator theorem, we get that \( J(f) \) is self-adjoint for \( f \in C^\infty(S^1, \mathbb{R}) \) and that all bounded functions of \( J(f) \) commute with all bounded functions of \( J(g) \) for \( f, g \in C^\infty(S^1, \mathbb{R}) \) with disjoint support.

Let \( I \) be a proper interval and let us define the von Neumann algebra

\[
\mathcal{B}(I) = \{e^{iJ(f)} : \text{supp } f \subset I\}''.
\]

The local net \( \mathcal{B}(I) \) restricted to \( \overline{\mathcal{B}(I)}\mathcal{O}_0 \) can be identified with the \( U(1) \)-current net \( \mathcal{A}_R \) on \( \mathcal{H}_{A_R} \), in particular we can identify \( \overline{\mathcal{B}(I)}\mathcal{O}_0 \cong \mathcal{H}_{A_R} \).

**Proposition 5.1.8.** Let \( I \) be a proper interval, then \( \mathcal{B}(I) \subset \text{Fer}_C^{U(1)}(I) \).

**Proof.** We see that \( \mathcal{B}(I) \) commutes with \( \text{Fer}_C(I') = \{c(g) : g \in L^2(I')\}'' \) because, for \( f, g \) with disjoint supports, \( c(g) \) commutes with \( J(f) \) on a core by Lemma 5.1.5 and therefore any spectral projection of \( c(g) \) commutes with \( J(f) \), and hence with any bounded functions of \( J(f) \).

Further because \( J(f) \) commutes by construction with the gauge action \( V(i) \) and is in particular even because \( V(\pi) = \Gamma \), it follows that \( \mathcal{B}(I) \) lies in the twisted commutant \( \text{Fer}_C(I')^{\perp} \). By twisted Haag duality it is \( \mathcal{B}(I) \subset \text{Fer}_C(I')^{\perp} = \text{Fer}_C(I) \) and therefore \( \mathcal{B}(I) = \mathcal{B}(I)^{U(1)} \subset \text{Fer}_C^{U(1)}(I) \).

Since the covariance has been seen, we have the following:

**Corollary 5.1.9.** \( \mathcal{B} \) is a subnet of \( \text{Fer}_C^{U(1)} \).

Now the following is straightforward.

**Proposition 5.1.10** (Algebraic version of Boson–Fermion Correspondence). The \( U(1) \)-fixed point subnet of the complex free Fermion net \( \text{Fer}_C \) is the \( U(1) \)-current net, i.e. \( \text{Fer}_C^{U(1)} = \mathcal{B} \equiv \mathcal{A}_R \).
5.2. A NEW FAMILY OF LONGO–WITTEN ENDOMORPHISMS ON $U(1)$–CURRENT NET

In this section we construct a new family of Longo–Witten endomorphisms for the $U(1)$–current net, using the Boson–Fermion correspondence (Proposition 5.1.10). While the first family from Section 4.2.1\(^2\) originally obtained in [LW11] is coming from inner symmetric functions, the here obtained family is coming from inner (not necessarily symmetric) functions. In other words: the first family comes from a real analogue of the Beurling–Lax thereom, while the second family comes from a complex analogue—which we describe in the following Section 5.2.1—applied to the complex Fermion net $\text{Fer}_C$ and restricted to the $(1)$–current net by the mentioned Boson–Fermion correspondence.

\(^2\)We remind that the $U(1)$–current net is the case $\dim F = 1$, i.e. $F = \mathbb{R}$.\n
**Proof.** Let us see $\mathcal{B}$ as a subnet of the Fermi net $\text{Fer}_C^{U(1)}$ on $\mathcal{H}_{\text{Fer}_C}^{U(1)} \equiv \mathcal{H}_{\cdot, 0}$. Further $\overline{\mathcal{B}(I)\mathcal{O}}$ does not depend on $I$ by the same proof of the Reeh-Schlieder property and is clearly a subspace of $\mathcal{H}_{\text{Fer}_C}^{U(1)} \equiv \mathcal{H}_{\cdot, 0}$.

In fact they coincide, since we have confirmed that $\text{tr}_{\mathcal{H}_{\mathcal{A}_0}}(\text{e}^{-\beta \mathcal{L}_0}) = \text{tr}_{\mathcal{H}_{\cdot, 0}}(\text{e}^{-\beta \mathcal{L}_0}) = p(t)$, where $e^{-\beta} = T$, namely, their conformal characters coincide (see also Section 3.2.3). \hfill\□

We finish this section by giving the parametrization in $x$–picture, where the action of the translation is more natural.

$$ f(x) = \frac{1}{\sqrt{2\pi}} \int \frac{d\theta(x)}{dx} e^{i\theta(x)/2} f_0(\theta(x)) $$

we identify $L^2(\mathbb{R}) = L^2(\mathbb{R}, dx)$ with $L^2(S^1) = L^2([0, 2\pi], d\theta/(2\pi))$ and therefore the space $\mathcal{H}_{\text{Fer}_C}^{1}$ is given by $P L^2(\mathbb{R}) \oplus P^\perp L^2(\mathbb{R})$ with $P : \hat{f}(p) \mapsto \Theta(p)\hat{f}(p)$ and it can be identified in “momentum space” with $L^2(\mathbb{R}, 2\pi dp) \oplus L^2(\mathbb{R}, 2\pi dq)$ by

$$ f(x) \mapsto P\hat{f}(p) \oplus \overline{P\hat{f}(-q)} \quad p, q > 0. $$

The field operators are defined for $f \in L^2(\mathbb{R})$ by $\psi(f) = a_P(f)$ and $\tilde{\psi}(f) = a_P(f)^*$. For $\Psi \in \mathcal{H}_{\text{Fer}_C}$ we write its components

$$ \Psi_{m,n} \in \mathcal{H}_{m,n} := L^2(\mathbb{R}_x^{m+n}, (2\pi)^{m+n} dp_1 \cdots dp_m dq_1 \cdots dq_n)_{-}, $$

where $-$ means the antisymmetrization within $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$. By using this notation $(\psi(f)\Omega_0)_{1,0}(p) = \hat{f}(p)$ and $(\tilde{\psi}(f)\Omega_0)_{0,1}(q) = \hat{f}(q)$. Further the bi-field $:\tilde{\psi}(f)\psi(g) = \hat{\psi}(f)\psi(g) - (\Omega_0, \hat{\psi}(f)\psi(g)\Omega_0)$ creates from the vacuum $\Omega_0$ a Fermionic $1+1$ particle state $\Psi_{f,g} := \psi(f)\psi(g)\Omega_0$ with $(\Psi_{f,g})_{1,1}(p, q) = -\hat{f}(q)\hat{g}(p)$ and it follows for $h \in C^\infty(\mathbb{R}, \mathbb{R})$ $(J(h)\Omega_0)_{1,1}(p, q) = -\frac{1}{2\pi} \hat{h}(p + q)$ holds which is obtained by taking a limit $\sum f_n(x)\bar{g}_n(y) \rightarrow h(x)\delta(x - y)$. We make the important observation that the $J(f)\Omega_0$ generate the one-particle space which we can identify with $\mathcal{H}_{\mathcal{A}_0}$ and this is obviously just a proper subspace of the Fermionic $1+1$-particle space $\mathcal{H}_{\text{Fer}_C}^{1,1}$. This is the reason why a second quantization unitary on the Fermionic Fock space, which restricts to the Bosonic subspace, in general does not have to be a second quantization unitary on the Bosonic subspace, but instead can be rather involved.
5.2.1. A complex analogue of the Beurling–Lax theorem. For a Borel function \( \varphi : \mathbb{R} \to \mathbb{C} \) we denote by \( \check{\varphi} \) the function \( \check{\varphi}(p) = \varphi(-p) \). Note that if \( \varphi \) is boundary value of an analytic function on the upper complex plane plane \( \mathbb{R} + i\mathbb{R}_+ \), then \( \check{\varphi} \) also has this property. It is useful to treat the operation \( \varphi \mapsto \check{\varphi} \) like a complex conjugation and define

\[
\tilde{\text{Re}} \varphi = \frac{1}{2} (\varphi + \check{\varphi}) \\
\tilde{\text{Im}} \varphi = \frac{1}{2i} (\varphi - \check{\varphi}).
\]

The following can be seen as a complex analogue of the Beurling–Lax theorem (Theorem 2.3.11): as the last statement of Proposition 2.3.12 can be seen as a real analogue of it.

**Definition 5.2.1.** We call \( \varphi : \mathbb{R} \to \mathbb{C} \) an inner function (on \( \mathbb{R} \)) if it a boundary value of an bounded analytic function \( \mathbb{R} + i\mathbb{R} \to \mathbb{C} \), such that \( |\varphi(p)| = 1 \) for almost all \( p \). The set of inner functions we denote by \( S_\mathbb{C} \).

**Proposition 5.2.2.** Let \( (H_0, T_0) \) be the unique irreducible standard pair with \( T_0(t) = e^{itP_0} \). Let \( (H, T) = (H_0 \oplus H_0, T_0 \oplus T_0) \). Then all \( V \in \mathcal{E}(H, T) \) commuting with the \( U(1) \) action

\[
U(\vartheta) = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}, \quad \vartheta \in \mathbb{R}/2\pi\mathbb{Z} \cong U(1)
\]

are of the form

\[
V = \begin{pmatrix} \tilde{\text{Re}} \varphi(P_0) & -\tilde{\text{Im}} \varphi(P_0) \\ \tilde{\text{Im}} \varphi(P_0) & \tilde{\text{Re}} \varphi(P_0) \end{pmatrix},
\]

where \( \varphi \in S_\mathbb{C} \).

A standard subspace \( K \subset H \) fulfills \( T(t)K \subset K \) for all \( t \geq 0 \) and \( U(\vartheta)K = K \) for all \( \vartheta \in \mathbb{R}/2\pi\mathbb{Z} \cong U(1) \) if and only if \( K = VH \) with a \( V \in \mathcal{E}(H, T) \) commuting with the \( U(1) \) action, in particular \( V \) comes from an inner function \( \varphi \in S_\mathbb{C} \).

**Proof.** By the characterization for the semigroup for reducible standard pairs (Theorem 2.3.10) we have that \( V \in \mathcal{E}(H_0 \oplus H_0, T_0 \oplus T_0) \) and \( V \) commutes with the\(^3 \) \( U(1) \) action if and only if

\[
V_0 = \begin{pmatrix} a(P_0) & -b(P_0) \\ b(P_0) & a(P_0) \end{pmatrix}
\]

and the matrix valued function

\[
p \mapsto \begin{pmatrix} a(p) & -b(p) \\ b(p) & a(p) \end{pmatrix}
\]

is symmetric and unitary form almost all \( p \) and boundary value of a bounded analytic function on \( \mathbb{R} + i\mathbb{R}_+ \to B(\mathbb{C}^2) \), i.e. in \( S_{\mathbb{C}}^{(2)} \). Further we have a \( * \)-isomorphism \( \Phi \) between the set of Borel functions \( \mathbb{R} \to \mathbb{C} \) and set of Borel functions \( \mathbb{R} \to B(\mathbb{C}^2) \) which are symmetric and commuting with the \( U(1) \) action. It is given by:

\[
\Phi : \varphi \mapsto \begin{pmatrix} \tilde{\text{Re}} \varphi & -\tilde{\text{Im}} \varphi \\ \tilde{\text{Im}} \varphi & \tilde{\text{Re}} \varphi \end{pmatrix} \quad \Phi^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto \varphi := a + ib.
\]

Finally every \( V_0 \) as above is characterized by an inner function \( \varphi \).

\(^3\)An operator commutes with \( U(t) \) if and only if it is of the form \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \).
For the second part if $V \in \mathcal{E}(H, T)$ commutes with $U(t)$ then $U(t)K = U(t)VK = VU(t)H = VH = K$. Let $K \subset H$ be a $U(1)$-invariant standard subspace with the asked property. We know that $K = VH$ by Proposition 2.3.12 and we need only to show that the operator $Z(t) = V^* U(t)V U(-t)$ fulfilling the cocycle relation $Z(t) \text{Ad}(U(t))(Z(s)) = Z(t+s) \text{gauge}$ equals 1. Using $S_K = V S_H V^*$ and the fact that $U(t)$ commutes with $S_H$ and $S_K$, we get $S_H = \text{Ad}(Z(t))(S_H)$ so $Z(t) \in \mathcal{E}(H, T)$ with $Z(t)H = H$. We conclude that $Z(t)$ comes from a matrix $\varphi_t \in \mathcal{O}(2)$ and by continuity is in $\text{SO}(2) \cong U(1)$. Together with the cocycle relation we get $Z(t) = U(mt)$, but this is equivalent with $U((m + 1)t) = V^* U(t)V$, so we get that the one-parameter groups $U((m + 1)t)$ and $U(t)$ are unitarily equivalent. Because the spectrum\footnote{We owe this argument to Roberto Longo.} of the generator of $U(t)$ is in $\pm 1$ we conclude $m = 0$ and therefore $Z(t) = 1$. \hfill \Box

5.2.2. Longo–Witten endomorphisms commuting with the gauge action. Now we can construct Longo–Witten endomorphism for the complex free Fermion net $\mathcal{F}_{\text{Fer}}$ commuting with the $U(1)$ gauge action. By the double construction Proposition 5.1.2 we can identify the one–particle space $\mathcal{H}^1_{\text{Fer}}$ with $\mathcal{H}_1 \oplus \mathcal{H}_2$. We remind that on $\mathcal{H}_1 \oplus \mathcal{H}_2$, we have a $U(1)$ gauge action given by

$$V(\vartheta) = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}, \quad \vartheta \in \mathbb{R}/2\pi \mathbb{Z} \cong U(1)$$

and we note that $(H(\mathbb{R}_+) \oplus H(\mathbb{R}_+), T \oplus T)$ is a Borchers pair (corresponding to $(\text{K}(\mathbb{R}_+), T)$) which we identify with $(H_0 \oplus H_0, T_0 \oplus T_0)$ and so that we can apply Proposition 5.2.2.

A convenient decomposition of $\mathcal{H}^1_{\text{Fer}}$ is given by the dual group $\mathbb{Z}$ of $U(1)$ and is given by

$$\mathcal{H}^1_{\text{Fer},+1} \oplus \mathcal{H}^1_{\text{Fer},-1} = P \mathcal{H}^1_{\text{Fer}} \oplus P \mathcal{H}^1_{\text{Fer}}$$

(by Proposition 5.1.2), where $\mathcal{H}_n = \{ x \in \mathcal{H} : V(x) = e^{i n x} \}$. Let us denote $\mathcal{E}^{U(1)}(\mathcal{F}_{\text{Fer}})$ the sub semigroup of Longo–Witten unitaries commuting with the $U(1)$ gauge action.

**Proposition 5.2.3.** A second quantization unitary $V = \Lambda(V_0)$

- (1) is in $\mathcal{E}(\mathcal{F}_{\text{Fer}})$ if and only if it comes from a unitary $2 \times 2$ matrix of inner functions $\varphi \in S^2$ as in Theorem 2.3.10. In this case $V_0 = \varphi(P_0)$ as unitary on $\mathcal{H}_1 \oplus \mathcal{H}_2$.

- (2) is in $\mathcal{E}^{U(1)}(\mathcal{F}_{\text{Fer}})$ if and only if it comes from a inner function $\varphi : \mathbb{R} \to \mathbb{C}$, i.e. boundary value of a bounded analytic function $\mathbb{R} + i \mathbb{R} \to \mathbb{C}$ with $|\varphi(p)| = 1$ for almost all $p \in \mathbb{R}$. In this case $V_0$ acts by $\varphi(P_0)$ on $\mathcal{H}^1_{\text{Fer},+1} \oplus \mathcal{H}^1_{\text{Fer},-1}$.

**Proof.** We have $\Gamma(V_0) \in \mathcal{E}(\mathcal{F}_{\text{Fer}})$ if and only if $V_0 \in \mathcal{E}(\text{K}(\mathbb{R}_+))$ (cf. Theorem 4.2.1) and the first statement is immediate from Theorem 2.3.10. The second follows from Proposition 5.2.2 and we note that $\xi \in \mathcal{H}^1_{\text{Fer},+1}$ correspond by the double construction (Proposition 5.1.2) to $\eta \oplus \pm \eta \in \mathcal{H}$ and $V_0$ acts on $\xi$ therefore by $(\text{Re} \varphi \pm i \text{Im} \varphi)(P_1)$. \hfill \Box

This gives by restriction a new family of Longo–Witten unitaries for the $U(1)$-current net $\mathcal{A}_{\mathbb{R}}$.

**Corollary 5.2.4.** For every $\varphi : \mathbb{R} \to \mathbb{C}$ inner symmetric function, i.e. boundary value of a bounded analytic function $\mathbb{R} + i \mathbb{R} \to \mathbb{C}$ with $|\varphi(p)| = 1$ for almost all $p \in \mathbb{R}$, there exist an element $V_\varphi \in \mathcal{E}(\mathcal{A}_{\mathbb{R}})$, given by the restriction of $\Lambda(\varphi(P_0) \oplus \varphi(P_0)) \restriction \mathcal{H}_{\mathcal{A}_{\mathbb{R}}}$.

We want to show, that these elements $V_\varphi \in \mathcal{E}(\mathcal{A}_{\mathbb{R}})$ can have a rather complicated action one the Bosonic Fock space $\mathcal{H}_{\mathcal{A}_{\mathbb{R}}}$ in the sense that they rarely are second quantization unitaries.

We remind that $\mathcal{H}^1_{\text{Fer}}$ can be identified with $L^2(\mathbb{R})$ as a real space. In $L^2(\mathbb{R})$ the function $\varphi(P_1)f$ is the function with Fourier transform $\varphi(p)\hat{f}(p)$ and we remark that it also follows directly from the Paley-Wiener theorem that $\varphi(P_1)$ leaves $L^2(\mathbb{R}_+)$ in $L^2(\mathbb{R})$ invariant for $\varphi$ inner. Further using that
the space $\mathcal{H}_{\text{Ferc}}$ decomposes in $\mathcal{H}_{\text{Ferc}} = \bigoplus_{m,n \in \mathbb{N}_0} \mathcal{H}_{m,n}$ like in (5.10) with the gauge action given by $V(\theta)\Psi_{m,n} = e^{i(m-n)\theta}\Psi_{m,n}$, the action of the Longo–Witten unitary $V_{\varphi} = \Lambda(\varphi(P_1))$ is given by

$$(V_{\varphi}\Psi)_{m,n}(p_1, \ldots, p_m, q_1, \ldots, q_n) = \varphi(p_1) \cdots \varphi(p_m)\varphi(-q_1) \cdots \varphi(-q_n)\Psi_{m,n}(p_1, \ldots, p_m, q_1, \ldots, q_n).$$

**Lemma 5.2.5.** Let $\theta : \Psi \in L^2(\mathbb{R}_+, pdp) \cong \mathcal{H}_{A,\text{b}}^1 \hookrightarrow \mathcal{H}_{1,1} \subset \mathcal{H}_{\text{Ferc}}$ be the embedding given by $\iota(\Psi)_{1,1}(p, q) = -\frac{1}{2\pi}\Psi(p + q)$. A second quantization Longo–Witten unitary $V_{\varphi}$ commuting with the gauge action $V(\cdot)$ satisfies $V_{\varphi}\mathcal{H}_{A,\text{b}}^1 \subset i\mathcal{H}_{A,\text{b}}^1$ if and only if $V_{\varphi} = V(\theta)T(\iota)$ for some $\theta \in \mathbb{R}/2\pi\mathbb{Z} \cong U(1)$ and $t \geq 0$.

**Proof.** The translations commute with the gauge action and it follows immediately that they leave $\mathcal{H}_{A,\text{b}}^1$ invariant. We note that $\varphi(p)\varphi(q)\varphi(p + q)$ belongs to $\mathcal{H}_{A,\text{b}}^1$ only if it can be written as a function of $g(p + q)$. This means that $\varphi(p)\varphi(q) = \varphi(p + q)$ for $p, q \geq 0$, where $\varphi$ is another function. Then, putting $q = 0$ and $p = 0$ respectively, we see that $\varphi(p)\varphi(0) = \varphi(p)$ for $p \geq 0$ and $\varphi(0)\varphi(-q) = \varphi(q)$ for $q \geq 0$, in particular $\varphi(0) = 1$. Multiplying the each side of these equations, one sees that $\varphi(p + q) = \varphi(p)\varphi(q)$ because $\varphi(0) = 1$. Then follows that $\varphi(p) = e^{i\kappa p}$ for some $\kappa \geq 0$, and $\varphi(p) = e^{i(\kappa p + \theta)}$ for some $\theta \in \mathbb{R}$. Finally a $\varphi$ is a Longo–Witten unitary only for $\kappa \geq 0$. The constant factor $e^{i\theta}$ corresponds to the factor $V(\theta)$. \hfill \square

**Theorem 5.2.6.** Let $\varphi$ be an inner function as above. The endomorphism implemented by the second quantization $V_{\varphi}$ of the operator constructed above restricts to the $U(1)$–current subnet. This restriction is implemented by a (Bosonic) second quantization operator if and only if $\varphi(p) = e^{i(\kappa p + \theta)}$ for some $\kappa \geq 0$ and $\theta \in \mathbb{R}$.

**Proof.** The operator $V_{\varphi}$ restricts to the subnet $\mathcal{A}_{A,\text{b}}$ by the general argument in Proposition 5.1.1. In the non-exponential case $\kappa$ cannot be implemented by a second quantization operator, since any second quantization operator preserves the particle number, while $V_{\varphi}$ does only for exponential $\varphi$ as we saw in Lemma 5.2.5, and a Longo–Witten endomorphism is uniquely implemented up to scalar (see Section 3.2.7). It is on the other hand clear using covariance that if $\varphi$ is exponential then the restriction of $V_{\varphi}$ equals $T(\kappa)$ which is a second quantization unitary. \hfill \square

**Remark 5.2.7.** By the construction Proposition 3.5.3 for every $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ inner function, we have a boundary net $\mathcal{A}_{\text{b}} = \mathcal{A}_{\text{b}}^1$. In the case where $\varphi$ is not exponential $V_{\varphi}$ does not come from second quantization—in contrast to the unitaries constructed from Section 4.2.1 originally constructed by Longo and Witten in [LW11]—and therefore gives completely new examples.

### 5.3. New Longo–Witten unitaries for Abelian current nets and Fermionic extensions

This section is rather a collection of remarks and is not part of [BT11].

We can use Proposition 5.2.3 to get also new for the net of Abelian currents $\mathcal{A}_F$ where $F$ is an Euclidean space with $n := \dim F$, namely it is $\mathcal{A}_F = \mathcal{A}_{\mathbb{R}^n}^\otimes \dim F$. For simplicity let in this section $F = \mathbb{R}^n$ with the standard scalar product. But then $\mathcal{A}_{\mathbb{R}^n}$ is isomorphic to the $U(1)^\otimes \mathbb{R}_n$ fixed point of the net $\mathcal{F}_C^\otimes$ by the Boson–Fermion correspondence (Proposition 5.1.10).

**Proposition 5.3.1.** Let $n \in \mathbb{N}$ and $\varphi = \text{diag}(\varphi_1, \ldots, \varphi_n)$ with $\varphi_i = e^{i\kappa_i \theta_i}$. Then there is an element $V_{\varphi} \in \mathcal{E}(\mathcal{A}_{\mathbb{R}^n})$ coming from the restriction $\Lambda(V_1) \otimes \cdots \otimes \Lambda(V_n) \in \mathcal{E}(\mathcal{F}_C^\otimes)$ to $\mathcal{A}_F$, where
\( V_i = \varphi(P_0) \oplus \hat{\varphi}(P_0) \) on \( \mathcal{H}_{\text{Fer}_{c,+1}} \oplus \mathcal{H}_{\text{Fer}_{c,-1}} \) as in Proposition 5.2.3. Every element \( V \in \mathcal{E}(\mathbb{A}_{1/2}) \) coming from a restriction of a second quantization unitary on \( \text{Fer}_{c}^{\otimes n} \) is of this form.

Proof. The second quantization unitaries \( V = \Gamma(V_0) \) with \( V \in \mathcal{E}(\text{Fer}_{c}^{\otimes n}) \) are given by \( V_0 \) of the form \( \varphi_i(P_0) \) with \( (\varphi_i) \in S^{(2n)} \) by Theorem 2.3.10. Now \( V \) restricts if and only if it commutes with \( U(1)^{\times n} \). Such an element commutes with the diagonal subgroup \( U(1) \subset U(1)^{\times n} \) if and only if it comes from a \( (n \times n) \)-matrix of functions which are boundary value of bounded analytic functions on \( \mathbb{R} + i\mathbb{R}^+ \) using the isomorphism (5.11) as in the proof of Proposition 5.2.2. That such an element commutes with \( U(1)^{\times n} \) is equivalent to the matrix being diagonal. \( \square \)

It makes sense to write \( \text{Fer}_{c} = A_{U(1),1} \cong A_{\mathbb{Z}} \) and write \( \text{Fer}_{c}^{\otimes n} = \text{Fer}_{c}^{\otimes n} \cong A_{U(0),1} \cong A_{\mathbb{Z}^n} \), even though these are Fermi nets.

Let \( L \) be a lattice, then on the \( L \) graded space

\[ \mathcal{H} = \bigoplus_{a \in L} \mathcal{H}_a \]

has an action \( V \) of the dual group \( \hat{L} \) which we identify with the torus \( \hat{T} = T_L := F/2\pi L^* \) given by \( V(x) \uparrow \mathcal{H}_a = e^{i\langle x, a \rangle} \) for \( x \in \hat{T} \). There is a correspondence between discrete subgroups \( G \subset \hat{T} \) and lattices \( M \) with \( \dim L = \dim M \) and \( M \subset L \), which is given by \( M \mapsto G_M = 2\pi M^*/2\pi L^* \subset F/2\pi L^* = \hat{T} \) and \( G \mapsto M_G := N_G^* \), where

\[ N_G := \bigcup_{[x] \in G} \frac{1}{2\pi} \cdot g + L^* \supset L^* , \]

which is well-defined, because \( G \subset F/2\pi L^* \).

We can also look into nets between \( A_{\mathbb{R}^n} \) and \( \text{Fer}_{c}^{\otimes n} \). We restrict here to local nets. A class comes from even lattices \( L \) of rank \( n \) which are \( L \subset \mathbb{Z}^n \). Then \( A_L \) equals \( \text{Fer}_{c}^{\otimes n} \) the fixed point with respect to the discrete group \( (\mathbb{R}^n/2\pi \mathbb{Z}^n)/(\mathbb{R}/2\pi L^*) \subset (\mathbb{R}^n/2\pi \mathbb{Z}^n) \cong U(1)^{\times n} \). The Longo–Witten endomorphism of Proposition 5.3.1 also restricts to this fixed points.

**Proposition 5.3.2.** Let \( n \in \mathbb{N} \) and \( \varphi = \text{diag}(\varphi_1, \ldots, \varphi_n) \) with \( (\varphi_i)_{1 \leq i \leq n} \in S_{c}^{\otimes n} \) then there is an element \( V_\varphi \in \mathcal{E}(A_L) \) coming from the restriction \( \Lambda(V_1) \otimes \cdots \otimes \Lambda(V_n) \in \mathcal{E}(\text{Fer}_{c}^{\otimes n}) \), where \( V_i = \varphi(P_0) \oplus \hat{\varphi}(P_0) \) on \( \mathcal{H}_{\text{Fer}_{c,+1}} \oplus \mathcal{H}_{\text{Fer}_{c,-1}} \) as in Proposition 5.2.3.

**Lemma 5.3.3.** Let \( A \) be the \( U(1) \) fixed point net of \( \text{Fer}_{c}^{\otimes n} \) the character \( \text{tr}_A(e^{(\eta, \Omega)} t^L) \) equals \( p(t) \cdot \text{tr}_{\text{SU}(N)}(e^{(\eta, \Omega)} t^L) \).

Proof. The character of the vacuum representation of \( \text{SU}(N)_1 \) is given by the Frenkel–Kac–Segal [Kac98, Theorem 5.6] construction and therefore coincides with the character of the vacuum representation of the \( A_{N-1} \) lattice vertex algebra, which is given by:

\[ \text{tr}_{\text{SU}(N)}(e^{(\eta, \Omega)} t^L) = (p(t))^{N-1} \sum_{x \in \mathbb{Z}^N} e^{\langle \eta, x \rangle} q^{\frac{1}{2}(x,x)} . \]

We note that the lattice \( A_{N-1} \) can be realized as \( \{ x \in \mathbb{Z}^N : x_1 + \cdots + x_N = 0 \} \) with the standard scalar product of \( \mathbb{R}^N \) (restricted to \( F_{A_{N-1}} := \{ x \in \mathbb{R}^N : x_1 + \cdots + x_N = 0 \} \)). The vacuum character of \( \text{Fer}_{c}^{\otimes N} \)
is
\[
\text{tr}_{\text{Fer}^n_C}(e^{\langle \eta, Q \rangle t^L}) = \prod_{i=1}^{N} \prod_{r \in \frac{1}{2} + \mathbb{N}_0} (1 + e^{\langle \eta, e_i \rangle f})(1 + e^{-\langle \eta, e_i \rangle f}) = \prod_{i=1}^{N} \left( p(t) \sum_{n \in \mathbb{Z}} e^{n \cdot \langle \eta, e_i \rangle t^L} \right) = (p(t))^N \sum_{x \in \mathbb{Z}^n} e^{\langle \eta, x \rangle t^L} \langle x, x \rangle.
\]

Let \( \eta \in \{ x \in \mathbb{R}^N : x_1 + \cdots + x_N = 0 \} \) then
\[
\text{tr}_{\text{Fer}^n_C}(e^{\langle \eta, Q \rangle t^L}) = (p(t))^N \sum_{x \in \mathbb{Z}^n \mid \langle x, x \rangle = 0} e^{\langle \eta, x \rangle t^L} \langle x, x \rangle = p(t) \cdot \text{tr}_{\text{SU}(N)}(e^{\langle \eta, t \rangle t^L}).
\]

\[\square\]

**Proposition 5.3.4.** The \( U(1) \) fixed point of the diagonal inclusion \( U(1) \subset U(1)^{\times n} \) of the Fermi net \( \text{Fer}^n_C \) is isomorphic to the tensor product net of the \( U(1) \)-current and the loop group net of \( \text{SU}(n) \) at level 1, i.e., \( (\text{Fer}^n_C)^{U(1)} \cong \mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\text{SU}(n),1} \).

**Proof.** The argument is the same as for the Boson–Fermion correspondence. One can check that \( \mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\text{SU}(n),1} \) is a subnet of \( \text{Fer}^n_C \) which is invariant under \( U(1) \). Equality follows then from equality of the characters and realizing that the net \( \mathcal{A}_{\text{SU},1} \) has the same character as its vertex operator algebra (see [Was98]). \( \square \)

**Proposition 5.3.5.** For \( n > 1 \) every \( \varphi = (\varphi_{ij})_{1 \leq i, j \leq n} \) with \( \varphi_{ij} \) boundary values of a bounded analytic function on \( \mathbb{R} + i\mathbb{R}_+ \), such that the matrix \( (\varphi_{ij}(p)) \) is unitary for almost all \( p \in \mathbb{R} \) gives an element \( V_\varphi \in \mathcal{E}(\mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\text{SU}(n),1}) \) coming from the restriction of a second quantization unitary in \( \mathcal{E}(\text{Fer}^n_C) \) and these are the only second quantization Longo–Witten unitaries which restrict to the fixed point.

**Proof.** This is a corollary of the proof of Proposition 5.3.1. \( \square \)

We want to finish this section by commenting on some constructions which are expected not to work. The inclusion of \( \mathcal{A}_{\text{SU}(n),1} \subset \text{Fer}^n_C \), also used in [Was98], was the first idea to obtain Longo–Witten unitaries from \( \text{Fer}^n_C \) by restriction. Although we have determined the class which restrict to \( \mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\text{SU}(n),1} \) by Proposition 5.3.5, we could not find non-trivial examples which leave the spaces \( \mathcal{A}_{\mathbb{R}} \Omega \) and \( \mathcal{A}_{\text{SU}(n),1} \Omega \) invariant and so would restrict to \( \mathcal{A}_{\text{SU}(n),1} \). We expect that such examples do not exist.
Longo–Witten endomorphisms and finite index inclusions

6.1. Using the dual Q–system

We want to study the situation where \( \mathcal{A} \) is a conformal net and \( \mathcal{B} \) a subnet with finite index \( \lambda \equiv [\mathcal{A}(I) : \mathcal{B}(I)] \) where we have \( \tilde{\eta} \) a Longo–Witten endomorphism of \( \mathcal{A} \) restricting to a Longo–Witten endomorphism \( \eta \) of \( \mathcal{B} \), i.e. \( \iota \circ \eta = \tilde{\eta} \circ \iota \) (we just write \( \eta = \tilde{\eta} \)). Let \( \mathcal{N} = \mathcal{B}(\mathbb{R}_+) \) and \( \mathcal{M} = \mathcal{A}(\mathbb{R}_+) \) and \( \iota \) the inclusion \( \iota(\mathcal{N}) \subset \mathcal{M} \) and \( (\theta, w, x) \) its dual Q–system. Locality of \( \mathcal{A} \) is equivalent with the eigenvalue condition \( \epsilon(\theta, \theta)x = x \), where \( \epsilon \) is the statistic operator (braiding). We remember that by the Bisognano–Wichmann property (Theorem 3.2.3) and Takesaki’s Theorem 2.1.11 there is a unique conditional expectation \( E : \mathcal{M} \rightarrow \mathcal{N} \) implemented by the Jones projection.

The conditional expectation can be given by the \( Q \)–system (\( \gamma \equiv \tilde{u}, v, \iota(w) \)) by \( \iota(E(m)) = \iota(w)^\ast \gamma(m)\iota(w) \) or equivalently \( E(m) = w^\ast \gamma(m)w \). Important is the dual \( Q \)–system (\( \delta, w, x \) \( \equiv (\tilde{u}, w, \tilde{v}) \)).

Let us define \( z_\theta = \lambda E(\tilde{\eta}(v)w^\ast) \) in other words \( z_\theta \in \mathcal{N} \) is the unique element such that \( \tilde{\eta}(v) = \iota(z_\theta) \cdot v \).

Lemma 6.1.1. It is \( z_\theta = \lambda^\frac{1}{2} \cdot w^\ast \cdot \tilde{\eta}(v) \in (\theta \eta, \eta \theta) \) and it holds \( z_\theta \cdot \theta(z_\theta) \cdot x = \eta(x) \cdot z_\theta \).

Proof. It is elementary to check that:

\[
z_\theta = \lambda \cdot E (\tilde{\eta}(v) \cdot v^\ast) \\
= \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot v^\ast \cdot w \\
= \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot x^\ast \cdot w \\
\]

\[
(w^\ast \cdot x = \lambda^\frac{1}{2} \cdot 1) = \lambda^\frac{1}{2} \cdot w^\ast \cdot \tilde{\eta}(v)
\]

and \( z_\theta \) is in \( (\theta \eta, \eta \theta) \) by

\[
\theta \eta = \tilde{\eta} \eta \overset{\tilde{\eta}(v)}{\rightarrow} \tilde{\eta} \gamma \iota = \theta \eta \theta \overset{w^\ast}{\rightarrow} \eta \theta.
\]

Now we compute:

\[
z_\theta \cdot \theta(z_\theta) \cdot x = \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \theta (w^\ast \cdot \tilde{\eta}(v)) \cdot x \\
\equiv \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \theta (w^\ast \cdot \tilde{\eta}(v)) \cdot x = \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \iota (w^\ast \cdot v) \cdot \tilde{\eta}(v) \]

\[
(x \in (\tilde{t}, \tilde{t}) \iota) = \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \theta (w^\ast \cdot \tilde{\eta}(v)) \cdot x \cdot \tilde{\eta}(v) \\
\equiv \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \cdot \tilde{\eta}(v) = \lambda \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \tilde{\eta}(v)
\]

\[
(v^\ast \cdot \iota(w) = \lambda^\frac{1}{2} \cdot 1) = \lambda^\frac{1}{2} \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \tilde{\eta}(v)
\]

\[
(\tilde{\eta}(v) \in (\tilde{t}, \tilde{t}) \iota \tilde{\eta}) = \lambda^\frac{1}{2} \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \tilde{\eta}(v) \\
\equiv \lambda^\frac{1}{2} \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \tilde{\eta}(v) = \lambda^\frac{1}{2} \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \tilde{\eta}(v)
\]

\[
(w^\ast \in (\theta \eta \tilde{\eta}, \eta \tilde{\eta})) = \lambda^\frac{1}{2} \cdot w^\ast \cdot \tilde{\eta}(v) \cdot \tilde{\eta}(v) \\
= \theta(\lambda) \cdot z_\theta.
\]
which finishes the proof.

The proof can also be done diagrammatically, namely \( \lambda^{-1/2}z_0 \) is represented by

\[
\begin{array}{c}
\tau_0 \\
\eta \\
\tilde{t} \\
= \\
\end{array}
\]

and then it can easily be checked that the diagram obtained from \( z_0 \cdot \theta(z_0) \cdot x \) and from \( \eta(x) \cdot z_0 \) can be “deformed” into each other.

We also note that in the case of Longo–Witten endomorphisms the translations \( \tau_t \) and \( \tilde{\tau}_t \) give endomorphisms of \( \mathcal{N} \) and \( \mathcal{M} \), respectively, for \( t \geq 0 \). They fulfill \( \tilde{\tau}_t = i\tau_t \). Therefore \( \tilde{\tau}_t \tilde{\eta} = \tilde{\eta} \tilde{\tau}_t \) implies that \( \tau_t \eta = \eta \tau_t \).

Let \( \tilde{\tau}_t(v) = u(t)^* \cdot v \) then \( \tilde{\eta} = \tilde{\tau}_t \tilde{\eta} \tilde{\tau}_{-t} \) gives

\[
z_0 = \tau_t(\eta(u(-t)^* \cdot z_0) \cdot u(t)^* = \eta(u(-t)^* \cdot \tau_t(z_0) \cdot u(t)^* \text{ for all } t,
\]

namely we compare

\[
\tilde{\tau}_t \tilde{\eta} \tilde{\tau}_{-t}(v) = \tilde{\tau}_t \tilde{\eta}(u(-t)^* \cdot v)
\]

\[
= \tilde{\tau}_t(\eta(u(-t)) \cdot z_0 \cdot v)
\]

\[
= \tau_t(\eta(u(-t)^* \cdot z_0) \cdot u(t)^* \cdot v
\]

\[
\tilde{\eta}(v) = z_0 \cdot v
\]

and we note that \( \tau_t(u(t)^*) = u(t)^* \), more general the cocycle condition \( u(t)\tau_t(u(s)) = u(s + t) \) holds.

**Lemma 6.1.2.** With the former assumption let \( \tilde{\eta} = \text{Ad } \tilde{V} \) with a unitary \( \tilde{V} \), then \( z_0 \) is unitary in \( \mathcal{N} \).

**Proof.** Let \( e \) be the projection on \( \overline{\mathcal{N} \Omega} \), we know that \( \tilde{V} \) and \( e \) commute (see Lemma 4.2.3) and that \( \eta = \text{Ad } V \) is unitarily implemented by \( V = e\tilde{V}e = e\tilde{V} = \tilde{V}e \). The conditional expectation \( E: \mathcal{M} \rightarrow \mathcal{N} \) extends to a conditional expectation \( \tilde{E}: \mathcal{N} \rightarrow \mathcal{N} \) and we can write \( u\eta^{-1}(\mathcal{N})v = \tilde{\eta}^{-1}(\mathcal{M}) \).

Then it is \( \eta^{-1}(z_0) = \tilde{\eta}^{-1}(v) \), i.e. \( \tilde{\eta}^{-1}(v) = \tilde{\eta}^{-1}(z_0^*) \).

It is \( \tilde{\eta}(v) = z_0v \in \tilde{\eta}(\mathcal{M}) \) and hence \( v = \tilde{\eta}^{-1}(z_0v) = \eta^{-1}(z_0^*) \) and \( v = \tilde{\eta}(\tilde{\eta}^{-1}(v)) = z_0^*z_0v \) and we conclude \( z_0 \) is unitary.

The next proposition tells us how one can extend an endomorphism and is similar to \( \alpha \)-induction (cf. \cite{LR95, BE98}).

**Proposition 6.1.3.** Let \( \iota(\mathcal{N}) \subset \mathcal{M} \) a irreducible finite index subfactor (of type III factors) with dual canonical \( Q \)-system \( (\theta, w, x) \). Let \( \eta \) be an endomorphism of \( \mathcal{N} \).

Given a unitary \( z_0 \in (\theta \eta, \eta \theta) \) with condition

\[
z_0 \cdot \theta(z_0) \cdot x = \eta(x) \cdot z_0
\]

there exists an extension \( \tilde{\eta} \) of \( \eta \), i.e. it holds \( \iota \circ \tilde{\eta} = \tilde{\eta} \circ \iota \) given by

\[
\tilde{\eta} = \iota^{-1} \circ \text{Ad } z_0^* \circ \eta \circ \iota.
\]
In particular \( \tilde{\eta}(v) = \iota(z_{\theta}) \cdot v \).

**Proof.** Let \( m = \iota(n)v \in M \) with \( n = \lambda \cdot E(mv^*) \in \mathcal{N} \). We first show that \( \tilde{\eta}(m) \) is well defined, i.e. that \( z_\theta^* \cdot \eta \circ \tilde{i}(m) \cdot z_\theta \equiv z_\theta^* \cdot \eta \circ \theta(n) \cdot z_\theta \cdot z_\theta^* \cdot \eta \circ \tilde{i}(v) \cdot z_\theta \) is in \( \tilde{i}(\mathcal{M}) \). From \( z_\theta \cdot \theta(z_\theta) \cdot x = \eta(x) \cdot z_\theta \) follows

\[
z_\theta^* \cdot \eta \tilde{i}(v) \cdot z_\theta \equiv z_\theta^* \cdot \eta(x) \cdot z_\theta = \theta(z_\theta) \cdot x
\]

and from \( z_\theta \in (\theta \eta, \eta \theta) \)

\[
z_\theta^* \cdot \eta \theta(n) \cdot z_\theta = z_\theta^* \cdot z_\theta \cdot \theta \eta(n) = \theta \eta(n) .
\]

In particular \( z_\theta^* \cdot \eta \tilde{i}(m) \cdot z_\theta = \theta(\eta(n) \cdot z_\theta) \cdot x \equiv \tilde{i}(\iota(\eta(n) \cdot z_\theta) \cdot v) \) and

\[
\tilde{\eta}(m) \equiv \tilde{\eta}(\iota(n)v) = \iota(\eta(n)z_\theta)v .
\]

The above calculation also shows that \( \tilde{\eta} = \iota \tilde{\eta} \) and setting \( m = v \) shows \( \tilde{\eta}(v) = \iota(z_\theta)v \).

**Proposition 6.1.4** (A sufficient condition.). Let \( \{\rho_i\} \) be a system of endomorphisms such that the sectors \( \{\rho_i\} \) are closed under fusion and

\[
[\theta] = \bigoplus_{i=0}^{n} n_i \cdot [\rho_i]
\]

coming from a subfactor with dual \( Q \)-system \((\theta, w, x)\). If there exists a family of unitaries \( z_i \in \text{Hom}(\rho_i \eta, \eta \rho_i) \) such that it holds:

\[
T \in \rho_i \rho_i \rho_i \implies \eta(T) \cdot z_i = z_j \cdot \rho_j (z_k) \cdot T
\]

then exists a unitary \( z_\theta \) with

\[
\eta(x) \cdot z_\theta = z_\theta \cdot \theta(z_\theta) \cdot x
\]

defined by

\[
z_\theta = \sum_{i=0}^{n} \sum_{k=1}^{n_i} \eta(w_{i,k}) \cdot z_i \cdot w_{i,k}^*
\]

where

\[
\theta = \sum_{i=0}^{n} \sum_{k=1}^{n_i} \text{Ad}(w_{i,k}) \circ \rho_i
\]

with \( (w_{i,k})_{k=1,\ldots,n_i} \) a ONB of \((\rho_i, \theta)\). Further if \( \tau_i(\eta(u_0(-t)) \cdot z_i) \cdot u_i(t) = z_i \) holds then also \( \tau_i(\eta(u_0(-t)) \cdot z_i) \cdot u_i(t) = z_\theta \) holds, where \( u_i(t) \rho_i \tau_i = \tau_i \rho_i u_i(t) \) and

\[
u_0(t) = \sum_{i=0}^{n} \sum_{k=1}^{n_i} \tau_i(w_{i,k}) \cdot u_i(t) \cdot w_{i,k}^*
\]
We suppress the second index and the sum over it in $w_{i,k}$ etc. Then it is easily calculated

$$\eta(x) \cdot z_0 = \sum_i \eta(x \cdot w_i) \cdot z_i \cdot w_i^*$$

$$= \sum_{i,k} \eta(w_j \cdot \rho_j(w_k)) \cdot \eta(w_j^* \cdot \theta(w_k^*) \cdot x \cdot w_i) \cdot z_i \cdot w_i^*$$

$$= \sum_{k} \eta(w_j) \cdot z_j \cdot \rho_j(\eta(w_k) \cdot z_k) \cdot w_j^* \cdot \theta(w_k^*) \cdot x \cdot w_i \cdot w_i^*$$

$$= \sum_k \eta(w_j) \cdot z_j \cdot w_j^* \cdot \theta(\eta(w_k) \cdot z_k \cdot w_k^*) \cdot x$$

$$= \sum_k \eta(w_j) \cdot z_j \cdot w_j^* \cdot \theta(\eta(w_k) \cdot z_k \cdot w_k^*) \cdot x$$

$$= z_0 \cdot \theta(z_0) \cdot x .$$

Unitarity follows from unitarity of $z_i$ and from $w_{i,k} w_{j,l} = \delta_{i,j} \delta_{k,l}$ and $\sum_i w_i \cdot w_i^* = 1$.

It further holds:

$$\tau_i(\eta(u_0(-t)) \cdot z_0 \cdot u_0(t)) = \sum_j \tau_i(\eta(\tau_{-i}(w_i)\eta(u_i(-t))\eta(w_i^*)\eta(w_j)z_jw_j^*)\tau_i(w_k)u_k(t))w_k^*$$

$$= \sum_j \eta(w_i)\tau_i(\eta(u_i(-t)) \cdot (z_j))u_j)w_k^*$$

$$= \sum_i \eta(w_i)z_i w_i^*$$

$$= z_0 .$$

\[ \Box \]

In this case $\tilde{\eta}$ acts on the charged intertwiners (see Subsection 2.2.4) $R_{ik}$ by $\tilde{R}_{ik} = (z_i R_{ik})$.

We can now sum up the extension of a Longo–Witten endomorphisms:

**Proposition 6.1.5.** Given an irreducible finite index inclusion of conformal nets $\mathcal{B} \subset \mathcal{A}$ and the $Q$–system of the inclusion $\mathcal{B}(\mathbb{R}_+) \subset \mathcal{A}(\mathbb{R}_+)$ where $\theta$ is a localized endomorphism of $\mathcal{B}$ localized in $\mathbb{R}_+$ and let $\eta$ be a Longo–Witten endomorphism of $\mathcal{B}$. If there is a unitary $z_0 \in (\theta \eta, \eta \theta)$, which fulfills

$$z_0 \cdot \theta(z_0) \cdot x = \eta(x) \cdot z_0$$

$$z_0 = \tau_i(\eta(u(-t)) \cdot z_0) \cdot u(t) = \eta(u(-t)) \cdot \tau_i(z_0) \cdot u(t) .$$

Then there is a Longo–Witten endomorphism $\tilde{\eta}$ of $\mathcal{A}$ given by

$$\tilde{\eta} = \tilde{\eta}^{-1} \circ \text{Ad} z_0^* \circ \eta \circ \tilde{\eta} .$$

Every extension arises in this way.
CHAPTER 7

Longo–Witten unitaries for Framed Vertex Operators

In [LW11] were given elements $V \in \mathcal{E}(\text{Vir}_\frac{1}{2})$ coming from inner symmetric functions $\varphi$ on the upper plane. In [KL06] a family of conformal nets with $c = n/2$ was constructed. The nets are finite index extensions of $\text{Vir}_\frac{1}{2} \otimes \text{Vir}_n$ for suitable $n$. If $n \geq 16$ there are examples which are not orbifolds (i.e. fixed points under a finite group) of free Fermionic nets, e.g. the conformal net associated with the $E_8$ ($n = 16$), the Leech lattice ($n = 48$) and the Moonshine net ($n = 48$) which is obtained by a twisted orbifold construction of the former. The question which arises is if and which of the Longo–Witten unitaries $V \in \mathcal{E}(\text{Vir}_\frac{1}{2} \otimes \text{Vir}_n)$ can be extend to the Longo–Witten unitaries of the bigger net using the techniques from Chapter 6. The orbifolds of free Fermionic nets are treated in Section 5.3. For the nets extending $\text{Vir}_\frac{1}{2} \otimes \text{Vir}_n$ which are not of this type, the problem is more complicated, because the extension includes the Ramond sector of the Ising model. Basically, the problem reduces to extending the endomorphism to this sector. This problem can be reduced to a problem on the one-particle space. We give some abstract criterion.

7.1. The real Fermion net and the Ising model

For the real Fermion it is convenient to use Araki’s self-dual CAR algebra (see Subsection 2.4.3). The real Fermion and the Ising model and its localized endomorphisms are studied in [Böc96b]. Let $\Gamma$ denote the complex conjugation on $\mathcal{K} := L^2(S^1)$. And let us consider the self-dual CAR algebra $\text{CAR}(\mathcal{K}, \Gamma)$.

**Definition 7.1.1.** A Bogoliubov operator $W$ is called **pseudolocalized** in $I \in I_\xi$ if for all $f \in L^2(S^1)$ it holds $(Wf)(z) = \sigma_\pm f(z)$ for $z \in I_\pm$ and $\sigma_\pm \in \{-1, 1\}$, where $I_\pm \in I_\xi$ such that $I_+ \cup I_- = I' \cap S^1 \setminus \{\xi\}$. If $\sigma_+ = \sigma_-$ then $W$ is called **even** and in the case $\sigma_+ = -\sigma_-$ **odd**.

A pseudolocalized operator induces a localized endomorphism of the even algebra.

**Definition 7.1.2** (Neveu–Schwarz/Ramond operator cf. [Böc96b (51)]). We define the Neveu–Schwarz polarization $P_{\text{NS}}$ and the Ramond operator $S_R$ by

$$P_{\text{NS}} = \sum_{r \in \mathbb{N}_0 + \frac{1}{2}} |e_{-r}\rangle \langle e_{-r}|$$

$$S_R = \frac{1}{2} |e_0\rangle \langle e_0| + \sum_{n \in \mathbb{N}} |e_{-n}\rangle \langle e_{-n}|$$

and both fulfill $\Gamma S \Gamma + S = 1$. 

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The Fermion algebra has a faithful and irreducible representation \( \pi_{NS} \) on the antisymmetric Fock space \( \Lambda(P_{NS}K) \), characterized by the state \( \omega_{NS} \) with

\[
\omega_{NS}(\psi(e_r)^*\psi(e_r)) = (e_r, P_{NS}e_r) = 0, \quad r \in \mathbb{N}_0 + \frac{1}{2},
\]

in particular \( \pi_{NS}(\psi(e_r))\Omega = 0 \) for \( r \in \mathbb{N}_0 + \frac{1}{2} \). We define two nets, where we by abuse of notation denote \( \pi_{NS}(\psi(f)) \) also by \( \psi(f) \):

\[
\begin{align*}
\mathcal{H}_{\text{Fer}} & := \Lambda(P_{NS}L^2(\mathbb{S}^1)) \\
\mathcal{H}_{\text{Vir}} & := \Lambda(P_{NS}L^2(\mathbb{S}^1)) \\
\mathcal{H}_{\text{Vir}_\frac{1}{2}} & := \Lambda(P_{NS}L^2(\mathbb{S}^1))^\mathbb{Z}_2.
\end{align*}
\]

There is a unitary action \( U_1 \) of \( \mathbb{Z}_2 \) on \( \mathcal{H}_{\text{Vir}} \) which restricts to an action of \( \mathbb{Z}_2 \) on \( \mathcal{H}_{\text{Vir}_\frac{1}{2}} \) and makes \( \mathcal{H}_{\text{Vir}_\frac{1}{2}} \) a conformal net. The net \( \mathcal{H}_{\text{Vir}_\frac{1}{2}} \) is called the \textbf{Ising net} and is isomorphic to the \textbf{Virasoro net} with \( c = 1/2 \) (cf. [MS90]). Let \( I \in \mathbb{Z} \) and define the \textbf{bi-fields} \( B_I(f, g) = \psi(f)\psi(g) \) with \( \mathrm{supp} \, f, \mathrm{supp} \, g \subset I \). The \( B_I \) is complex linear in both arguments and fulfills:

\[
\begin{align*}
2B_I(f, f) &= (\Gamma f, f) \cdot 1 \\
2B_I(f, g)B_I(g, h) &= (\Gamma g, g)B_I(f, h) \\
B_I(f, g)^* &= B_I(\Gamma g, \Gamma f).
\end{align*}
\]

The algebra \( \mathcal{H}_{\text{Vir}_\frac{1}{2}} \) can easily be seen to equal the von Neumann algebra generated by all bi-fields \( B_I(f, g) \) with \( f, g \in L^2(I) \subset L^2(\mathbb{S}^1) \).

7.1.1. \textbf{Localized endomorphisms.} The localized endomorphisms of \( \mathcal{H}_{\text{Vir}_\frac{1}{2}} \) are given by restrictions of Bogoliubov endomorphisms of \( \mathcal{H}_{\text{Vir}} \). The net \( \mathcal{H}_{\text{Vir}} \) is completely rational and the \( \mu \)-index of this net is 4. It has 3 sectors with conformal weights 0, \( 1/16 \) and \( 1/2 \) (cf. [Böc96a, KL06]). We sum up the properties of the representation and the fusion rules in Table 1 and 2.

<table>
<thead>
<tr>
<th>sector</th>
<th>conformal weight</th>
<th>statistical dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>vacuum</td>
<td>( \rho_0 )</td>
<td>0</td>
</tr>
<tr>
<td>Ramond type</td>
<td>( \rho_{\frac{1}{16}} )</td>
<td>( \frac{1}{16} )</td>
</tr>
<tr>
<td>Fermionic</td>
<td>( \rho_{\frac{1}{2}} )</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

Table 1. Sectors of \( \mathcal{H}_{\text{Vir}_\frac{1}{2}} \)

The Fermionic sector \( \rho_{\frac{1}{2}} \) is defined as the restriction of the Bogoliubov endomorphism \( \rho_W \) with Bogoliubov operator \( W = |h\rangle\langle h| - 1 \), where \( h = \Gamma h \) with \( ||h||^2 = 2 \) and \( \mathrm{supp} \, h \subset I \). We have
\[
\begin{array}{c|cc}
\times & 0 & \frac{1}{16} & \frac{1}{4} \\
\hline
0 & 0 & \frac{1}{16} & \frac{1}{4} \\
\frac{1}{16} & \frac{1}{16} & 0 + \frac{1}{2} & \frac{1}{16} \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 \\
\end{array}
\]

Table 2. Fusion rules of \( \text{Vir}_{\frac{1}{2}} \)

\[
\psi(h) = \psi(\Gamma h) = \psi(h)^* \text{ and } \psi(h)^2 = 1,
\]
which implies

\[
\begin{align*}
\psi(h)\psi(f)\psi(h)^* &= \{\psi(h), \psi(f)\} - \psi(f)\psi(h)\psi(h) \\
&= (h, f)\psi(h) - \psi(f) \\
&= \psi([h| (h-1)f) \\
&= \psi(Wf)
\end{align*}
\]

and in particular \( \text{Ad}_{\psi(h)} = \rho_{\frac{1}{2}} \) so \( \rho_{\frac{1}{2}} \) is implemented on the Fock space by \( \psi(h) \) and localized in \( I \).

We define the localized endomorphisms for the Ramond sector \( \rho_{\frac{1}{2}} \). For this we split the circle in three parts

\[
I_2 = e^{i(-\frac{\pi}{2}, \frac{\pi}{2})} \subseteq \mathbb{R} \\
I_- = e^{i(-\infty, -1)} \subseteq \mathbb{R} \\
I_+ = e^{i(1, \infty)} \subseteq \mathbb{R}
\]

and in particular we construct \( \rho_{\frac{1}{2}} \) localized in \( I_2 \) and we restrict the net to \( S^1 \setminus \{ -1 \} \subseteq \mathbb{R} \). We define an orthonormal basis of \( L^2(I_2) \) by

\[
e^{(2)}_\alpha(z) = \begin{cases} 
\sqrt{2}z^{2\alpha} & z \in I_2 \\
0 & z \in S^1 \setminus I_2
\end{cases}, \quad \alpha \in \frac{1}{2}\mathbb{Z}.
\]

Let \( P_{I_+}, P_{I_-} \) be the projections on \( L^2(I_+) \) and \( L^2(I_-) \), respectively. Then we define Bogoliubov operators by

\[
W = P_{I_-} - P_{I_+} + \frac{i}{\sqrt{2}} \left( \left| e_{n+\frac{1}{2}}^{(2)} \right\rangle \left\langle e_0^{(2)} \right| - \left| e_{n-\frac{1}{2}}^{(2)} \right\rangle \left\langle e_0^{(2)} \right| \right),
\]

\[
+ i \sum_{n=1}^{\infty} \left( \left| e_{n+\frac{1}{2}}^{(2)} \right\rangle \left\langle e_n^{(2)} \right| - \left| e_{n-\frac{1}{2}}^{(2)} \right\rangle \left\langle e_{n-1}^{(2)} \right| \right) \\
W' = P_{I_-} - P_{I_+} + i \sum_{n=1}^{\infty} \left( \left| e_{n+\frac{1}{2}}^{(2)} \right\rangle \left\langle e_n^{(2)} \right| - \left| e_{n-\frac{1}{2}}^{(2)} \right\rangle \left\langle e_{n-1}^{(2)} \right| \right).
\]

The localized endomorphism is finally given by the restriction \( \rho_W \) to the even part of \( \rho_W \) or, equivalently using \( W' \) instead of \( W \).

**Lemma 7.1.3** ([Böc96b, Lemma 3.10.]). \( W^* P_{NS} W - S_R \) is Hilbert–Schmidt.
7.1.2. Longo–Witten unitaries for $\text{Vir}_{\frac{1}{2}}$. Every $\varphi$ inner symmetric function (cf. [LW11]) gives a Longo–Witten unitary by the Bogoliubov operator $V = \varphi(P_1)$ with $U(\tau(t)) = e^{itP_1}$ (translations) on $\mathcal{K}$. Because $U(\tau(t))$ commutes with the basis projection $P_{\NS}$ also $V$ do and therefore $V$ restricts to a unitary on $P_{\NS}\mathcal{K}$ and $V$ is canonical quantized by $V = \Lambda(V_1)$. The restriction to the even part $V^+$ gives a Longo–Witten unitary $V^+ \in \mathcal{E}(\text{Vir}_{\frac{1}{2}})$. 

7.1.3. Cocycles for second quantization Longo–Witten unitaries of $\text{Vir}_{\frac{1}{2}}$. Let us fix some $h = \Gamma h$, $\|h\| = 2$ with supp $h \subset \mathbb{R}_+$, $U = \psi(h)$. For a second quantization unitary $\Lambda(V)$ we denote the restriction to the even space $\mathcal{H}_{\text{Vir}_{\frac{1}{2}}}$ by $V_1$ and $\eta = \text{Ad} V_1$. Note that for $V \in \mathcal{E}(P_{\NS}L^2(\mathbb{R}_+))$ this defines a Longo–Witten endomorphism of $\text{Vir}_{\frac{1}{2}}(\mathbb{R}_+)$, we assume this from now on.

7.1.4. Cocycles for the Fermionic sector. This section is basically warm-up and everything is more or less trivial. Namely, we give an extension of Longo–Witten endomorphisms defined of the even part of the CAR algebra (the Ising net) to the full one, but the Longo–Witten endomorphisms were originally defined on the CAR algebra, so we just recover the original one.

**Lemma 7.1.4.** Let $z = B(Vh, h) \in \text{Vir}_{\frac{1}{2}}(\mathbb{R}_+)$ and for $t \in \mathbb{R}$ let $u_t = B(h_t, h)$. Then holds:

1. $z \in \text{Hom}_{\text{Vir}_{\frac{1}{2}}(\mathbb{R}_+)}(\rho_{1/2} \eta, \eta \rho_{1/2})$.
2. $\text{Ad} u_t \circ \rho_{1/2} = \tau_t \circ \rho_{1/2} \circ \tau_{-t}$, in other words $u_t \in \text{Hom}_{\text{Vir}_{\frac{1}{2}}(\mathbb{R}_+ + \text{min}(0, t))}(\rho_{1/2}, \tau_t \rho_{1/2} \tau_{-t})$
3. $z \rho_{1/2}(\eta(\psi(g))) = \psi(Vh)\psi(h)\psi(h)\psi(g)V^*\psi(h)$
4. $z u_t^* = \eta(u_t^*)\tau_t(z)$.

**Proof.** Here we denote the Bogoliubov operator and the implementing unitary both by $V$. Then we show (1) by

$$z \rho_{1/2}(\eta(\psi(g))) = \psi(Vh)\psi(h)\psi(h)\psi(g)V^*\psi(h)$$

$$= \psi(Vh)V\psi(h)\psi(h)$$

$$= V\psi(h)V^*\psi(h)\psi(h)$$

$$= V\psi(h)\psi(h)\psi(h)\psi(h)V^*\psi(h)$$

$$= V\psi(h)\psi(h)\psi(h)\psi(h)V^*\psi(h)$$

$$= \eta(\rho_{1/2}(\psi(g)))z$$

and (2) goes analogously. (3) follows from

$$z \rho_{1/2}(\eta(\psi(g))) = B(Vh, h)\psi(h)B(Vh, h)\psi(h)$$

$$= \psi(Vh)\psi(h)\psi(h)\psi(Vh)\psi(h)\psi(h)$$

$$= \psi(Vh)\psi(Vh)$$

$$= 1,$$

because $\Gamma V = \Gamma V$. The lhs of (4) reads $B(Vh, h)B(h, h_t) = \frac{1}{2}(h, h)B(Vh, h_t) = B(Vh, h_t)$ and the rhs $\eta(B(h, h_t)\tau_t(B(Vh, h)) = B(Vh, h_t)B(h_t, h) = \frac{1}{2}(h_t, h_t)B(Vh, h_t) = B(Vh, h_t)$ and therefore holds equality. □
There is another way to show implicitly that these cocycles exist. Namely, we define as Bogoliubov operators by

$$Z = V(WV^* W^*) = \eta(W)W^*,$$

where as before $h = \Gamma h$ with $||h||^2 = 2$ and $W_{\frac{1}{2}} = |h\rangle\langle h| - 1 = W^*_{\frac{1}{2}}$.

It holds $\rho_{Z} \rho_{W_{\frac{1}{2}}} \eta(x) = \eta \rho_{W_{\frac{1}{2}}}(x)$, namely

$$\rho_{Z} \rho_{W_{\frac{1}{2}}} \eta(\psi(f)) = \psi(\eta Z W_{\frac{1}{2}} V f)$$

$$= \psi(V(W_{\frac{1}{2}}V^* W^*_{\frac{1}{2}}) W_{\frac{1}{2}} V f)$$

$$= \psi(VW_{\frac{1}{2}} V^* W^*_{\frac{1}{2}}) V f)$$

$$= \psi(VW_{\frac{1}{2}} f)$$

$$= \eta \rho_{W_{\frac{1}{2}}} \eta(\psi(f)).$$

We check that the expression

$$W^* P_{NS} W_{\frac{1}{2}} - V^* W^*_{\frac{1}{2}} P_{NS} W_{\frac{1}{2}} V = |h\rangle\langle h| P |h\rangle\langle h| - |h\rangle\langle h| P - P |h\rangle\langle h| + P$$

$$- V^* (|h\rangle\langle h| P |h\rangle\langle h| - \{ |h\rangle\langle h| , P \} + P) V^*$$

$$= (h, Ph)(|h\rangle\langle h| - |V^* h\rangle\langle V^* h|)$$

$$= (h, Ph)(|h\rangle\langle h| - |V^* h\rangle\langle V^* h| , P)$$

$$= X - \{ |h\rangle\langle h| - |V^* h\rangle\langle V^* h| , P \}$$

is Hilbert–Schmidt, namely the anticommutator $\{ \cdots , P \}$ is as a finite sum of rank one operators and obviously Hilbert–Schmidt. $X$ is Hilbert–Schmidt because the integral kernel is in $L^{2}(\mathbb{R}^2, d\rho dq)$, namely

$$K_X(p, q) = (h, Ph) \cdot \hat{h}(p) \hat{h}(q) (1 - \varphi(p)\varphi(q)) \in L^{2}(\mathbb{R}^2, d\rho dq) \bigoplus_{L^{2}(\mathbb{R}^2, d\rho dq)}$$

So by the Shale–Stinespring quantization criterion there is a unitary $Q_{Z}$, such that $\rho_{Z}(x) = \text{Ad} \ Q_{Z}(x)$. We denote by $z_{\frac{1}{2}}$ the restriction of $z$ to the even space and note that $z_{\frac{1}{2}} \in \langle \rho_{\frac{1}{2}}, \eta, \eta \rangle$ holds.

### 7.1.5. Cocycle for the Ramond sector

We want to construct similar cocycles for the Ramond sector. This is much more involved because the Ramond state is naturally expressed in the circle picture, while the Longo–Witten unitaries have a natural action in the real line picture.

For the Bogoliubov operators holds

$$W^* W = 1 \quad \text{and} \quad W W^* = 1 - \frac{1}{2} \left| e_{0}^{(2)} + e_{-\frac{1}{2}}^{(2)} \right|$$

$$W^* W' = 1 \quad \text{and} \quad W' W^* = 1 - \left| e_{0}^{(2)} \right|^2.$$
Let us now denote by \( x = 1/\sqrt{2}(e_{1/2}^{(2)} + e_{-1/2}^{(2)}) \) or \( x' = e_{0}^{(2)} \). We define
\[
Z = V(WV^*W^* + |x\rangle\langle x|) = \eta(W)W^* + V|x\rangle\langle x|
\]
\[
Z' = V(WV^*W^* + |x'\rangle\langle x'|) = \eta(W'W^* + V|x'\rangle\langle x'|
\]
with \( \eta = \text{Ad } V \). Note that \( \ker W^* = Cx \) and \( \ker W'^* = Cx' \) and that \( \text{Im } W = \mathcal{H} \oplus Cx \) and \( \text{Im } W' = \mathcal{H} \oplus Cx' \). We claim that \( Z, Z' \) are unitaries, namely
\[
ZZ^* = V(WV^*W^* + |x\rangle\langle x|)(WVW^* + |x\rangle\langle x|)V^* = V(WV^*W^*WVW^* + |x\rangle\langle x|)V^* = V(WW^* + |x\rangle\langle x|)V^* = 1
\]
\[
Z'Z = (WVW^* + |x\rangle\langle x|)V^*V(WVW^* + |x\rangle\langle x|) = (WVW^* + |x\rangle\langle x|)(WV^*W^* + |x\rangle\langle x|) = (WVW^*WV^*W^* + |x\rangle\langle x|) = (WW^* + |x\rangle\langle x|) = 1.
\]

It holds \( \rho_Z \rho_W \eta(x) = \eta \rho_W(x) \), namely
\[
\rho_Z \rho_W \eta(f) = \eta(\rho_W(f)) = \psi(ZWVf) = \psi(V(WV^*W^* + |x\rangle\langle x|)WVf) = \psi(VV^*W^*WVf) = \psi(VWf) = \eta \rho_W(\psi(f))
\]
where we also showed, that \( ZWV = VW \) holds.

**Lemma 7.1.5.** For all \( f \in L^2(\mathbb{S}^1) \) we have \( Zf(z) = f(z) \) for \( z \in I_- \equiv (-\infty, -1) \subset \mathbb{R} \).

**Proof.** We claim that \( Zf(z) = f(z) \) for \( z \in I_- \). Let us write \( L^2(\mathbb{S}^1) = L^2(I_-) \oplus L^2(I'_-) \). Then by \( VL^2(I'_-) \subset L^2(I'_-) \) and \( Wf(z) = f(z) \) for \( z \in I_- \) we have
\[
V = \begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 0 & W_{22} \end{pmatrix}
\]
\[
VV^* = \begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{21}^* & V_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
and we calculate
\[
VV^*W^* = \begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W_{22} \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{21}^* & V_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & |x\rangle\langle x| \end{pmatrix}
\]
in particular \( Zf(z) = f(z) \) for \( z \in I_- \). \( \square \)

**Proposition 7.1.6.** If \( S_R V^*(1 - S_R) \) is Hilbert–Schmidt, then exists a \( Q_Z \in \text{Vir}_{1/2}(\mathbb{R}_+) \) with:
\[
Q_Z \in \text{Hom}(\rho_{1/16}, \eta \rho_{1/16})
\]
A characterization for which inner functions the Hilbert–Schmidt condition is fulfilled is still an open problem. We expect a similar result as in the case of the extensions of Abelian currents, namely that the inner function \( \varphi \) has to be regular (in some sense) at \( p = 0 \) and leave this as an open conjecture.

### 7.2. The net \( \mathcal{A} \) and the framed vertex operator construction

Now we want to apply our results to understand how to extend Longo–Witten endomorphisms from tensor products of the Ising model to the nets constructed in [KL06]. We start with the Virasoro net with \( c = 1/2 \), i.e. the Ising net denoted by Vir\(_{\frac{1}{2}}\). We remember that this net is completely rational, has \( \mu \)-index 4, and 3 sectors with conformal weights 0, 1/16 and 1/2.

Let us consider the net Vir\(_{\frac{1}{2}} \otimes \text{Vir}_{\frac{1}{2}}\). This net is also completely rational with \( \mu \)-index 16. It has 9 irreducible sectors, since each sector is a tensor product of two sectors of Vir\(_{\frac{1}{2}}\). We label these sectors by

\[
\lambda_{0,0}, \lambda_{0,1}, \lambda_{\frac{1}{16}}, \lambda_{\frac{1}{8}}, \lambda_{\frac{2}{3}}, \lambda_{\frac{1}{16}}, \lambda_{\frac{1}{8}}, \lambda_{\frac{2}{3}}, \lambda_{\frac{1}{16}}, \lambda_{\frac{1}{8}}, \lambda_{\frac{2}{3}}.
\]

We can apply the simple current extension to the net Vir\(_{\frac{1}{2}} \otimes \text{Vir}_{\frac{1}{2}}\) and the sectors \( \{\lambda_{0,0}, \lambda_{\frac{1}{16}}\} \) with \( G = \mathbb{Z}_2 \). The obtained net is denote by \( \mathcal{A} \) and just equals the extension of Vir\(_{\frac{1}{2}} \otimes \text{Vir}_{\frac{1}{2}}\) with the field \( \psi_1 \psi_2 \) and is isomorphic to Fer\(_{-}\), the even part of the complex Fermion net Fer\(_{\mathbb{C}}\).

The dual canonical endomorphism \( \theta \) of the subfactor \( (\text{Vir}_{\frac{1}{2}} \otimes \text{Vir}_{\frac{1}{2}})(I) \subset \mathcal{A}(I) \) for some \( I \in \mathcal{I} \) is given by \( [\theta] = [\text{id}] \oplus [\lambda_{\frac{1}{2}}] \). The net \( \mathcal{A} \) has \( \mu \)-index 4, which follows directly from Proposition 7.2.1.

**Proposition 7.2.1** ([KLM01, Proposition 24.]). Let \( \mathcal{A} \subset \mathcal{B} \) be an inclusion of completely rational conformal nets on \( S^1 \). Then it holds

\[
\mu_{\mathcal{A}} = [\mathcal{B} : \mathcal{A}]^2 \mu_{\mathcal{B}}.
\]

The sectors of \( \mathcal{A} \) are obtained by \( \alpha \)-induction from sectors of Vir\(_{\frac{1}{2}} \otimes \text{Vir}_{\frac{1}{2}}\). The following is shown in [KL06].

**Proposition 7.2.2** (DHR sectors of the net \( \mathcal{A} \)). The net has 4 DHR sectors \( \{\lambda_{0}, \lambda_{\frac{1}{2}}, \lambda_{1}, \lambda_{-\frac{1}{2}}\} \), i.e. \( \{\lambda_{\mu}\}_{\mu \in \frac{1}{2}\mathbb{Z}/2\mathbb{Z}} \) with fusion rules given by \( \frac{1}{2}\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_4 \).

We have the identification (see e.g. [BJL02])

\[
\text{CAR}(K_1, \Gamma_1) \otimes \text{CAR}(K_2, \Gamma_2) \rightarrow \text{CAR}(K_1 \oplus K_2, \Gamma_1 \oplus \Gamma_2)
\]

\[
A(f) \otimes 1 \mapsto A(f \oplus 0)
\]

\[
1 \otimes A(g) \mapsto A(0 \oplus g),
\]
i.e. $\text{Fer}_C$ can identify the twisted tensor product of two copies of $\text{Fer}_R$:

$$\text{Fer}_R \otimes \text{Fer}_R \longrightarrow \text{Fer}_C$$

where $\psi_1(f) = \psi(f) \otimes 1$ and $\psi_2(f) = 1 \otimes \psi(f)$, in particular $\{\psi_1, \psi_2\} = 0$.

We need the following classification result. A diffeomorphism covariant conformal net $\mathcal{A}$ with central charge $c = 1$ is said to satisfy the **spectrum condition** if a degenerate representation of the Virasoro net other than the vacuum representation appears in the vacuum representation of $\mathcal{A}$ if $\mathcal{A} \neq \text{Vir}_{c=1}$.

If $\mathcal{A} \supset \text{Vir}_{c=1}$ fulfills the spectrum condition, then it is one out of the following list ([Xu05]):

- $\mathcal{A}_{\text{SU}(2),1}^G$ where $G \subset SO(3)$ is a closed subgroup.
- $\mathcal{A}_{\sqrt{n} \mathbb{Z}} \equiv \mathcal{A}_{\text{U}(1),2n}$ where $n$ is not the square of an integer. The fusion rules are $\mathbb{Z}_{2n}$ with sectors $\{\pi_{i,j}\}_{i,j \in \mathbb{Z}_{2n}}$ and $d_i = 1$. Note that $\mathcal{A}_{\text{U}(1),2} = \mathcal{A}_{\text{SU}(2),1}$.
- $\mathcal{A}_{\sqrt{n} \mathbb{Z}} \equiv \mathcal{A}_{\text{U}(1),2n}$ where $n$ is not the square of an integer.
- $\mathcal{A}_{\sqrt{n} \mathbb{Z}} \equiv \mathcal{A}_{\text{U}(1),2n}$ where $n$ is not the square of an integer.

We note that $\mathcal{A}_{\text{SU}(2),1}^{(1)}$ equals the $\text{U}(1)$–current net $\mathcal{A}_{\mathbb{R}}$.

The following is an open problem in [KL06].

**Proposition 7.2.3.** The net $\mathcal{A}$ equals the net $\text{Fer}_C^{\mathbb{Z}_2}$ and the net $\mathcal{A}_{\mathbb{Z}}$ associated with the even lattice $2\mathbb{Z}$ (corresponding to $N = 2$ in [BMT88]).

**Proof.** By construction the net is the even part of $\text{Fer}_C$. We can use the argument in [KL06] using the classification of Xu of nets with central charge $c = 1$. What was missing was prove that the spectrum condition holds. But in Proposition 5.1.10 we proved that the $\text{U}(1)$–current net $\mathcal{A}_{\mathbb{R}}$ is the $\text{U}(1)$–current net of $\text{Fer}_C$ so we get $\mathcal{A}_{\mathbb{R}} \subset \mathcal{A}$ and $\mathcal{A}$ fulfills in particular the spectrum condition because $\mathcal{A}_{\mathbb{R}}$ fulfills it or one can look directly the proof to get that $\mathcal{A}_{\text{U}(1)_4}$ in Xu’s notation is the only extension of $\mathcal{A}_{\mathbb{R}}$ having the asked fusion rules. 

With the identification $\mathcal{A} \equiv \text{Fer}_C^{\mathbb{Z}_2}$ is the fixed point under the automorphism $\alpha(\cdot) = \rho_{-1}(\cdot)$. Further we can identify Bogoliubov automorphism via

$$\rho_V \otimes \rho_V \equiv \rho_V \otimes V,$$

where $V \otimes V$ acts on $K \oplus K$.

Let $\eta = \rho_V^*$ be a Longo–Witten endomorphism, i.e. the restriction of $\rho_V$ to the even part with

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

that means that for $f \in L^2(\mathbb{R}) \equiv K$ (here we use the real line parametrization of the one-particle space):

$$\overline{V_{ij} f(p)} = \varphi(p) \hat{f}(p),$$

where $\varphi_{ij}$ is boundary value of a bounded analytic function on $\mathbb{R} + i\mathbb{R}_+$ with $\varphi_{ij}(-p) = \overline{\varphi_{ij}(p)}$ and $(\varphi_{ij}(p))$ is a unitary $(2 \times 2)$–matrix for almost all $p > 0$. Further let $\beta = \rho_W^*$ with $W = W \oplus W$. We know that restriction to the even space $\beta^\tau \equiv [\lambda_{-1}] \oplus [\lambda_{-1}]$ decompose in the two Ramond like representations of $\mathcal{A}$. The Fermionic sector $\lambda_1$ is given by the restriction of $\rho_W$ with $W_1 := (|h\rangle\langle h| - 1) \oplus 1$. We note
in the diagonal case $V = \text{diag}(V_1, V_2)$ we get the same Longo–Witten endomorphism then starting with $\check{V}_i \in \mathcal{E}(\text{Vir}_{1/2})$ coming from the restriction to the even part of $\Lambda(V_i)$ and extending $\check{V}_1 \otimes \check{V}_2 \in \mathcal{E}(\text{Vir}_{1/2} \otimes \text{Vir}_{1/2})$ to a unitary in $\mathcal{E}(A)$.

Similar like before we define

$$Z = V(\check{W}V^*\check{W}^* + P)$$

with $P = 1 - \check{W}\check{W}^*$ the projection on the kernel of $\check{W}^*$.

Let us assume that $P_{NS}Z(1 - P_{NS})$ is Hilbert–Schmidt. Then exists a unitary $z := Q_Z \in (\beta\eta, \eta\beta)$ as above.

We show that there also exist unitaries $z_{\pm \frac{1}{2}} \in (\lambda_{\pm \frac{1}{2}}, \eta, \eta\lambda_{\pm \frac{1}{2}})$. Let $w_{\pm} \in (\lambda_{\pm \frac{1}{2}}, \beta)$ be isometries. By inequivalence of $\lambda_{\pm \frac{1}{2}}$, they are orthonormal, i.e. $w_{\pm}^*w_{\pm} = \delta_{ij} \cdot 1$ and it holds $f_+ + f_- = 1$ with $f_{\pm} = w_{\pm}w_{\pm}^*$, i.e. $\beta = \text{Ad} w_+ \circ \lambda_{\pm \frac{1}{2}} + \text{Ad} w_- \circ \lambda_{\pm \frac{1}{2}}$.

We define

$$z_{\pm \frac{1}{2}} = \eta(w_{\pm})^* \cdot z \cdot w_{\pm} \in (\lambda_{\pm \frac{1}{2}}, \eta, \eta\lambda_{\pm \frac{1}{2}}).$$

The finite dimensional $C^*$-algebra $C := \beta(A(1^\prime)) \cap \Lambda(A(1^\prime))$ is spanned by $\{f_{\pm}\}$ and is two dimensional. It is straightforward to check that an algebra spanned by the projections $\psi(e_{\pm})\psi(e_{\mp})$ with $\{e_{\pm}\}$ an orthonormal basis of $\ker W^*$ satisfying $\Gamma e_{\pm} = e_{\mp}$ is a two dimensional subalgebra of $\Lambda(A(1^\prime))$ commuting with $\beta(A(1^\prime))$ if and therefore equals $C$. We may and do choose $e_{\pm}$ such that $f_{\pm} = \psi(e_{\pm})\psi(e_{\mp})$. Because $e_{\pm} \in \ker W^*$ it is

$$Ze_{\pm} = V(WV^*W^* + 1 - WW^*) = Ve_{\pm}$$

and therefore $z \cdot f_{\pm} \equiv \rho_j(f_{\pm}) \cdot z = \eta(f_{\pm}) \cdot z$. We note that this implies $\eta(f_{\pm})^* \cdot z \cdot f_{\mp} = 0$ and therefore $\eta(w_{\pm})^* \cdot z \cdot w_{\mp} = 0$; and that $\eta(f_{\pm})^* \cdot z \cdot f_{\mp} = \eta(f_{\pm}) \cdot z$ gives by restriction a unitary $f_{\pm} \mathcal{H}_A \to \eta(f_{\pm})\mathcal{H}_A$. Therefore $z_{\pm \frac{1}{2}}$ is unitary.

**Proposition 7.2.4 (Properties of $(z_1, z_{\pm \frac{1}{2}})$).** For $i \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\}$ and $\rho_i$ localized endomorphisms of $A$ localized in $\mathbb{R}_+$ we have unitaries $z_i \in \text{Hom}_{A(\mathbb{R})}(\rho_i, \eta, \rho_i)$ such that for $T \in (\rho_i, \rho_j, \rho_k)$ it holds

$$\eta(T) \cdot z_i = z_j \cdot \rho_j(z_k) \cdot T,$$

$$\tau_i(\eta(u_i(-t)) \cdot z_i) = z_i.$$

**Proof.** We note that $z_1$ is constructed in the same way as for the Fermionic sector of $\text{Vir}_{1/2}$. The operators are all implementing Bogoliubov endomorphisms and are unique up to a phase, so the properties can be checked on the one-particle space. \hfill $\square$

### 7.3. Lattice extensions of $A^{\otimes n}$

We consider the net $A^{\otimes n}$. The sectors and fusion rules are given by the group

$$\left(\frac{1}{2}\mathbb{Z}/2\mathbb{Z}\right)^n \cong (\mathbb{Z}_4)^n$$

in the obvious identification. Let us consider an even lattice $L \supset (2\mathbb{Z})^n$ of rank $n$. It is integral and therefore it follows $(2\mathbb{Z})^n \subset L \subset \left(\frac{1}{2}\mathbb{Z}\right)^n$. Let us denote by $A_L$ the simple current extension of $A^{\otimes n}$ by the subgroup $G_L := L/(2\mathbb{Z})^n \subset \left(\frac{1}{2}\mathbb{Z}/2\mathbb{Z}\right)^n \cong (\mathbb{Z}_4)^n$. Note that $A_L$ is local because $h_L = \frac{1}{2}g^2$ (mod 4)
is integer for all \( g \in G_L \) by the assumption that \( L \) is even. The dual canonical endomorphism \( \theta_L \) of the inclusion \( \mathcal{A}^{\otimes n}(I) \subset \mathcal{A}_L(I) \) is given by

\[
[\theta_L] = \bigoplus_{g \in G_L} [\lambda_{g_1}] \otimes \cdots \otimes [\lambda_{g_n}], \quad g = (g_1, \ldots, g_n) \text{ with } g_i \in \frac{1}{2} \mathbb{Z}/2\mathbb{Z}.
\]

The question is which \( V_1 \otimes \cdots \otimes V_n \in \mathcal{E}(\mathcal{A}^{\otimes n}) \) coming from second quantization extend to \( \mathcal{A}_L \). This problem is trivial if \( L \setminus \mathbb{Z}^n = \emptyset \) (in this case it is just a fixed point net of a Fermionic net) and we already gave the answer in Section 5.3. The interesting case is if \( L \) contains products of Ramond sectors \( \lambda_{\pm \frac{1}{2}} \). Note that because \( h_{\lambda_{\pm \frac{1}{2}}} = \frac{1}{8} \) from locality—or more abstractly from evenness of \( L \)—follows that \( n \geq 8 \). An example is the \( E_8 \) and the Leech lattice.

**Example.** The \( E_8 \) lattice can be parametrized as subgroup of \( \left( \frac{1}{2} \mathbb{Z} \right)^8 \subset \mathbb{R}^8 \) by

\[
\Gamma_8 = \left\{ x \in \left( \frac{1}{2} \mathbb{Z} \right)^8 : \sum_{i=1}^{8} x_i = 0 \text{ mod 2} \right\} \quad \text{or}
\]

\[
\Gamma_8 = \left\{ x \in \mathbb{Z}^8 : \sum_{i=1}^{8} x_i = 0 \text{ mod 2} \right\} \cup \left\{ x \in (\mathbb{Z} + 1)^8 : \sum_{i=1}^{8} x_i = 1 \text{ mod 2} \right\}.
\]

The problem of extending a \( V \in \mathcal{E}(\mathcal{A}^{\otimes n}) \) to some \( \tilde{V} \in \mathcal{E}(\mathcal{A}_L) \) reduces by Proposition 6.1.4 to find cocycles \( z_g \in \mathcal{A}^{\otimes n}(\mathbb{R}_+) \) with \( z : \lambda_g \eta \to \eta \lambda_g \) for \( g \in G_L \), where \( \psi_g \in \mathcal{A}_L(\mathbb{R}_+) \) are charged intertwiners \( \psi_g x = \lambda_g(x) \psi_g \), which fulfills asked the properties. Because of the tensor structure it reduces to the problem of finding cocycles for the net \( \mathcal{A} \).

We can sum the construction up:

**Proposition 7.3.1.** The Longo–Witten unitaries \( V_1 \otimes \cdots \otimes V_n \in \mathcal{E}(\mathcal{A}^{\otimes n}) \) with \( V_1 \) coming from restriction of second quantization unitary \( \Lambda(V_{1,0}) \) extends to the local extension \( \mathcal{A}_L \) with \( L \supset (2\mathbb{Z})^n \) an even lattice if: for every \( 1 \leq i \leq n \) where there is a \( g \in G_L \) with \( g_i = \pm \frac{1}{2} \) the operator \( (S_R \otimes S_R)V_{i,0}(1 - (S_R \otimes S_R)) \) is Hilbert–Schmidt.
CHAPTER 8

Conclusions and outlook

By exploiting the explicit construction of a family of conformal nets containing loop group nets of simply laced groups at level 1, namely conformal nets associated with lattices, we have obtained semigroup elements of the Longo–Witten semigroup $E(A)$. These elements give rise to new models in BQFT, i.e. local, time-translation covariant nets on Minkowski half-plane.

The level 1 loop group models can also be embedded in free Fermi nets, which could lead to different elements of the semigroup, coming from restrictions of second quantization unitaries.

It would be desirable to analyze also the semigroup for loop group models at higher level. These loop group nets are subnets of the tensor product of level 1 nets and one could ask if the here obtained endomorphisms restrict to these subnets. By applying the coset and orbifold construction one obtains new nets and should also get new semigroup elements. A simple example using the orbifold construction we have given.

A new group of Longo–Witten endomorphisms for the $U(1)$–current net was obtained in Chapter 5 using the Boson–Fermion correspondence, which we established for this purpose in the algebraic framework. The result can be used to find new Longo–Witten endomorphisms of Abelian current nets and certain Lattice extensions, but the last case includes just includes finite group orbifolds of free Fermionic theories. It should be stressed that this Longo–Witten unitaries coming from Fermionic second quantization are (besides the trivial ones) different from the one obtained from Bosonic second quantization.

In Chapter 6 we gave some abstract criterion how Longo–Witten unitaries restrict and extend through irreducible finite index subnets. The result is: the possibility of extending a Longo-Witten endomorphism (and therefore the implementing unitary) from an irreducible finite index subnet $A$ of $B$ is equivalent with the existence of a certain “cocycle” $z_\theta(\eta,\eta\theta)$ where $\theta$ is the dual canonical endomorphism of the inclusion $A(\mathbb{R}^+) \subset B(\mathbb{R}^+)$. We applied this result to understand also more complicated extensions of (subnets of) free Fermionic nets in Chapter 7 using the framed vertex operator construction. The problem reduces to a condition for the existence of a cocycle for the Ramond sector which can be formulated on the one-particle space using a second quantization criterion. An open problem is the characterization of the for this extension admissible inner functions.

Regarding the Longo–Witten semigroup $E(A)$ in general, remarkable questions and applications arise. An example is the mystery relation between elements of semigroup and integrable models with factorizing S-matrix [ZZ79] on two dimensional Minkowski space, constructed in the operator algebraic setting in [Lec08]. Both of them take inner symmetric (or scattering) functions as an input, but at the moment a deeper relation is not yet found. Some relations are presented in the Chapter 10.

Noteworthy applications of the Longo–Witten semigroup can be noticed in deformations of chiral conformal nets, where the endomorphisms $\text{Ad} V$ associated with $V \in E(A)$ bring deformations of chiral CQFT’s on two dimensional Minkowski space. Particularly, in [Tan12a] the endomorphisms are used for a family of deformations of the $U(1)$-current net and the Ising net which are both second
quantization nets. In this point the question that arises is if such deformations also exist for the endomorphisms of the conformal nets associated with lattices (obtained in this work), or more generally for any Longo–Witten endomorphism. In Chapter 9 we will see that for the new class of Longo–Witten endomorphisms obtained for the $U(1)$–current net in Chapter 5 this construction can be generalized and give interesting deformation of the free Bosonic net.
Part III

Construction of models on 2D Minkowski space
Construction of a family of massless and interacting wedge-local nets with non-factorizing S-matrix

In this chapter we use the family of Longo–Witten endomorphisms obtained by the Boson–Fermion correspondence in Chapter 5 to construct a family of S-matrices for the asymptotic algebra of the conformal field theory $\mathcal{A} = \mathcal{A}_R \otimes \mathcal{A}_R$ on Minkowski space with chiral parts the $U(1)$–current $\mathcal{A}_R$. More precisely, the S-matrix is constructed on the bigger Fermionic theory and then restricted. By this we obtain a family of asymptotically complete (in the sense of waves) nets with asymptotic net $\mathcal{A}$ and a S-matrix $S$, which in general acts complicated on the Bosonic Fock space, in the sense that it does not preserve the Fock space particle number. The result was obtained in [BT11].

9.1. Generalization to Fermi nets

In this section we generalize the construction of massless interacting models of Section 3.7 to Fermi nets.

**Definition 9.1.1.** Let $\mathcal{F}_+ \otimes \mathcal{F}_-$ a chiral net on $\mathcal{H}_+ \otimes \mathcal{H}_-$. A unitary $S \in \mathcal{B}(\mathcal{H}_+ \otimes \mathcal{H}_-)$ is called a scattering operator for $\mathcal{F}_+ \otimes \mathcal{F}_-$ if

1. $S$ commutes with $T_+ \otimes T_-$
2. $S(\xi \otimes \Omega_-) = \xi \otimes \Omega_-$ for $\xi \in \mathcal{H}_+$ and $S(\Omega_+ \otimes \eta) = \Omega_+ \otimes \eta$ for $\eta \in \mathcal{H}_-$
3. $x \otimes 1$ commutes with $\text{Ad}^S(1 \otimes x')$ for $x \in \mathcal{F}_+(\mathbb{R}_-)$ and $x' \in \text{Ad}^Z_+(\mathcal{F}_+(\mathbb{R}_+))$
4. $\text{Ad}^S(1 \otimes y) \text{commutes with} \ 1 \otimes y'$ for $y \in \mathcal{A}_-(\mathbb{R}_+)$ and $y' \in \text{Ad}^Z_-(\mathcal{F}_-(\mathbb{R}_-))$

the semigroup of scattering matrices is denoted $S(\mathcal{F}_+ \otimes \mathcal{F}_-)$

**Proposition 9.1.2.** Let $\mathcal{F}_+ \otimes \mathcal{F}_-$ a graded local chiral net and on $\mathcal{H}_+ \otimes \mathcal{H}_-$ and $S$ scattering operator $S$ for $\mathcal{F}_+ \otimes \mathcal{F}_-$. Then $(\mathcal{M}_S, T, \Omega)$ with

$$\mathcal{M}_S = (\mathcal{F}_+(\mathbb{R}_-) \otimes 1) \lor \text{Ad}^S(1 \otimes \mathcal{F}_-(\mathbb{R}_+))$$

$$T(t, x) = T_0((t - x)/\sqrt{2}) \otimes T_0((t + x)/\sqrt{2})$$

$$\Omega = \Omega_0 \otimes \Omega_0.$$

is an asymptotically complete Borchers triple with S-matrix $S$.

**Proof.** As in [Tan12a], the conditions on $T$ and $\Omega$ are automatic because they are just tensor products of objects for Fermi nets. Similarly, the condition that $\text{Ad} T(a) \mathcal{M}_S \subset \mathcal{M}_S$ for $a \in W_R$ is easily seen from the assumption that $T$ commutes with $S$ and the covariance of Fermi nets.

What remains is the cyclicity and separating property of $\Omega$ for $\mathcal{M}_S$. Cyclicity is immediate because we have

$$\mathcal{M}_S \Omega \supset \{x \otimes 1 \cdot S(1 \otimes y)S^* \cdot \Omega : x \in \mathcal{F}_+(\mathbb{R}_-), y \in \mathcal{F}_-(\mathbb{R}_+)\}$$

$$= \{x \otimes y \cdot \Omega : x \in \mathcal{F}_+(\mathbb{R}_-), y \in \mathcal{F}_-(\mathbb{R}_+)\}.$$
by the assumed property of $S$, and the latter set is total in $\mathcal{H}_+ \otimes \mathcal{H}_-$ by the Reeh–Schlieder property for Fermi nets. As for the separating property, we define:

$$\mathcal{M}_S^1 := \{ \text{Ad} S(x' \otimes 1), 1 \otimes y' : x' \in \text{Ad} Z_+(\mathcal{F}_+(\mathbb{R}_+)), y' \in \text{Ad} Z_-(\mathcal{F}_-(\mathbb{R}_-)) \}''.$$ 

By an analogous proof, one sees that $\Omega$ is cyclic for $\mathcal{M}_S^1$. Furthermore, $\mathcal{M}_S$ and $\mathcal{M}_S^1$ commute by assumption. Hence $\Omega$ is separating for $\mathcal{M}_S$. In other words, $(\mathcal{M}_S, T, \Omega)$ is a Borchers triple.

It is immediate that $\Phi_{\text{out}}^q(x \otimes 1) = x \otimes 1$ and $\Phi_{\text{in}}^q(\text{Ad} S(1 \otimes y)) = \text{Ad} S(1 \otimes y)$ (the latter follows since $S$ commutes with $T$). Similarly, we have $\Phi_{\text{in}}^w(\text{Ad} S(x' \otimes 1)) = \text{Ad} S(x' \otimes 1)$ and $\Phi_{\text{out}}^w(1 \otimes y') = 1 \otimes y'$. From this, one concludes that the Borchers triple $(\mathcal{M}_S, T, \Omega)$ is asymptotically complete and its $S$-matrix is $S$.

\[\square\]

### 9.2. Restriction of wedge-local nets

We consider a wedge local net given by a Borchers triple $(\mathcal{M}, T, \Omega)$ on a Hilbert space $\mathcal{H}_\mathcal{M}$. It will be interesting to study also subalgebras $\mathcal{N} \subset \mathcal{M}$ and when they give rise to another Borchers triple. Let us denote $\mathcal{H}_\mathcal{N} = \mathcal{N}\overline{\Omega}$.

**Proposition 9.2.1.** Let $(\mathcal{M}, T, \Omega)$ be a Borchers triple and $\mathcal{N} \subset \mathcal{M}$ such that the subspace $\mathcal{H}_\mathcal{N}$ is invariant under $T$ and $\text{Ad} T(\mathcal{N}) \subset \mathcal{N}$ for all $a \in W_\mathcal{M}$. Then $(\mathcal{N}, T \upharpoonright \mathcal{H}_\mathcal{N}, \Omega)$ is a Borchers triple on $\mathcal{H}_\mathcal{N}$.

**Proof.** The components $\mathcal{N}, T$ and $\Omega$ naturally restricts to $\mathcal{H}_\mathcal{N}$. The conditions on $T$ and $\Omega$ are trivial, even restricted to $\mathcal{H}_\mathcal{N}$. The cyclicity of $\Omega$ is immediate from the definition of $\mathcal{H}_\mathcal{N}$. Since $\Omega$ is already separating for $\mathcal{M}$, so is also for $\mathcal{N}$. Endomorphic action of $T$ on $\mathcal{N}$ is in the hypothesis. \[\square\]

**Definition 9.2.2.** Let $(\mathcal{M}, T, \Omega)$ be a Borchers triple. We call $(\mathcal{N}, T, \Omega)$ a Borchers subtriple of $(\mathcal{M}, T, \Omega)$ with $\mathcal{N} \subset \mathcal{M}$, if

- the subspace $\mathcal{H}_\mathcal{N}$ is invariant under $T$
- $\text{Ad} T(a)(\mathcal{N}) \subset \mathcal{N}$ for all $a \in W_\mathcal{M}$
- $\text{Ad} \Delta_\mathcal{N} = \mathcal{N}$ where $\Delta_\mathcal{M}$ is the modular operator of $\mathcal{M}$ with respect to $\Omega$

We recall that a strictly local Borchers triple give rise to a (strictly) local net.

**Proposition 9.2.3.** Let $(\mathcal{N}, T, \Omega)$ be a Borchers subtriple of a Borchers triple $(\mathcal{M}, T, \Omega)$, then $(\mathcal{N}, T \upharpoonright \mathcal{H}_\mathcal{N}, \Omega)$ is a Borchers triple on $\mathcal{H}_\mathcal{N}$.

**Proof.** The components $\mathcal{N}, T$ and $\Omega$ naturally restricts to $\mathcal{H}_\mathcal{N}$. The conditions on $T$ and $\Omega$ are trivial, even restricted to $\mathcal{H}_\mathcal{N}$. The cyclicity of $\Omega$ is immediate from the definition of $\mathcal{H}_\mathcal{N}$. Since $\Omega$ is already separating for $\mathcal{M}$, so is also for $\mathcal{N}$. Endomorphic action of $T$ on $\mathcal{N}$ is in the hypothesis. \[\square\]

The following proposition shows that the concept of a subtriple corresponds to the notion of a subnet.

**Proposition 9.2.4.** Let $(\mathcal{M}, T, \Omega)$ be a strictly local Borchers triple, then every subtriple $(\mathcal{N}, T, \mathcal{M})$ of it is again strictly local Borchers triple when restricted to $\mathcal{H}_\mathcal{N}$.

**Proof.** Since $\mathcal{N}$ is invariant under the modular automorphism $\text{Ad} \Delta_\mathcal{M}$, there is a conditional expectation $E$ from $\mathcal{M}$ onto $\mathcal{N}$ which preserves the state $(\Omega, \mathcal{M})$ and is implemented by the projection $P_\mathcal{N}$ by Takesakis Theorem 2.1.11.

We have to show that $\Omega$ is cyclic for the relative commutant $\mathcal{N} \cap \text{Ad} T(a)(\mathcal{N})'$ on the subspace $\mathcal{H}_\mathcal{N}$. Let us denote $\mathcal{M}_{0,a} := \mathcal{M} \cap \text{Ad} T(a)(\mathcal{M})'$. We claim that $E(\mathcal{M}_{0,a})$ is contained in $\mathcal{N} \cap \text{Ad} T(a)(\mathcal{N})'$. 

\[\square\]
Indeed, by the definition of $E$, the image of $E$ is contained in $\mathcal{N}$. Furthermore, if $x \in \mathcal{M}_{0,a}$, $y \in \text{Ad} \, T(a)(\mathcal{N}) \subset \text{Ad} \, T(a)(\mathcal{M})$, then

$$E(x)y = E(xy) = E(yx) = yE(x),$$

hence they commute and the image $E(\mathcal{M}_{0,a})$ lies in the relative commutant. Now we have

$$\overline{\left(\mathcal{N} \cap \text{Ad} \, T(a)(\mathcal{N})\right)} \supset \overline{E(\mathcal{M}_{0,a})}\Omega \supset \mathcal{P}_N \mathcal{M}_{0,a}\Omega = \mathcal{H}_N$$

by the assumed strict locality of $(\mathcal{M}, T, \Omega)$. \hfill $\square$

Let $\mathcal{B}$ be an asymptotically complete local Poincaré covariant net on 2D Minkowski space with vacuum vector $\Omega$ and representation $U$ of the Poincaré group. We denote by $T$ the restriction of $U$ to the translations.

One can define the (out-) asymptotic algebra $\mathcal{B}_+ \otimes \mathcal{B}_-$ and the scattering operator $S$ which is a unitary operator.

It is possible to recover the original net $\mathcal{B}$ by the formula

$$\mathcal{B}(W_R) = (\mathcal{B}(\mathbb{R}_-) \otimes 1) \vee \text{Ad} \, S(1 \otimes \mathcal{B}_-(\mathbb{R}_+))$$

and using covariance and intersections. In particular $(\mathcal{M}, T, \Omega)$ with $\mathcal{M} := \mathcal{B}(W_R)$ is an asymptotically complete strictly local Borchers triple, the one associated with the net $\mathcal{B}$.

Here we exhibit a simple way to construct subtriples. Let $\mathcal{A}_{\pm}$ local Möbius covariant subnets of $\mathcal{B}_{\pm}$, respectively. If we define

$$\mathcal{B}(W_R) = (\mathcal{A}(\mathbb{R}_-) \otimes 1) \vee \text{Ad} \, S(1 \otimes \mathcal{A}_-(\mathbb{R}_+))$$

the $(\mathcal{N}, T, \Omega)$ defines a Borchers subtriple of $(\mathcal{M}, T, \Omega)$. Indeed, conditions regarding $\mathcal{N}$, $T$ and $\Omega$ are immediate. To check the invariance of $\mathcal{N}$ under $\text{Ad} \, \Delta^R_M$, it is sufficient to note that $S$ and $\Delta^R_M$ commute ([Tan12a, Lemma 2.4] check, cf. [Buc75]) and that $\mathcal{A}_+(\mathbb{R}_-)$ and $\mathcal{A}_-(\mathbb{R}_+)$ are preserved by $\text{Ad} \, \Delta^R_M$ by the Bisognano–Wichman property (Theorem 3.2.3) for conformal nets.

The trouble is, however that such Borchers triples in general do not need to be asymptotically complete. The out states span the subspace $\mathcal{A}_+(\mathbb{R}_-)\Omega \otimes \mathcal{A}_-(\mathbb{R}_+)\Omega$. It is easy to see that this coincides with $\mathcal{H}_N = (\mathcal{N} \Omega \otimes \mathcal{H}_0)$ if and only if $S^* \mathcal{H}_N = \mathcal{H}_N$.

It is worthwhile to have a general condition to assure that the subnets are asymptotically complete to have a clear-cut scattering theory, which is only available for asymptotically complete Borchers triples.

We study the special case of compact group fixpoints. Let $\mathcal{B}_0$ be a conformal net or Fermi net on $\mathcal{H}_0$ with an action of a compact group $G$ by inner symmetries implemented by $V_g$, $g \in G$ and we denote by $\mathcal{A}_0$ the fixed point net $\mathcal{B}_0^G$. We suppose that we are in the situation of Proposition 9.1.2, so that we have a unitary operator $S$ on $\mathcal{H}_0 \otimes \mathcal{H}_0$ and that $(\mathcal{M}_S, T, \Omega)$ is a Borchers triple, where

$$\mathcal{M}_S = (\mathcal{B}_0(\mathbb{R}_-) \otimes 1) \vee \text{Ad} \, S(1 \otimes \mathcal{B}_0(\mathbb{R}_+))$$

$$T(t, x) = T_0((t - x)/\sqrt{2}) \otimes T_0((t + x)/\sqrt{2})$$

$$\Omega = \Omega_0 \otimes \Omega_0.$$

**Proposition 9.2.5.** If $S$ commutes with $V_g \otimes V_g'$ for every $g, g' \in G$, then the triple $(\mathcal{N}_S, T, \Omega)$ with

$$\mathcal{N}_S = (\mathcal{A}_0(\mathbb{R}_-) \otimes 1) \vee \text{Ad} \, S(1 \otimes \mathcal{A}_0(\mathbb{R}_+))$$

$$T(t, x) = T_0((t - x)/\sqrt{2}) \otimes T_0((t + x)/\sqrt{2})$$

$$\Omega = \Omega_0 \otimes \Omega_0.$$


is a Borchers subtriple giving (by restriction to $\mathcal{H}_{N^5} = \overline{\mathcal{N}_5 \Omega}$) rise to an asymptotically complete Borchers triple with asymptotic algebra $A_0 \otimes A_0$, and scattering operator $S \upharpoonright \mathcal{H}_{N^5}$.

Proof. As remarked above, $(\mathcal{N}_5, T, \Omega)$ is a Borchers triple on $\mathcal{H}_{N^5}$, hence the only thing to be proven is asymptotic completeness. We show that the subspace $\mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}_0(\mathbb{R}_+) \Omega$ is invariant under $S$.

We claim that $\mathcal{A}_0(\mathbb{R}_- \mathcal{O}) \otimes \mathcal{A}_0(\mathbb{R}_+) \Omega$ coincides with the subspace $\mathcal{H}_0^G$ of invariant vectors under $\{V_g\}_{g \in G}$. Indeed, for any $x \in A_0$, the averaging $\int_G V_g x \Omega \Omega_g d\gamma_g = \left( \int_G \alpha_g(x) d\gamma_g \right) \Omega_0$ gives a projection onto $\mathcal{H}_0^G$. By the Reeh-Schlieder property, any vector in $\mathcal{H}_0^G$ can be approximated by vectors in $\mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}_0(\mathbb{R}_+) \Omega$. The converse inclusion is obvious.

Now it is easy to see that $\mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}_0(\mathbb{R}_+) \Omega = \mathcal{H}_0^G \otimes \mathcal{H}_0^G$. This is the space of invariant vectors under the action $\{V_g \otimes V_g' : g, g' \in G\}$. Since $S$ commutes with $V_g \otimes V_g'$ by assumption, this subspace is preserved under $S$. Then, as remarked before, $\mathcal{H}_{N^5}$ coincides with $\mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}_0(\mathbb{R}_+) \Omega$ and we obtain the asymptotic completeness.

The statement on $S$-matrix is immediate from the definition and by Proposition 9.1.2.

9.3. Interacting wedge-local nets

9.3.1. Construction of scattering operators. In the previous section we saw that, in the basis $\{e_n + e_{-n}, e_n - e_{-n}\}$ the matrix operator

\[
\begin{pmatrix}
    a(p) & ib(p) \\
    -ib(p) & a(p)
\end{pmatrix}
\]

implements a Longo-Witten endomorphism if $a$ is symmetric and $b$ is antisymmetric, and after the simultaneous diagonalization it becomes

\[
\begin{pmatrix}
    \varphi(p) & 0 \\
    0 & \bar{\varphi}(p)
\end{pmatrix}
\]

where $\varphi$ is an inner function and $\bar{\varphi}(p) = \varphi(-p)$ (note that if $\varphi$ extends to an analytic function $\varphi(z)$ on $\mathbb{H}$, then $\bar{\varphi}(z) = \varphi(-z)$ also extends to $\mathbb{H}$, hence $\bar{\varphi}$ is again an inner function). By the same argument one sees that $\begin{pmatrix}
    \bar{\varphi}(p) & 0 \\
    0 & \varphi(p)
\end{pmatrix}$ implements an endomorphism since $\bar{\varphi} = \varphi$.

With respect to the basis after diagonalization, we split the Hilbert space $\mathcal{H}_{\text{Ferz}}^1 = \mathcal{H}_+ \oplus \mathcal{H}_-$ and the generator of translation $P_1 := P_+ \oplus P_-$. Then the tensor product space can be written as follows:

$$
\mathcal{H}_{\text{Ferz}}^1 \otimes \mathcal{H}_{\text{Ferz}}^1 = (\mathcal{H}_+ \otimes \mathcal{H}_+) \oplus (\mathcal{H}_+ \otimes \mathcal{H}_-) \oplus (\mathcal{H}_- \otimes \mathcal{H}_+) \oplus (\mathcal{H}_- \otimes \mathcal{H}_-)
$$

According to this decomposition into a direct sum of four subspaces, we define an operator

$$
M_\varphi := \varphi(P_+ \otimes P_+) \oplus \bar{\varphi}(P_+ \otimes P_-) \oplus \varphi(P_- \otimes P_+) \oplus \bar{\varphi}(P_- \otimes P_-).
$$

Then this restricts to the subspace $\mathcal{H}_{\text{Ferz}}^1 \otimes \mathcal{H}_+ = (\mathcal{H}_+ \oplus \mathcal{H}_-) \otimes \mathcal{H}_+$ and it is $\varphi(P_+ \otimes P_+) \oplus \bar{\varphi}(P_- \otimes P_+)$. or we can decompose it with respect to the spectral measure of $P_+$:

$$
\int_{\mathbb{R}_+} \left( \begin{pmatrix}
    \varphi(pP_+) & 0 \\
    0 & \bar{\varphi}(pP_-)
\end{pmatrix} \right) \otimes dE_+(p).
$$

Similarly, the restriction to $\mathcal{H}_{\text{Ferz}}^1 \otimes \mathcal{H}_-$ is written as

$$
\int_{\mathbb{R}_-} \left( \begin{pmatrix}
    \bar{\varphi}(pP_+) & 0 \\
    0 & \varphi(pP_-)
\end{pmatrix} \right) \otimes dE_-(p).
$$
Using the two-point set \( \mathbb{Z}_2 = \{+, -\} \) we define

\[
\varphi_+(p, +) := \varphi(p), \quad \varphi_+(p, -) := \varphi(p), \quad \varphi_-(p, +) := \varphi(p), \quad \varphi_-(p, -) := \varphi(p).
\]

By defining the spectral measure \( E_1 = E_+ \oplus E_- \) on \( \mathcal{H}^1 \), \( M_\varphi \) can be simply written as

\[
M_\varphi = \int_{\mathbb{R}_+ \times \mathbb{Z}_2} \begin{pmatrix} \varphi_+(pP_+, \iota) & 0 \\ 0 & \varphi_-(pP_-, \iota) \end{pmatrix} \otimes dE_1(p, \iota),
\]

where \( \iota = \pm \).

As in [Tan12a], we construct the scattering matrix first on the unsymmetrized Fock space, then restrict it to the antisymmetric space. For the operator \( A \) on \( \mathcal{H}^1_{\text{Fer}} \otimes \mathcal{H}^1_{\text{Fer}} \), we denote by \( A_{i,j}^{m,n} \) on \((\mathcal{H}^1_{\text{Fer}})^{\otimes m} \otimes (\mathcal{H}^1_{\text{Fer}})^{\otimes n}\) an operator which acts only on the \( i \)-th factor in \((\mathcal{H}^1_{\text{Fer}})^{\otimes m}\) and \( j \)-th factor in \((\mathcal{H}^1_{\text{Fer}})^{\otimes n}\) as \( A \). As a convention, \( A_{i,j}^{m,n} \) equals to the identity operator if \( m \) or \( n \) is 0. Let us denote simply:

\[
\tilde{\varphi}(p, \iota) := \begin{pmatrix} \varphi_+(p, \iota) & 0 \\ 0 & \varphi_-(p, \iota) \end{pmatrix},
\]

\[
\tilde{\varphi}(P_1, \iota) := \begin{pmatrix} \varphi_+(P_+, \iota) & 0 \\ 0 & \varphi_-(P_-, \iota) \end{pmatrix}.
\]

From the observation above, it is straightforward to see that

\[
(M_\varphi)_{i,j}^{m,n} = \int \left( 1 \otimes \cdots \otimes \tilde{\varphi}(p_jP_{1,i}, \iota_j) \otimes \cdots \otimes 1 \right)_{i\text{-th}} \otimes dE_1(p_1, \iota_1) \otimes \cdots \otimes dE_1(p_n, \iota_n)
\]

(the case where \( m \) or \( n \) is 0 is treated separately). Then we define, as in [Tan12a],

\[
S_\varphi^{m,n} := \prod_{i,j} (M_\varphi)_{i,j}^{m,n},
\]

\[
S_\varphi := \bigoplus_{m,n} S_\varphi^{m,n}.
\]

Let \( \mathcal{H}^\Sigma_{\text{Fer}} \) be the unsymmetrized Fock space based on \( \mathcal{H}^1_{\text{Fer}} \). Note that \( S_\varphi \) is defined on \( \mathcal{H}^\Sigma_{\text{Fer}} \otimes \mathcal{H}^\Sigma_{\text{Fer}} \), and it naturally restricts to \( \mathcal{H}_{\text{Fer}} \otimes \mathcal{H}^\Sigma_{\text{Fer}}, \mathcal{H}^\Sigma_{\text{Fer}} \otimes \mathcal{H}_{\text{Fer}} \) and \( \mathcal{H}_{\text{Fer}} \otimes \mathcal{H}_{\text{Fer}} \). This \( S_\varphi \) will be interpreted as the scattering matrix. In order to confirm this, we have to take the spectral decomposition of \( S_\varphi \).
only with respect to the right or left component. Namely,

\[
S_\varphi := \bigoplus_{m,n} \prod_{i,j} (\mathcal{M}_\varphi)_{i,j}^{m,n} \\
= \bigoplus_{m,n} \prod_{i,j} \left\{ \int \left( 1 \otimes \cdots \otimes \varphi(p_j P_1, t_j) \otimes \cdots \otimes 1 \right) i-th \right\} \otimes dE_1(p_1, t_1) \otimes \cdots \otimes dE_1(p_n, t_n) \\
= \bigoplus_{m,n} \int \prod_{i,j} \left( 1 \otimes \cdots \otimes \varphi(p_j P_1, t_j) \otimes \cdots \otimes 1 \right) i-th \right\} \otimes dE_1(p_1, t_1) \otimes \cdots \otimes dE_1(p_n, t_n) \\
= \bigoplus_n \int \prod_j \left( \varphi(p_j P_1, t_j) \right)^{\otimes m} \otimes dE_1(p_1, t_1) \otimes \cdots \otimes dE_1(p_n, t_n) \\
= \bigoplus_n \int \prod_j \Lambda(\varphi(p_j P_1, t_j)) \otimes dE_1(p_1, t_1) \otimes \cdots \otimes dE_1(p_n, t_n),
\]

where the integral and the product commute in the third equality since the spectral measure is disjoint for different values of \( p \)'s and \( t \)'s, and the sum and the product commute in the fifth equality since the operators in the integrand act on mutually disjoint spaces, namely on \( (\mathcal{H}_{\text{Fer}_C}^1)^{\otimes m} \otimes \mathcal{H}_{\text{Fer}_C}^2 \) for different \( m \).

In the final expression, all operators appearing in the integrand are the second quantization operators, thus this formula naturally restricts to the partially antisymmetrized space \( \mathcal{H}_{\text{Fer}_C}^1 \otimes \mathcal{H}_{\text{Fer}_C}^2 \).

Now we define

\[
\mathcal{M}_\varphi = (\text{Fer}_C(\mathbb{R}_-) \otimes 1) \lor \text{Ad} S_\varphi(1 \otimes \text{Fer}_C(\mathbb{R}_+)) \\
T(t, x) = T_0((t - x)/\sqrt{2}) \otimes T_0((t + x)/\sqrt{2}) \\
\Omega = \Omega_0 \otimes \Omega_0.
\]

as in Proposition 9.1.2. As the net \( \text{Fer}_C \) is Fermionic by nature, the interpretation of the scattering theory of [Buc75] is not clear. Nevertheless, we can show the following by an almost same proof as in [Tan12a, Lemma 5.2, Theorem 5.3].

**Lemma 9.3.1.** \( S_\varphi \) is a scattering operator for \( \text{Fer}_C \otimes \text{Fer}_C \) like in Definition 9.1.1, and in particular the triple \((\mathcal{M}_\varphi, T, \Omega)\) is a Borchers triple.

**Proof.** To apply Proposition 9.1.2, it is immediate that \( S_\varphi \) commutes with translation since it is defined through the spectral measure as above. It preserves \( \mathcal{H}_{\text{Fer}_C} \otimes \Omega_0 \) and \( \Omega_0 \otimes \mathcal{H}_{\text{Fer}_C} \) pointwise, since these subspaces correspond to the case where \( m \) or \( n \) is 0 in the above decomposition and \( S_\varphi \) acts as the identity operator by definition. What remains to show is the commutation property.

As we saw above, the operator \( S_\varphi \) can be written as

\[
S_\varphi = \bigoplus_n \int \prod_j \Lambda(\varphi(p_j P_1, t_j)) \otimes dE_1(p_1, t_1) \otimes \cdots \otimes dE_1(p_n, t_n).
\]

The point is that the operators which appear in the integrand implement Longo–Witten endomorphisms as we saw above since \( p_j \geq 0 \) in the support of the integration.
Let \( x' \in \text{Fer}_C(\mathbb{R}_+ \mathbb{C}) \) and consider \( x' \otimes 1 \) as an operator on \( \mathcal{H}_{\text{Fer} C} \otimes \mathcal{H}_{\text{Fer} C}^\mathbb{C} \). We have

\[
\text{Ad} S_{\varphi}(x' \otimes 1) = \bigoplus_n \int \text{Ad} \left( \prod_j \Lambda(\varphi(p_j)P_1,t_j) \right) \left( x' \otimes \text{d}E_1(p_1,t_1) \otimes \cdots \otimes \text{d}E_1(p_n,t_n) \right).
\]

Although this formula is not closed on \( \mathcal{H}_{\text{Fer} C} \otimes \mathcal{H}_{\text{Fer} C}^\mathbb{C} \), the left hand side obviously restricts there. One sees that the integrand remains in \( \text{Fer}_C(\mathbb{R}_+ \mathbb{C}) \).

Recall the operator \( Z_0 \) which gives the graded locality of \( \text{Fer}_C \). One has to remind that \( Z_0 = \frac{1+\eta_0}{1-\eta_0} \), where \( \eta_0 = \Lambda(-1) \), hence \( Z_0 \) commutes with any second quantization operator. Then by the disintegration above (and the corresponding disintegration with respect to the left component), it is easy to see that \( Z_0 \otimes 1 \) commutes with \( S_{\varphi} \).

Let us check the commutation property of the assumptions in Proposition 9.1.2. Note that \( \text{Ad} Z_0(x) \otimes 1 \) and \( \text{Ad} Z_0(x) \in \text{Fer}_C(\mathbb{R}_+ \mathbb{C}) \) for \( x \in \text{Fer}_C(\mathbb{R}_- \mathbb{C}) \). Since \( Z_0 \otimes 1 \) and \( S_{\varphi} \) commute as we saw above, to prove the first commutation relation, it is enough to show that \( [\text{Ad} Z_0(x) \otimes 1, \text{Ad} S_{\varphi}(x' \otimes 1)] = 0 \) for \( x \in \text{Fer}_C(\mathbb{R}_- \mathbb{C}) \) and \( x' \in \text{Fer}_C(\mathbb{R}_+ \mathbb{C}) \). As operators acting on \( \mathcal{H}_{\text{Fer} C} \otimes \mathcal{H}_{\text{Fer} C}^\mathbb{C} \), this is done by the above disintegration of \( \text{Ad} S_{\varphi}(x' \otimes 1) \). Then both operators naturally restrict to \( \mathcal{H}_{\text{Fer} C} \otimes \mathcal{H}_{\text{Fer} C}^\mathbb{C} \), and we obtain the claim (cf. [Tan12a, Lemma 5.2, Theorem 5.3]). The second commutation relation for Proposition 9.1.2 can be proven analogously. \( \square \)

Finally we arrive at a new family of interacting Borchers triples with asymptotic algebra \( A_R \otimes A_R \).

**Theorem 9.3.2.** Let us define

\[
\mathcal{N}_{\varphi} = (\text{Fer}_C^U(1)(\mathbb{R}_- \mathbb{C}) \otimes 1) \vee \text{Ad} S_{\varphi}(1 \otimes \text{Fer}_C^U(1)(\mathbb{R}_+ \mathbb{C}))
\]

\[
T(t,x) = T_0((t-x)/\sqrt{2}) \otimes T_0((t+x)/\sqrt{2})
\]

\[
\Omega = \Omega_0 \otimes \Omega_0.
\]

Then the triple \( (\mathcal{N}_{\varphi}, T, \Omega) \), restricted to \( \mathcal{N}_{\varphi} \Omega \), is an asymptotically complete, interacting Borchers triple with the asymptotic algebra \( A_R \otimes A_R \) and the scattering operator \( S_{\varphi} |_{\mathcal{N}_{\varphi} \Omega} \). It also holds that \( \mathcal{N}_{\varphi} \Omega = \mathcal{A}_R(I_+ \times I_+) \Omega \) for arbitrary intervals \( I_+, I_- \).

**Proof.** Substantial arguments are already done: In Lemma 9.3.1 we constructed Borchers triples with \( \text{Fer}_C \otimes \text{Fer}_C \) as the asymptotic algebra. We have seen in Section 5.1.5 the \( U(1) \)-current net \( \mathcal{A}_R \) is the fixed point subnet of \( \text{Fer}_C \) with respect to the action of \( U(1) \). From the construction in Section 9.3.1 and Theorem 5.2.6 it is easy to see that \( S_{\varphi} \) commutes with the product action of the inner symmetries. Then all the statements of the Theorem follow from the general consideration of Proposition 9.2.5 \( \square \)

9.3.2. **Action of the S-matrix on the 1+1 particle space.** In this section we want to analyze the action of the S-matrix of the models constructed in Section 9.3.1 on the 1+1 particle space \( \mathcal{H}_{A_R}^1 \otimes \mathcal{H}_{A_R}^1 \), i.e. one left and one right moving particle, where we use the word particle in the sense of Fock space excitations. We note that on the \( n+0 \) and \( 0+n \) particle spaces \( \mathcal{H}_n \otimes \mathcal{O}_0 \) and \( \mathcal{O}_n \otimes \mathcal{H}_n \), respectively, the S-matrix \( S \) acts trivially. A typical vector in \( \mathcal{H}_{A_R}^1 \otimes \mathcal{H}_{A_R}^1 \) is of the form \( \Psi := J(f) \Omega_0 \otimes J(g) \Omega_0 \) which we express as the function \( \Psi(p, \bar{p}) = \hat{f}(p) \hat{g}(\bar{p}) \). The embedding \( \iota : L^2(\mathbb{R}_+, pdp) \otimes L^2(\mathbb{R}_+, pd\bar{p}) \cong \mathcal{H}_{A_R}^1 \otimes \mathcal{H}_{A_R}^1 \hookrightarrow \mathcal{H}_{\text{Fer} C} \otimes \mathcal{H}_{\text{Fer} C} \) is given by \( \iota(\Psi)(1,1,t)(p, q, \bar{p}, \bar{q}) = \frac{1}{\sqrt{\gamma}} \Psi(p + q, \bar{p} + \bar{q}) \).

We have an analogue of Lemma 5.2.5

**Proposition 9.3.3.** Let \( \varphi \) be some inner function. The unitary \( S_{\varphi} \) satisfies \( S_{\varphi} \mathcal{H}_{A_R}^1 \otimes \mathcal{H}_{A_R}^1 \subset \mathcal{H}_{A_R}^1 \otimes \mathcal{H}_{A_R}^1 \) if and only if \( \varphi(p) = e^{i(kp + \theta)} \).
9. CONSTRUCTION OF A FAMILY OF MASSLESS AND INTERACTING WEDGE-LOCAL NETS

Proof. The action of $S_\varphi$ on $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by

$$S_\varphi \Psi(p + q, \bar{p} + \bar{q}) = \varphi(p \cdot \bar{p}) \tilde{\varphi}(q \cdot \bar{q}) \tilde{\varphi}(p \cdot \bar{q}) \varphi(q \cdot \bar{q}) \Psi(p + q, \bar{p} + \bar{q})$$

which is again in $\mathcal{H}_{A_8}^1 \otimes \mathcal{H}_{A_8}^1$ if it can be written as a function $\tilde{\Psi}(p + q, \bar{p} + \bar{q})$, in particular if $\varphi(p \cdot \bar{p}) \tilde{\varphi}(q \cdot \bar{q}) \varphi(q \cdot \bar{q}) = \tilde{\varphi}(p + q, \bar{p} + \bar{q})$. Setting $\bar{p} = 1$ and $\bar{q} = 0$, we have $\varphi(p) \tilde{\varphi}(q) = \tilde{\varphi}(p + q, 1)$. The rest follows as Lemma 5.2.5. \qed

Remark 9.3.4. In the case $\varphi(p) = e^{ipp}$, one gets the models obtained in [DT11] using warped convolution.

Proposition 9.3.5. Let $e$ be the projection on $\mathcal{H}_{A_8}^1 \otimes \mathcal{H}_{A_8}^1$, then $eS \varphi e = \tilde{\varphi}(P_{A_8} \otimes P_{A_8})$, where $\tilde{\varphi}$ is boundary value of an analytic function in $\mathbb{H}$ with $|\tilde{\varphi}(p)| \leq 1$ and $P$ is the generator of translation restricted to the one-particle space (which gives rise the irreducible standard pair).

Proof. It can be checked that

$$(e_0 f)(p, q) = \frac{1}{p + q} \int_0^{p+q} f(p + q - x, x) dx$$

is the projection on $\mathcal{H}_{A_8}^1 \subset \mathcal{H}_{1,1}$. Then the action of $eS \varphi$ on a $f \in \mathcal{H}_{A_8}^1 \otimes \mathcal{H}_{A_8}^1$ can be calculated to be $\varphi_2(P \otimes 1, 1 \otimes P)$ with

$$\varphi_2(p, q) = \frac{1}{P \cdot q} \int_0^p \int_0^q \varphi((p - x) \cdot (q - y)) \varphi(x \cdot y) \tilde{\varphi}((p - x) \cdot y) \tilde{\varphi}(x \cdot (q - y)) dy dx$$

and it is easy to check that with $\tilde{\varphi}(p) := \varphi_2(p, 1)$ it holds $\varphi_2(p, q) = \tilde{\varphi}(p \cdot q)$ for all $p, q > 0$. That $|\tilde{\varphi}(p)| \leq 1$ can be checked directly or follows from the fact that $S_\varphi$ is unitary. \qed

Remark 9.3.6. It is a general feature of asymptotically complete Borchers triples with asymptotic algebra $A_R \otimes A_R$ that the restriction of the scattering matrix $S$ to $eS e$ is a functional calculus of $P \otimes P$. Indeed, both $e$ and $S$ commute with the translation $T$, but $T$ is maximally abelian when restricted to $\mathcal{H}_{A_8}^1 \otimes \mathcal{H}_{A_8}^1$, hence there is a function $\varphi_S$ such that $eS e = \varphi_S(P \otimes 1, 1 \otimes P)$. Furthermore, both $e$ and $S$ commute with boosts, so does $\varphi_S$ and one obtains the form $eS e = \varphi'_S(P \otimes P)$.

We note that the proof above shows that $|\tilde{\varphi}(M^2/2)|$ is the probability that an improper state in $\mathcal{H}_{A_8}^1 \otimes \mathcal{H}_{A_8}^1$ with mass $M^2$ is scattered elastically in the sense of Fock space particles, where

$$\tilde{\varphi}(p) = \frac{1}{p} \int_0^p \int_0^1 \varphi((p - x)(1 - y)) \varphi(xy) \tilde{\varphi}((p - x)y) \tilde{\varphi}(x(1 - y)) dy dx$$.

As we discussed in 5.1.2, the Hilbert space of the $U(1)$-current net, and hence the tensor product of two copies of it, admit the bosonic Fock space structure, hence we can consider the particle number. Although we admit that this concept does not have an intrinsic meaning, we claim that it is possible to interpret this as the number of massless particles.

An evidence comes from the comparison with massive cases. In [Lec08] Lechner has constructed a family of massive interacting models parametrized by so-called scattering functions, and later he reinterpreted them as deformations of the massive free field [Lec11]. If one applies the same deformation procedure to the derivative of the massless free field whose net is $A_R \otimes A_R$ (with scattering functions satisfying $S_2(0) = 1$), one obtains the Borchers triples with $A_R \otimes A_R$ as the asymptotic net constructed in [Tan12a]. That the two deformation schemes are equivalent was recently shown in [LST12]. Hence the models in [Tan12a] should be considered as the massless versions of the models in [Lec08]. Likewise, it can be said that the models constructed in this chapter are the deformed (in an appropriate sense) version of the massless free field.
In massive case, there is a mass gap in the spectrum of the spacetime translation and the one-particle space of the Fock space has an intrinsic meaning. In massless case, such an intrinsic interpretation is lost but there is still the Fock space structure. Thus we think that, if the two-particle space in the Fock structure is not preserved by the S-matrix, as in the case where $\varphi$ is not exponential (see Proposition 9.3.3), then it represents massless particle production.
CHAPTER 10

Massive and massless factorizing models

10.1. General results

In this section we give some general connections between nets on the light ray $\mathbb{R}$ and nets on Minkowski space.

10.1.1. Half-ray local nets. A half-ray local net is the 1D counter part to a wedge-local net.

**Definition 10.1.1.** A 1D Borchers triple on a Hilbert space $\mathcal{H}$ is a triple $(\mathcal{M},T,\Omega)$ of a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, a strongly continuous unitary representation $T$ of $\mathbb{R}$ on $\mathcal{H}$ and a unit vector $\Omega \in \mathcal{H}$, such that

- $\text{Ad} T(t)(\mathcal{M}) \subset \mathcal{M}$ for all $t \geq 0$.
- The generator of the translations is positive.
- $\Omega$ is a unique (up to a phase) invariant vector under $T$ and is cyclic and separating for $\mathcal{M}$.

We associate a net on the family of half-rays $W_1 = \{a + \mathbb{R}_\pm : a \in \mathbb{R}\}$ by

$$
A(\mathbb{R}_+ + a) \equiv A(a, \infty) = \text{Ad}(T(a))(\mathcal{M}),
$$

$$
A(\mathbb{R}_- + a) \equiv A(-\infty, a) = \text{Ad}(T(a))(\mathcal{M}').
$$

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$. It is called **standard**, if there is a vector $\Omega \in \mathcal{H}$, which is cyclic and separating for $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M} \cap \mathcal{N}'$. If $\Omega$ is cyclic and separating for $\mathcal{N}$ and $\mathcal{M}$ and it holds $\text{Ad} \Delta_{\mathcal{M},\Omega}^{-it}(\mathcal{N}) \subset \mathcal{N}$ for all $t \geq 0$, then the inclusion is called **half-sided modular inclusion**.

**Theorem 10.1.2** (Wiesbrock, see [Wie94, Thm 4.], [GLW98, Cor. 1.7.]). Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a standard half-sided modular inclusion, then there is a unique (not necessarily irreducible) local Möbius covariant net with $A(\mathbb{R}_+) = \mathcal{M}$ and $A(\mathbb{R}_- + 1) = \mathcal{N}$.

We remark there was a gap in the proof [Wie93] which was fixed in [AZ05].

**Theorem 10.1.3.** There is a one-to-one correspondence between

1. half-sided modular inclusions $(\mathcal{N} \subset \mathcal{M}, \Omega)$,
2. 1D Borchers triple $(\mathcal{M}, T, \Omega)$ without the assumption that $\Omega$ is the unique translation invariant vector.

**Proof.** Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a half-sided modular inclusion, then there exists a $T(t)$ leaving $\Omega$ invariant and having positive generator given by the closure of $\frac{1}{2\pi}(\log \Delta_{\mathcal{M},\Omega} - \log \Delta_{\mathcal{M},\Omega})$, cf. [Lon08a, Theorem 6.3.3.].
Define $\mathcal{N} = T(1) M T(-1)$ then by Borchers Theorem (Theorem 2.1.9)

$$
\begin{align*}
\text{Ad} \Delta^{-i\mu}_{\mathcal{M}, \Omega}(\mathcal{N}) &= \text{Ad} \Delta^{-i\mu}_{\mathcal{M}, \Omega} \circ \text{Ad} T(1)(\mathcal{M}) \\
&= \text{Ad} T(e^{2\pi i}) \circ \text{Ad} \Delta^{-i\mu}_{\mathcal{M}, \Omega}(\mathcal{M}) \\
&= \text{Ad} T(e^{2\pi i} - 1)(\mathcal{N}) \\
&\subseteq \mathcal{N}
\end{align*}
$$

for $t \geq 0$. \hfill \Box

**Theorem 10.1.4.** There is a one-to-one correspondence between:

1. strong additive local Möbius covariant nets $\mathcal{A}$ on $\mathbb{S}^1$ and
2. strictly local 1D Borchers triple $(\mathcal{M}, T, \Omega)$.

**Proof.** Let $(\mathcal{M}, T, \Omega)$ be a strictly local 1D Borchers triple, then set $\mathcal{N} = T(1)$ and $(\mathcal{N} \subseteq \mathcal{M}, \Omega)$ is a half-sided modular inclusion by Borchers theorem and standard by strict locality.

Let $\mathcal{N} \subseteq \mathcal{M}$ be a standard half-sided modular inclusion, then there exists a unique strong additive local Möbius covariant net characterized by $\mathcal{M} = \mathcal{A}(0, \infty)$ and $\mathcal{N} = \mathcal{A}(1, \infty)$ by the Wiesbrock Theorem.

Let $\mathcal{A}$ be a strongly additive local Möbius covariant net on $\mathbb{S}^1$, then it is Haag dual on $\mathbb{R}$. Set $\mathcal{M} = (\mathcal{A}(0, \infty), T(t) = U(\tau(t)), \Omega)$ is a strictly local 1D Borchers triple by Möbius covariance. \hfill \Box

**10.1.2. Interplay between massive models and 1D models.** We remember the notion of a (2D) Borchers triple $(\mathcal{M}, U, \Omega)$ on a Hilbert space $\mathcal{H}$. It consists of of a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$, a strongly continuous unitary representation $T$ of $\mathbb{R}^2$ on $\mathcal{H}$ and a unit vector $\Omega \in \mathcal{H}$, such that

- $\text{Ad} U(t, x)(\mathcal{M}) \subset \mathcal{M}$ for all $(t, x) \in \mathbb{R}_+$,
- The joint spectrum of the generator of the translations is contained in the forward light cone $\mathcal{V}_+$.
- $\Omega$ is a unique (up to a phase) invariant vector under $U$ and is cyclic and separating for $\mathcal{M}$.

From this point on we use $U$ for the representation of space-time translations, while $T$ for translations on the real line and further for 1D Borchers triple $(\mathcal{M}, T, \Omega)$ and for (2D) Borchers triple $(\mathcal{M}, U, \Omega)$. If we speak just about a Borchers triple we mean a 2D Borchers triple. Let us remark that for a 2D (or 1D) Borchers triple $(\mathcal{M}, T, \Omega)$ (or $(\mathcal{M}, U, \Omega)$) the algebra $\mathcal{M}$ is a factor of type III$_1$ (cf. [Dri75], [Lon79, Cor. 3]) with exception of the trivial case $\mathcal{M} = \mathbb{C}$. Let us call a Borchers triple $(\mathcal{M}, U, \Omega)$ on $\mathcal{H}$ **massive** if there are no massless excitations, i.e. the Hilbert subspaces

$$
\mathcal{H}_\pm = \{ \xi \in \mathcal{H} : U(t, \pm t)\xi = \xi \text{ for } t \in \mathbb{R} \} \subset \mathcal{H},
$$

equal $\mathbb{C}\Omega$.

**Definition 10.1.5.** Let $(\mathcal{M}, T, \Omega)$ be a 1D Borchers triple on $\mathcal{H}$, by $\mathcal{E}(\mathcal{M}, T) \equiv \mathcal{E}(\mathcal{M}, T, \Omega)$ we denote the **semigroup of Longo–Witten unitaries**, i.e. unitaries $V \in \mathcal{U}(\mathcal{H})$ with $V\Omega = \Omega$ commuting with the translations $T(t)$ and implementing endomorphisms of $\mathcal{M}$, i.e., $\text{Ad} V(\mathcal{M}) \subset \mathcal{M}$.

We note that also Remark 3.5.2 applies and that this coincides with the former notion of Longo–Witten unitaries for conformal nets by setting $\mathcal{M} = \mathcal{A}(\mathbb{R}_+)$ and taking the translations $T(t) = U(\tau(t))$.

Now we want to make the simple observation that 1D Borchers triple and (2D) Borchers triple are connected by one-parameter groups whose restrictions to $t \geq 0$ are one-parameter semigroups of Longo–Witten unitaries.
Proposition 10.1.6. Let \((\mathcal{M}, T, \Omega)\) be a 1D Borchers triple, and \((V(t))_{t \in \mathbb{R}}\) a unitary one-parameter group with negative generator with \(V(t) \in \mathcal{E}(\mathcal{M}, T)\) for \(t \geq 0\) which only leaves multiples of \(\Omega\) invariant. Then with \(U(t, x) = T\left(\frac{x+t}{2}\right)V\left(\frac{x-t}{2}\right)\) we get a massive Borchers triple \((\mathcal{M}, U, \Omega)\) and hence a wedge-local Poincaré covariant net on 2D Minkowski space.

Proof. We just have to define an action \(U\) of \(\mathbb{R}\) implementing endomorphisms of \(\mathcal{M}\) for \(x \in \mathcal{W}_R\). We choose \(U(t, t) = T(t)\) and \(U(-t, t) = V(t)\) which is well-defined as if \([T(t), V(s)] = 0\) by definition. For \((2t, 2x) \in \mathcal{W}_R\) it is \(t + x, x - t > 0\) and therefore

\[
\text{Ad} U(2t, 2x)\mathcal{M} = \text{Ad}(V(x-t)T(t+x))\mathcal{A}(\mathbb{R}_+) \\
\subset \text{Ad} V(x-t)\mathcal{M} \\
\subset \mathcal{M}.
\]

By irreducibility and the fact that \(V\) leaves only \(\mathbb{C}\Omega\) pointwise invariant it follows that \(\mathcal{H}_\pm = \mathbb{C}\Omega\) \(\Box\)

Also the converse is true.

Proposition 10.1.7. Let \((\mathcal{M}, U, \Omega)\) be a Borchers triple then with \(T(t) = U(t, t)\) the triple \((\mathcal{M}, T, \Omega)\) is a 1D Borchers triple and \(U(t, x) \in \mathcal{E}(\mathcal{M}, T)\) for \((t, x) \in \mathcal{W}_R\). In particular, \(V(t) = U(-t, t) \in \mathcal{E}(\mathcal{M}, T)\) for \(t \geq 0\) and the unitary one-parameter group has negative energy.

So we conclude.

Corollary 10.1.8. There is a one-to-one correspondence between

1. 1D Borchers triples \((\mathcal{M}, T, \Omega)\) with \((V(s))_{s \in \mathbb{R}}\) a unitary one-parameter group with negative generator, such that \(V(s) \in \mathcal{E}(\mathcal{M}, T)\) for \(s \geq 0\) and \(V(s)\) leaves only multiples of \(\Omega\) invariant; and

2. massive Borchers triple \((\mathcal{M}, U, \Omega)\)

given by \(U(t, x) = T\left(\frac{x+t}{2}\right)V\left(\frac{x-t}{2}\right)\) and conversely \(T(t) = U(t, t), V(s) = U(-s, s)\).

In a similar way, one can restrict to the other light-ray using the opposite wedge.

10.1.3. Intersection property and Möbius covariance.

Proposition 10.1.9. Let \(\mathcal{A}\) be a local Möbius covariant net on \(\mathbb{R}\), and \((V(t))_{t \in \mathbb{R}}\) a unitary one-parameter group with negative generator with \(V(t) \in \mathcal{E}(\mathcal{A})\) for \(t \geq 0\) and \(V(s)\) leaving only multiples of \(\Omega\) invariant. Then with \(U(t, x) = T\left(\frac{x+t}{2}\right)V\left(\frac{x-t}{2}\right)\) we get a strictly local Borchers triple \((\mathcal{M} = \mathcal{A}(\mathbb{R}_+), U, \Omega)\) and hence a local Poincaré covariant net on 2D Minkowski space.

Proof. We choose \(\mathcal{M} = \mathcal{A}(\mathbb{R}_+)\) and \(T(t) = U(t))\) and obtain a Borchers triple by Proposition 10.1.6. We have to check strict locality. If \(V \in \mathcal{E}(\mathcal{A})\) then \(\text{Ad} V^* \mathcal{A}(\mathbb{R}_-) \subset \mathcal{A}(\mathbb{R}_-),\) namely \(\text{Ad} V^* \mathcal{A}(\mathbb{R}_-) \subset \mathcal{A}(\mathbb{R}_-) \Leftrightarrow \mathcal{A}(\mathbb{R}_-) \subset \text{Ad} V \mathcal{A}(\mathbb{R}_-) \Leftrightarrow \mathcal{A}(\mathbb{R}_+) \supset \text{Ad} V \mathcal{A}(\mathbb{R}_+).\)

Let \(a, b > 0\). We check for the double cone \(\mathcal{O} = D((-a, -a), (-b, b))\). It is \(\mathcal{A}(\mathcal{O}) = \mathcal{A}(-a, \infty) \cap V(b)\mathcal{A}(\mathbb{R}_-) V(-b).\) We claim that \(\mathcal{A}(-a, 0) \subset \mathcal{A}(\mathcal{O})\) which is equivalent with

\[
\mathcal{A}(-\infty, -a) \vee (V(b)\mathcal{A}(\mathbb{R}_-) V(-b))' \subset \mathcal{A}(-\infty, -a) \vee \mathcal{A}(\mathbb{R}_+),
\]

i.e. \((V(b)\mathcal{A}(\mathbb{R}_-) V(-b))' \subset \mathcal{A}(0, \infty) \Leftrightarrow V(b)\mathcal{A}(\mathbb{R}_-) V(-b) \supset \mathcal{A}(\mathbb{R}_-) \Leftrightarrow V(-b)\mathcal{A}(\mathbb{R}_-) V(b) \subset \mathcal{A}(\mathbb{R}_-).\) \(\Box\)

In particular we have a construction from local Möbius covariant net \(\mathcal{A}\) to a local Poincaré covariant net on 2D Minkowski space if we have a one-parameter Longo–Witten semigroup with negative generator for \(\mathcal{A}\).
This explains somehow why such Longo–Witten unitaries are hard to construct. For example such a semigroup exists for the $U(1)$–current net but seems to not exist for its Buchholz–Mack–Todorov extensions, because of the failure of the Hölder continuity of the inner function $e^{-i/p}$.

**Proposition 10.1.10** (cf. [BLM11]). Let $(\mathcal{M}, T, \Omega)$ be a 1D Borchers triple, then there is a strongly additive, local Möbius covariant net $\mathcal{A}$ on $\mathbb{S}^1$ on $\mathcal{H}_A = \bigvee_{a>b} \text{Ad} T(a)(\mathcal{M}^\prime) \cap \text{Ad} T(b)(\mathcal{M})\Omega$, whose representation of $U$ of $\text{Mob}$ extends the representation of $T(i)$ $\upharpoonright \mathcal{H}_A$ and $\mathcal{A}(\mathbb{R}_+^\times) = e_A \mathcal{M} e_A$, where $e_A$ is the projection on $\mathcal{H}_A$.

![Diagram](image.png)

**Figure 1.** lightlike intersection

Using this proposition from a Borchers triple $(\mathcal{M}, U, \Omega)$, we get a local Möbius covariant net by

$$\mathcal{A}(a, b) = \text{Ad} U(a, a)(\mathcal{M}) \cap \text{Ad} U(b, b)(\mathcal{M}),$$

$$\mathcal{H}_A := \bigvee_{\mathbb{I} \in \mathbb{R}} \mathcal{A}(\mathbb{I})\Omega,$$

which is as a net on the half-rays is subnet of the net associated with the 1D Borchers triple $(\mathcal{M}, T, \Omega)$ with $T(a) = U(a, a)$. The local algebra $\mathcal{A}(a, b)$ is given by the intersection of two touching wedges as in Figure 1. We remark that it can happen that $\mathcal{A}$ is the trivial net, i.e. $\mathcal{A}(\mathbb{I}) = \mathbb{C} \cdot 1$. Interesting is the case if $\mathcal{A}(\mathbb{R}_+^\times) = \mathcal{M}$ (or equivalently the 1D Borchers triple $(\mathcal{M}, T, \Omega)$ is strictly local). We then say the massive net associated with $(\mathcal{M}, U, \Omega)$ has the **light strip property**. We will see that for the free massive net this is the case, but not much is known in the general case, in particular it is unclear if there can be interacting models with this property.

### 10.2. Models

**10.2.1. Borchers pairs and 2D Wigner representations.** In this subsection we show that from a irreducible standard pair we can obtain a representation of the 2D Poincaré group. Everything could be done abstractly by using Borchers commutation relations, but we rather give a proof using an explicit representation to get in contact with models constructed in the literature.

Let $U$ be the irreducible positive-energy representation of the the 2D proper Poincaré group $\mathcal{P}_+$ with mass $m > 0$ on a Hilbert space denoted by $\mathcal{H}_m$. We can identify $\mathcal{H}_m = L^2(\mathbb{R}, d\theta)$ and the action
is given by

\[ p_m(\theta) = m(\cosh \theta, \sinh \theta) \]

\[ (p^0, p^1, x^0, x^1) = p^0 x^0 - p^1 x^1 \]

\[ (U(x, \lambda)f)(\theta) = e^{ipm(\theta)/2} \psi(\theta - \lambda) \]

\[ J\psi(\theta) = \overline{\psi(\theta)} \]

where \( J = U(-I) \) is the anti-unitary representation of \( (x^0, x^1) \mapsto (-x^0, -x^1) \). We remind that we can associate a standard space \( H(V_{W_R}) \) with the right wedge using modular localization \[BGL02\], namely \( H(V_{W_R}) = \ker(1 - S_{\mu}) \) is the standard space associated with \( S = J\Delta^{\mu} \), where \( \Delta^{\mu} = U(0, -2\pi i) \).

For the irreducible Borchers pair is convenient to take the restriction to the translation subgroup \( \{T(t)\}_{t \in \mathbb{R}} \) of the lowest weight 1 positive energy representation of the Möbius group \( \text{Möb} \) on \( \mathcal{H} \) and the standard subspace \( H = H(\mathbb{R}_+) \). It can be represented on \( \mathcal{H} = L^2(\mathbb{R}_+, p dp) \) by

\[ (T(t)f)(p) = e^{ipt} f(p) \]

\[ (\Delta^{\mu}f)(p) = e^{-2\pi t f(e^{-2\pi t p})} \]

\[ Jf(p) = \overline{f}(p) \]

such that \( (J, \Delta) \) are the modular objects for \( H \).

**Proposition 10.2.1.** Let \( (H, T) \) be the irreducible Borchers pair and \( V_m(s) = e^{-im^2 x^0/p} \), where \( T(a) = e^{iaP} \). Then \( U(x, \lambda) = T(\frac{1}{2}(x^0 + x^1))V_m(\frac{1}{2}(x^1 - x^0))\Delta^{-1/2} \) gives the mass \( m \) representation and \( H \) is identified with \( H(W_R) \).

**Proof.** We show using the explicit parametrization. First we note that

\[ R_m : L^2(\mathbb{R}_+, p dp) \longrightarrow L^2(\mathbb{R}, d\theta) \]

\[ f \longrightarrow (\theta \mapsto me^{-\theta}f(me^{-\theta})) \]

defines a unitary, namely

\[ (R_m f, R_m g)_{L^2(\mathbb{R}_+, p dp)} = \int_{\mathbb{R}} R_m f(\theta)R_m g(\theta) d\theta \]

\[ = \int_{\mathbb{R}} \overline{f(e^{-\theta + \ln m})} g(e^{-\theta + \ln m}) e^{-2\theta + 2\ln m} d\theta \]

\[ = \int_{\mathbb{R}} \overline{f(\theta)} g(\theta) e^{-2\theta} d\theta \]

\[ = \int_{\mathbb{R}} \overline{f(\theta)} g(\theta) e^{-2\theta} d\theta \]

\[ = (f, g)_{L^2(\mathbb{R}_+, p dp)} \]

shows unitarity. Then using

\[ T(\frac{1}{2}(x^0 + x^1))V_m(\frac{1}{2}(x^1 - x^0))\Delta^{-1/2} f(p) = e^{i\frac{1}{2}(s^0 + s^1)p + \frac{1}{2}(s^0 - s^1)p} f(e^4 p) \]
we get:

\[
RT\left(\frac{1}{2}(x^0 + x^1)\right)\mathcal{V}_m\left(\frac{1}{2}(x^1 - x^0)\right)\Delta^{-i\frac{\pi}{2}} f(\theta)
= me^{-\theta \lambda} e^{i\frac{\pi}{2}(x^0 + x^1)} e^{-x + i\frac{\pi}{2}(x^0 - x^1)} e^{\theta} f(me^{-\theta + \lambda})
= e^{i\mathcal{P}_m(\theta, x)} Rf(\theta - \lambda)
= U(x, \lambda)Rf(\theta),
\]

in particular \(RT\left(\frac{1}{2}(x^0 + x^1)\right)\mathcal{V}_m\left(\frac{1}{2}(x^1 - x^0)\right)\Delta^{-i\frac{\pi}{2}} = U(x, \lambda)R\). \(J\) acts in both representation by complex conjugation, so it holds also \(RJ = JR\). 

\[\square\]

\textbf{10.2. Massive free field in 2D and the \(U(1)\)-current.} Let \(U_m\) be the Wigner representation of \(P_+\) on 2D Minkowski space and let us denote the standard space associated with the right wedge by \(H = H(W_R)\) obtained by modular localization. The Borchers triple of the massive scalar field with mass \(m\) is defined by Bosonic second quantization \((\mathcal{M} = R(H), U(x) = \Gamma(U_m(x, 0), \Omega))\) on \(H = e^{H_1}\), where \(R(H) = \{W(f) : f \in H\}''\).

Let \(U_1\) be the highest weight 1 representation of \(\text{Möb}^+\) on \(H_1\), then we get a net of standard subspace \(H(I)\). In the same way like before the \(U(1)\)-current net \(A_{\mathbb{R}}\) is obtained by second quantization \(A_{\mathbb{R}}(I) = R(I)\) and it corresponds to the 1D Borchers triple \((A(\mathbb{R}_+), R(H(\mathbb{R}_+)), \Gamma(T_1))\) because it is known to be strong additive. So it actually does just come from the second quantization of the irreducible standard pair \((H_1, T_1)\). So we can identify \((H, T)\) and \(V_m\) with \(H(W_R)\) like in Proposition 10.2.1 and by functoriality we get an identification of the second quantization nets, in other words \(\Gamma(R_m)\) with \(R_m\) from Proposition 10.2.1 gives equivalence of the Borchers triple of the massive free field \(\mathcal{V}_m\) which can be seen as a model with two particles and \(\Delta\)-symmetric Fock space \(\mathcal{V}_m(s)\) obtained from the 1D Borchers triple and the Longo–Witten unitary \(\Gamma(V_m(s))\).

We get the correspondence

\((U(1)\text{-current net}) \iff \text{(Massive scalar free field in 1+1D with mass } m)\)

via the Longo–Witten one-parameter semigroup of unitaries \(V_m(s) := \Gamma(e^{-im^2s/P})\).

In Section 5.2 another family of Longo–Witten unitaries for the \(U(1)\)-current is constructed, which comes from the Fermion–Boson correspondence. Namely, the \(U(1)\)-current net is the \(U(1)\) gauge fixed point \(A_{\mathbb{R}} = \text{Fer}_{\mathbb{C}}^{U(1)}\) of the complex free Fermion net \(\text{Fer}_{\mathbb{C}}\). The restriction to the \(U(1)\)-current of \(V_{\text{Fer}_{\mathbb{C}}, m}(s) = \Gamma(e^{-im^2s/P})\) (the Fermionic second quantization) gives another one-parameter semigroup of Longo–Witten unitaries with negative energy for the \(U(1)\)-current net. The corresponding massive model is the \(U(1)\) gauge fixed point of either a free massive Fermionic net, or the Borchers triple \((\text{Fer}_{\mathbb{C}}(\mathbb{R}_+), U, \Omega)\) with \(U(t, x) = T\left(\frac{im^2}{2P}\right) V_m\left(\frac{t - x}{2}\right)\), which can be seen as a model with two particles and the diagonal factorizing \(S\)-matrix \(S_2 = -1\), similar to the massive Ising model.

\textbf{10.3. Second Quantization of Borchers pairs on } \(S\)-\text{symmetric Fock space}

First we need to introduce some notation. For \(i_1 \neq \ldots \neq i_r \leq n\) and \(A \in B(\mathcal{H}^\otimes r)\) we define \(A_{i_1, \ldots, i_r} = A_{i_1, \ldots, i_r}^n \in B(\mathcal{H}^\otimes n)\) by

\[F(A \otimes \text{id}_{\mathcal{H}^{r-1}})F^*,\]
where \( F : \xi_1 \otimes \cdots \otimes \xi_n \mapsto \xi_{i_1} \otimes \cdots \otimes \xi_{i_r} \otimes \xi_{i_{r+1}} \otimes \cdots \otimes \xi_{i_n} \) with \( \{i_1, \ldots, i_r\} \cup \{i_{r+1}, \ldots, i_n\} = \{1, \ldots, n\} \). For example
\[
(A \otimes B)_{ij} = \begin{cases} 1 \cdots A_{i-th} \otimes 1 \otimes B_{j-th} \otimes 1 & i < j \\ 1 \cdots B_{j-th} \otimes 1 \otimes A_{i-th} \otimes 1 & j < i 
\end{cases}
\]
or
\[
(A_1 \otimes \cdots \otimes A_r)_{\sigma(1),\ldots,\sigma(r)} = A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(r)}.
\]

10.3.1. Two particle scattering operators for Borchers pairs. Let \( \mathcal{H} \) be a Hilbert space and \((H, T)\) a standard pair. We write \( T_1 = T \otimes 1 \) and \( T_2 = 1 \otimes T \).

Definition 10.3.1. The flip operator \( F_{12} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \) is given by \( F_{12}(\xi \otimes \eta) = \eta \otimes \xi \).

For \( x \in \mathcal{H} \) let us denote by \( \langle x \rangle \) the continuous linear functional given by the Riesz anti isomorphism, i.e.
\[
\langle x \rangle : \mathcal{H} \to \mathbb{C} : y \mapsto \langle x, y \rangle.
\]
This gives linear map from \( \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes (n-1)} \) denoted by \( \langle x \rangle_i \) for \( 1 \leq i \leq n \), where the subscript \( i \) denotes like before that \( \langle x \rangle \) just acts on the \( i \)-th factor, in other words it is defined by \( \langle x \rangle_i : f_1 \otimes \cdots \otimes f_n = (x, f_i) \cdots f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_n \).

Definition 10.3.2. A two particle scattering operator is a unitary operator \( S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \) such that
\begin{enumerate}
\item[(1)] Reflection property: \( S_{21} = S^* \).
\item[(2)] Yang–Baxter–property: \( S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12} \).
\item[(3)] Translation invariance: \( [S, T_1] = [S, T_2] = 0 \).
\item[(4)] \( (\Delta_H^U \otimes \Delta_H^U)S = S(\Delta_H^U \otimes \Delta_H^U) \) and \( S(J \otimes J) = (J \otimes J) S^* \).
\item[(5)] The operator \( A^S_f \) defined by
\[
\langle h \rangle \mapsto \langle J_H f \rangle S(g \otimes h)
\]
is self-adjoint for all \( f, g \in H \).
\end{enumerate}

Lemma 10.3.3. Let \((H, T)\) be a Borchers triple, \( S \) be a two particle scattering operator for \((H, T)\) and \( \tilde{S} = S_{12} S_{13} \cdots S_{1,n+1} \). Then the operator \( A^S \in \mathcal{B}(\mathcal{H}^{\otimes n}) \) given by
\[
\langle h \rangle \mapsto \langle J_H f \rangle S(f \otimes g)
\]
is self-adjoint for all \( f, g \in H \).

Proof. We can write as
\[
S = \sum_{i,j} V_{q,ij} \otimes dE(q)_{ij} = \sum_{i,j} \int \frac{1}{2} (V_{q,ij} + V_{q,ji}) \otimes (dE(q)_{ij} + dE(q)_{ji}) + \sum_{i < j} \int \frac{1}{2} (V_{q,ij} + V_{q,ji}) \otimes \left( dE(q)_{ij} - dE(q)_{ji} \right)
\]
\[
= \sum_k V_{q,k} \otimes dE(q)_k
\]
\[
S^* = \sum_k V^*_q \otimes dE(q)\kappa.
\]
Then by assumption we have
\[
\sum_k \int (J_H f, \bar{V}_{q,k} \bar{g}) \bar{dE}(q)_k = A^S_{f,g} = (A^S_{f,g})^* = \sum_k \int (J_H f, \bar{V}_{q,k} \bar{g}) d\bar{E}(q)_k.
\]
So \( A^S_{f,g} = (A^S_{f,g})^* \) for all \( f, g \in H \) is equivalent with \( \bar{V}_{q,k} H \subset H \) for almost all \( q \). But this is then true for all \( q \) by translation invariance and implies the equality of
\[
A^S_{f,g} = \sum_{k_1, \ldots, k_n} \int (J_H f, \bar{V}_{q_1, k_1} \cdots \bar{V}_{q_n, k_n} \bar{g}) \bar{dE}(q_1)_{k_1} \otimes \cdots \otimes \bar{dE}(q_n)_{k_n}
\]
and
\[
(A^S_{f,g})^* = \sum_{k_1, \ldots, k_n} \int (J_H f, \bar{V}_{q_1, k_1} \cdots \bar{V}_{q_n, k_n} \bar{g}) d\bar{E}(q_1)_{k_1} \otimes \cdots \otimes d\bar{E}(q_n)_{k_n},
\]
because \( \bar{V}_{q_1, k_1} \cdots \bar{V}_{q_n, k_n} H \subset H \). \( \square \)

The standard representation of the unique irreducible standard pair \((H, T)\) is given as follows. We realize \((H, T)\) on \( H = L^2(\mathbb{R}) \) and \( T(t) = e^{itP} \), where \( Q = \ln P \) with
\[
(e^{itQ} f)(q) = e^{itq} f(q)
\]
\[
(\Delta^{-it} f)(q) = f(q + 2\pi s)
\]
(cf. [LW11]). The operator \( J \) can be identified with the complex conjugation, i.e. \( J f = \overline{f} \). A \( f \in L^2(\mathbb{R}) \) is in \( H \) if and only if \( f \) admits an analytic continuation on the strip \( \mathbb{R} + i(0, \pi) \), such that for every \( a \in (0, \pi) \) it is: \( f(\cdot + ia) \in L^2(\mathbb{R}) \) with boundary value \( f(q + i\pi) = \overline{f(q)} \).

10.3.1.1. **Classification of scattering operators for the unique irreducible standard pair \((H, T)\).**

We get the well–known scattering functions from the abstract definition.

**Definition 10.3.4 (Scattering function).** A scattering function is bounded Borel function \( \psi : \mathbb{R} + i[0, \pi] \to \mathbb{C} \) analytic in \( \mathbb{R} + i(0, \pi) \) which is

1. reflection symmetric \( \overline{\psi(q)} = \psi(-q) \) for \( q \in \mathbb{R} \),
2. symmetric \( \psi(q) = \psi(q + i\pi) \) for \( q \in \mathbb{R} \) and
3. inner \( \psi(q) = \psi^{-1}(q) \) for \( q \in \mathbb{R} \);

or equivalently a Borel function on \( \phi : \mathbb{R} \to \mathbb{C} \) which is boundary value of a bounded analytic function on \( \mathbb{R} + i\mathbb{R}_+ \) and which is

1. reflection symmetric \( \overline{\phi(p)} = \phi(p^{-1}) \) for \( p \in \mathbb{R} \setminus \{0\} \),
2. symmetric, \( \phi(p) = \phi(-p) \) for \( p \in \mathbb{R} \) and
3. inner \( \phi(p) = \phi^{-1}(p) \) for \( p \in \mathbb{R} \).

**Proposition 10.3.5.** Let \((H, T)\) be the irreducible standard pair, let \( \psi \) be a scattering function, then \( S = \psi(Q_1 - Q_2) \) with \( Q = \log P \) and \( Q_1 = Q \otimes 1, Q_2 = 1 \otimes Q \) is a two–particle scattering operator and all arise in this way.

**Proof.** Let \( \psi \) be a scattering function and \( S = \psi(Q_1 - Q_2) \). Innerness \( \psi(q)^{-1} = \overline{\psi(q)} \) implies that \( S \) unitary. Reflection symmetry implies:
\[
[S^* f](q_1, q_2) = \overline{\psi(q_2 - q_1) f(q_1, q_2)} = \overline{\psi(q_1 - q_2) f(q_1, q_2)} = [S f](q_1, q_2),
\]
i.e. $S_{21} = S^*$. The Yang–Baxter identity is trivial, because the $S_{ij}$ commute. Translation invariance is trivial. Further

$$[(\Delta_H^u \otimes \Delta_H^u)S]f(q_1, q_2) = \psi(q_1 - 2\pi t - (q_2 - 2\pi t))f(q_1 - 2\pi t, q_2 - 2\pi t)$$
$$= \psi(q_1 - q_2)f(q_1 - 2\pi t, q_2 - 2\pi t)$$
$$= [S(\Delta_H^u \otimes \Delta_H^u)f](q_1, q_2),$$

i.e. $[S, \Delta_H^u \otimes \Delta_H^u] = 0$ and by $[J_H, Q] = 0$ follows $(J_H \otimes J_H)S = S(J_H \otimes J_H)$. Finally we write

$$\psi(Q_1 - Q_2) = \int \psi(Q_1 - q) \otimes dE(q)$$
$$=: \int V_q \otimes dE(q)$$

and the operator $V_q \in \mathcal{E}(H, T)$. This implies for $f, g \in H$ that $(J_H f, V_q g) \in \mathbb{R}$ and therefore

$$A_{f,g} = \int (J_H f, V_q g) dE(q) = \int (V_q g, J_H f) dE(q) = A^*_f g.$$

Now let $S \in \mathcal{S}_2(H, T)$. The commutation with $T_1, T_2$ means that $T = \phi(Q_1, Q_2)$ and unitary implies $\phi = \phi^{-1}$. $\text{Ad} \Delta^{-1}(\phi(Q_1, Q_2)) = \phi(Q_1 + 2\pi t, Q_2 + 2\pi t)$ implies that $S = \psi(Q_1 - Q_2)$ with $\psi(q) = \phi(q, 0)$. Then $\psi(Q_1 - Q_2) = S^* = S_{21} = \psi(Q_2 - Q_1) = \psi(p) = \psi(-p)$. From $A_{f,g} = A^*_f g$ for all $f, g \in H$ follows that $\text{Im}(J_H f, \psi(q - q) g) = 0$ for almost all $q$, in other words $V_q H \subset H$ for almost all $q \in \mathbb{R}$ (then for all $q \in \mathbb{R}$). By [LW11 Lem. A.4.] and the proof of [LW11 Thm 2.3.] follows that $\psi$ is boundary value of a bounded analytic function on $\mathbb{R} + i(0, \pi)$ with $\psi(q + i\pi) = \psi(q)$ for almost all $q \in \mathbb{R}$.

### 10.3.2. $S$-symmetric Fock space.

**Proposition 10.3.6.** Let $\mathcal{H}$ be a Hilbert space and $F = F_{12} : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ the canonical flip operator given by $F(a \otimes b) = b \otimes a$. Then there is a one-to-one correspondence of

1. unitary operators $S$ on $\mathcal{H} \otimes \mathcal{H}$ satisfying $S_{21} = S^*$ and the Yang-Baxter equation

$$S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12},$$

2. unitary involutions (i.e. self-adjoint) operators $\Phi$ on $H \otimes H$, such that the braiding relation

$$\Phi_{12} \Phi_{23} \Phi_{12} = \Phi_{23} \Phi_{12} \Phi_{23}$$

holds, and

3. series $(D_n : \mathcal{H} \rightarrow \mathcal{H})_{n=2, 3, ...}$ of unitary representations of the symmetric group compatible with inclusions, i.e. $m > n$

$$U_m(\pi) = U_n(\pi) \otimes 1^{\otimes m-n} \quad (\pi \in S_n \subset S_m).$$

The correspondence given by $\Phi = F \circ R$ and $D_n$ is defined by $D_n(\tau_j) = \Psi_{ij}$, where $1 \leq j \leq n - 1$ and $\tau_j$ is the transposition of $j \leftrightarrow j + 1$.

**Proof.** Given unitary $R$ with $FRF = R^*$, define $\Phi := FR$ and therefore $\Phi^* = R^* F = FR = \Phi$. Is on the other hand a unitary involution $\Phi$ given, defining $R := F \Phi$ we get $R^* = \Phi F = R^{-1}$ and
\[ FRF = FF\Phi F = \Phi F = R^*. \] It is obvious that \( F_{12}F_{23}F_{12} = F_{23}F_{12}F_{23} \) holds. For \( F \) and \( S \) we get the commutation relation \( S_{12}F_{23} = F_{23}S_{13} \) and therefore

\[
\begin{align*}
\Phi_{23}\Phi_{12}\Phi_{23} &= F_{23}S_{21}F_{12}S_{12}F_{23}S_{23} \\
&= F_{23}F_{12}F_{23} \circ S_{12}S_{13}S_{23} \\
\Phi_{12}\Phi_{23}\Phi_{12} &= F_{12}S_{12}F_{23}S_{23}F_{12}S_{12} \\
&= F_{12}F_{23}F_{12} \circ S_{23}S_{13}S_{12}
\end{align*}
\]

For \( \tau_i \) the transposition of the \( i \)-th and \( i + 1 \)-th element, we define \( D_n(\tau_i) = \Phi_{i,i+1} \), which gives a representation of

\[ S_n = (\tau_1, \ldots, \tau_{n-1} : \tau_i\tau_{i+1}\tau_i = \tau_i + 1 \tau_i\tau_{i+1} \text{ and } \tau_i\tau_j = \tau_j\tau_i \text{ for } |i - j| \geq 2) \]

by the properties of \( \Phi \). Given \( \{D_n\} \) we set \( \Phi := D_2(\tau_1) \).

\[ \square \]

10.3.3. \textbf{\( S \)-symmetric Fock functor.} For a double \((\mathcal{H}, S)\) of a Hilbert space and a unitary \( S \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \) fulfilling \( S_{21} = \tau S \tau = S^* \) and the Yang–Baxter identity, i.e. Properties (1) and (2) of Definition 10.3.2 we associated the Fock space \( \mathcal{F}_{H,S} \) given by

\[ \mathcal{F}_{H,S} = P_S \mathcal{F}_{H}, \]

where \( P_S \) is the projection

\[ P_S \upharpoonright \mathcal{H}^\otimes n = \sum_{\sigma \in S_n^\times} D_{\sigma} \]

and

\[ \mathcal{F}_H = \mathcal{C} \Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^\otimes n \]

the full Fock space over \( \mathcal{H} \). For \(|A| \leq 1\), such that \([A \otimes A, S] = 0\) there is an operator \( \Gamma(A) \in \mathcal{B}(\mathcal{F}_{H,S}) \).

The construction is functorial, from the additive (by taking direct sums) category with

\textbf{Objects:} Doubles \((\mathcal{H}, S)\) of a Hilbert space and a unitary \( S \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \) fulfilling \( S_{21} = \tau S \tau = S^* \) and the Yang–Baxter relation.

\textbf{Morphisms:} Contractions \( A : (\mathcal{H}_1, S_1) \to (\mathcal{H}_2, S_2) \) with \((A \otimes A)S_1 = S_2(A \otimes A)\)

to the multiplicative (by taking tensor products) category of Hilbert spaces with contractions, which is given by

\[ (\mathcal{H}, S) \mapsto \mathcal{F}_{H,S} \]

\[ A : (\mathcal{H}_1, S_1) \to (\mathcal{H}_2, S_2) \mapsto \Gamma(A) = 1 \oplus \bigoplus_{n=1}^{\infty} A^\otimes n : \mathcal{F}_{H_1,S_1} \to \mathcal{F}_{H_2,S_2}. \]

We note that \( \Gamma(A) \) is well defined because from \((A \otimes A)S_1 = S_2(A \otimes A)\) follows that \( P_1\Gamma(A) = \Gamma(A)P_2 = P_1\Gamma(A)P_2 \) where \( P_i \) is here the projection from \( \mathcal{F}_{H_i} \) onto \( \mathcal{F}_{H_i,S_i} \). It preserves adjoints

\[ \Gamma(A^*) = \Gamma(A)^* \]

namely they are preserved on the full Fock space and \((A \otimes A)S_1 = S_2(A \otimes A)\) is equivalent with \((A^* \otimes A^*)S_2 = S_1(A^* \otimes A^*)\) due to \( S_i^* = S_{21} \). In particular, \( \Gamma(U) \) is unitary if \( U \) is unitary. There is a natural isomorphism

\[ N : \mathcal{F}_{H_1 \oplus H_2, S_1 \oplus S_2} \cong \mathcal{F}_{H_1, S_1} \otimes \mathcal{F}_{H_2, S_2} \]

\[ N\Gamma(A_1 \oplus A_2) = \Gamma(A_1) \otimes \Gamma(A_2)N. \]
For a antilinear with \((A \otimes A)S = S^*(A \otimes A)\) we define
\[
\hat{F}(A) = 1 \oplus \bigoplus_{n=1}^\infty F_{1\ldots n} A^\otimes n,
\]
where \(F_{1\ldots n} f_1 \otimes \cdots \otimes f_n = f_n \otimes \cdots \otimes f_1\). This is well-defined, namely we have
\[
F_{1\ldots n} A^\otimes n D_n(\tau_i) = F_{1\ldots n} \Phi_{i+1} A^\otimes n = \Phi_{n-i,n-i+1} F_{1\ldots n} A^\otimes n = D_n(\tau_{n-i}) F_{1\ldots n} A^\otimes n
\]
hence \(\hat{F}(A) P_S = P_S \hat{F}(A)\). This can also be formulated as
\[
\hat{F}(A) \uparrow P_S H^\otimes n = A^\otimes n F_{1\ldots n} = A^\otimes n \bigoplus_{1 \leq i < j \leq n} S_{ij} = \bigoplus_{1 \leq i < j \leq n} S_{ij} A^\otimes n.
\]
Namely, \(f \in H^\otimes n P_S f\) is \(S\)-symmetric in the sense that we have \(F_{i,i+1} P_S f = S_{i,i+1} \Psi_{i,i+1} P_S f\). From this one can show that on \(H^\otimes n\) holds: \(F_{1\ldots n} P_S = \prod_{1 \leq i < j \leq n} S_{ij} P_S\).

10.3.4. Second quantization on \(S\)-symmetric Fock space. For \(f \in H\) let \(b(f)\) be the creation operator on the subspace \(\mathcal{F}_H^0\) of finite particles of \(\mathcal{F}_H\), given by \(b(f) \uparrow H^\otimes n = \sqrt{n+1} \cdot f \otimes \cdots\). Then its adjoint is given on \(\mathcal{F}_H^0\) by \(b(f)^* \uparrow H^\otimes n = \sqrt{n} \cdot (f|_1 \cdot \cdot)\), namely
\[
(h_0 \otimes \cdots \otimes h_m, b(f)g_1 \otimes \cdots \otimes g_n)
= \delta_{mn} \sqrt{n+1} (h_0 \otimes \cdots \otimes h_m, f \otimes g_1 \otimes \cdots \otimes g_n)
= \delta_{mn} \sqrt{m+1} \cdot (f, h_0)(h_1 \otimes \cdots \otimes h_m, g_1 \otimes \cdots \otimes g_n)
= (b(f)h_0 \otimes \cdots \otimes h_m, g_1 \otimes \cdots \otimes g_n).
\]

Let \(D\) be the vectors with finite particle number, i.e. \(\psi \in \mathcal{F}_H^{0,S}\) with \(P_S H^\otimes n = 0\) for all \(n > n_0\).

We define on \(\mathcal{F}_H^{0,S}\) the compressed operators \(a(f) = P_S b(f) P_S\) and define the Segal type field \(\phi(f) = a(f) + a(f)^*\) on \(D\) which is symmetric. Note that \(f \mapsto \psi(f)\) is just real linear.

Lemma 10.3.7 (cf. [Lec06 Lemma 4.1.3.]). Let \(N\) be the particle operator. On \(\Psi \in D\) holds
\[
\|a(f)\Psi\| \leq \|\Psi\| \cdot \|(N+1)^\frac{1}{2}\Psi\|,
\]
\[
\|a(f)^*\Psi\| \leq \|\Psi\| \cdot \|N^\frac{1}{2}\Psi\|.
\]

Proof. On the unsymmetrized Fock space \(\mathcal{F}_H\) with \(N\Psi_n = n\Psi_n\) one checks \(b(g)^* b(f) \Psi_n = (g, f)(N+1)^2 \Psi_n\) and gets \(b(g)^* b(f) = (g, f)(N+1)\) on \(D\). Hence \(\|b(f)\Psi\|^2 = \|(N+1)^2 \Psi\|^2 \cdot \|f\|^2\)

which implies \(\|b(f)(N+1)^{-\frac{1}{2}}\| = \|f\|\). But then also the adjoint \((N+1)^{-\frac{1}{2}} b(f)^* = b(f)^* N^{-\frac{1}{2}}\) has this norm. Then the bounds follow from \(a(f)^* = P b(f)^* P\).

Lemma 10.3.8. It holds:

1. \((\phi, f)\) is essentially self-adjoint on \(D\).
2. \(f \mapsto \phi(f)\) is strongly continuous on \(D\).
3. \(f \mapsto e^{\phi(f)}\) is strongly continuous (where \(\phi(f)\) here is the self-adjoint extension).
4. Let \(U \in \mathcal{U}(H)\) with \([U \otimes U, S] = 0\), then \(\Gamma(U) \phi(f) \Gamma(U) = \phi(U f)\) on \(D\).
5. If \(H\) is cyclic then \(D\) is cyclic for the polynomial algebra of \(\psi(f)\) with \(f \in H\).

Proof. Like in [Lec06, Proposition 4.2.2]. For \(\Psi_n \in D\) with \(N\Psi_n = n\Psi_n\) we get for \(c_f = 2\|f\|\) with the bounds of Lemma 10.3.7 \(\|\phi(f)\Psi_n\| \leq \sqrt{n+1} \cdot c_f \cdot \|\Psi_n\|\). Iteratively we get
\[
\|\phi(f)^k \Psi_n\| \leq \sqrt{(n+1) \cdot \cdots \cdot (n+k)} c_f^k \|\Psi_n\|
\]
and for every $t > 0$ we have

$$
\sum_{k=0}^{\infty} \frac{\|\phi(f)\psi_k\|^k}{k!} \leq \|\psi_0\| \sum_{k=0}^{\infty} \sqrt{\frac{(n+k)!}{n!}} \frac{1}{k!} (c_f \cdot t)^k \leq \infty.
$$

By Nelson’s Theorem (Theorem 2.1.3) $\phi(f)$ is essentially self-adjoint on $\mathcal{D}$.

For the continuity (cf. [RS75 Thm X.41]) for $\psi \in P_S \mathcal{H}^{\otimes k}$ and $f_n \to f$ a sequence in $\mathcal{H}$ we get

$$
\|\phi(f_n)\psi - \phi(f)\psi\| \leq \|c(f_n)\psi - c(f)\psi\| \leq \sqrt{2} \sqrt{k+1} \|\psi\|
$$

so $\phi(f_n)\psi \to \phi(f)\psi$ and thus $\phi(f_n)$ converges strongly to $\phi(f)$ on $\mathcal{D}$. Since $\mathcal{D}$ is a core for $\phi(f)$ it holds that $e^{it\phi(f_n)} \to e^{it\phi(f)}$ strongly.

Let $U \in \mathcal{U}(\mathcal{H})$ with $[U \otimes U, S] = 0$, then $U$ commutes with $P_S$. For $x \in \mathcal{H}^{\otimes n}$ we get

$$
\Gamma(U) a(f) \Gamma(U)^* x = \sqrt{n+1} U^{\otimes(n+1)} P_S (f \otimes U^{\otimes n} x)
$$

$$
= \sqrt{n+1} P_S (U f \otimes x)
$$

$$
= a(U f) x
$$

and $\Gamma(U) a(f)^* \Gamma(U^*) = (\Gamma(U) a(f) \Gamma(U^*))^* = a(U f)^*$.

The cyclicity can be shown inductively, namely by applying $\psi(f)$ on $\Omega$ one can show that one obtain a total set in $P_S \mathcal{H}^{\otimes n}$.

We also denote the self-adjoint closure of $\phi(f)$ by $\phi(f)$ and define for every real subspace $H \subset \mathcal{H}$ the von Neumann algebra

$$
\mathcal{M}_S(H) = \left\{ e^{i\phi(f)} : f \in H \right\}^\prime\prime \subset \text{B}(\mathcal{F}_S).
$$

This can be seen as a generalization of the CCR and CAR algebra.

**Proposition 10.3.9.** Let $(\mathcal{H}, S)$ like before and $K, H \subset \mathcal{H}$ real subspaces:

1. $K \subset H$ then $\mathcal{M}_S(K) \subset \mathcal{M}_S(H)$
2. $\mathcal{M}_S(K) = \mathcal{M}_S(H)$ if and only if $\overline{K} = \overline{H}$
3. Let $U \in \mathcal{U}(\mathcal{H})$ with $[U \otimes U, S] = 0$, then $\Gamma(U) \mathcal{M}(H) \Gamma(U)^* = \mathcal{M}(UH)$.
4. If $H$ is cyclic then $\Omega$ is cyclic for $\mathcal{M}_S(H)$.

**Proof.** The first statement is clear and the second follows from continuity. The covariance with respect to unitaries with $[U \otimes U, S] = 0$ follows from the covariance of $\phi(f)$. Let $f_1, \ldots, f_n \in H$ and let $E_k(t)$ be the spectral projection of the selfadjoint operator $\phi(f_k)$ on the spectral values $[-t, t]$. Then $F_k(t) := \phi(f_k) \in \mathcal{M}$ for all $t > 0$ and $F_k(t) \to \phi(f_k)$ strongly on $\mathcal{D}$ and hence $F_1(t) \cdots F_n(t) \Omega$ converges to $\phi(f_1) \cdots \phi(f_n) \Omega$ for $t \to \infty$. The cyclicity of $\Omega$ for $\mathcal{M}$ follows from the cyclicity of $\Omega$ for $\phi$.

10.3.5. $S$–symmetric second quantization of Borchers pairs and modular theory. In this section we are interested in the construction of 1D Borchers triples from a standard pair $(H, T)$ on $\mathcal{H}$. It will turn that for all $S \in \mathcal{S}_2(H, T)$ it is possible to construct a 1D Borchers triple on the “twisted Fock space” $\mathcal{F}_{H, S}$.

Before we turn to the von Neumann algebras we first need commutation relation of the Segal field $\phi(f)$ with the “reflected Segal field” $J\phi(f)J$. One can think of $\phi(f)$ for $f \in T(a)H$ as a field localized in a right half-ray $\mathbb{R}_+ + a$ and of $\phi'(g) := J\phi(J_{g^2})J$ as a field localized in the left half-ray $\mathbb{R}_- - b$ for $g \in T(-b)H'$. 

Lemma 10.3.10. Let \((H,T)\) be a standard pair and \(S \in \mathcal{S}_2(H,T)\) two particle scattering operator and \(\phi(f)\) the operator on \(\mathcal{D} \subset \mathcal{F}_{\mathcal{H},S}\) and \(J = \hat{\Gamma}(J_H)\). For \(f,g \in H\) the commutator \([J\phi(g)J,\phi(f)]\) vanishes on \(\mathcal{D}\).

**Proof.** For \(x \in P_S \mathcal{H}^{\otimes n}\)

\[
Ja(g)^* Jx = Ja(f)^* F_{1\ldots n} J_{H}^{\otimes n} x
\]

\[
= \sqrt{n} \cdot F_{1\ldots(n-1)} J_{H}^{\otimes(n-1)} \langle f \mid 1 \rangle F_{1\ldots n} J_{H}^{\otimes n} x
\]

\[
= \sqrt{n} \cdot F_{1\ldots(n-1)} J_{H}^{\otimes(n-1)} \langle f \mid 1 \rangle F_{1\ldots n} J_{H}^{\otimes n} x
\]

\[
= \sqrt{n} \cdot F_{1\ldots(n-1)} J_{H}^{\otimes(n-1)} \langle f \mid 1 \rangle F_{1\ldots n} J_{H}^{\otimes n} x
\]

\[
= \sqrt{n} \langle J_H f \mid x \rangle
\]

holds. From \(Ja(g)^* J \upharpoonright \mathcal{H}_n = \sqrt{n} \langle J_H g \mid \rangle\) follows

\[
[Ja(g)^* J, a(f)]^* \Psi_n = \sqrt{n} (Ja(g)^* J \langle f \mid 1 \rangle - a(f) \langle J_H g \mid \Psi_n \rangle^* \Psi_n
\]

\[
= \sqrt{(n-1)n} (\langle jg \mid \rangle_{n-1} \langle f \mid 1 \rangle - \langle f \mid 1 \rangle \langle J_H g \mid \rangle_{n} \Psi_n
\]

\[
= \sqrt{(n-1)n} (\langle f \mid 1 \rangle \langle jg \mid \rangle_{n} - \langle f \mid 1 \rangle \langle J_H g \mid \rangle_{n} \Psi_n
\]

\[
= 0
\]

and therefore also \([Ja(g)J, a(f)] = -[Ja(g)^* J, a(f)^*]^* = 0\) on \(\mathcal{D}\). We calculate the mixed commutator to be

\[
Ja(g)^* Ja(f)^* \Psi_n = Ja(g)^* J \frac{1}{\sqrt{n + 1}} \sum_{i=1}^{n+1} X_{i1} (f \otimes \Psi_n)
\]

\[
= \sum_{i=1}^{n+1} \langle jg \mid \rangle_{n+1} X_{i1} (f \otimes \Psi_n)
\]

\[
a(f)Ja(g)^* J \Psi_n = a(f) \sqrt{n} \langle J_H g \mid \rangle \Psi_n
\]

\[
= \sum_{i=1}^{n} X_{i1} (f \otimes (J_H g \mid \rangle \Psi_n))
\]

\[
= \sum_{i=1}^{n} (J_H g \mid \rangle_{n+1} X_{i1} (f \otimes \Psi_n)
\]

Finally, restricted to \(\mathcal{H}_n\) holds:

\[
[Ja(g)J, \phi(f)] = [Ja(g)^* J, a(f)] + [Ja(g)J, a(f)^*]
\]

\[
= 0
\]

for \(f, g \in H\) because of Lemma 10.3.3. \(\square\)

Proposition 10.3.11. Let \((H_1, T_1)\) be a Borchers pair with finite multiplicity on \(\mathcal{H}\) and \(S \in \mathcal{S}_2(H_1, T_1)\), then for the von Neumann algebra \(\mathcal{M}_S(H_1) = \{e^{\phi(f)} : f \in H_1\}''\) on \(\mathcal{F}_{\mathcal{H},S}\) holds:

(1) \(T(t)M(H_1)T(-t) \subset M(H)\) for \(t \geq 0\), where \(T(t) = \Gamma(T_1(t))\).

(2) \(\Omega \in \mathcal{F}_{\mathcal{H},S}\) is cyclic and separating for \(M(H_1)\).
(3) $\Delta^H_{M_H(H)} \Omega = \Gamma(\Delta^H_{H_1})$ and $J_{M_H(H)} \Omega = \hat{J}(J_{H_1})$.

(4) $\Omega$ is up to phase unique translation invariant vector in $\mathcal{F}_{H,S}$.

**Proof.** (1): For $t \geq 0$ it holds $\text{Ad} \Gamma(T(t)) \mathcal{M}(H) = \mathcal{M}(T(t)H) \subset \mathcal{M}(H)$, because $T(t)H \subset H$.

(2): We define $M_2 := \{ e^{it\hat{J}/2} : f \in H \}$. Analogous to $\mathcal{M} = M_S(H)$ cyclic follows that $M_2$ is cyclic, so that $\Omega$ is separating for $\mathcal{M}$ can be shown by proving $[\mathcal{M}, M_2] = \{ 0 \}$.

To show that $\mathcal{M}$ and $M_2$ commute we need to use energy bounds. Let $P_0 = d \Gamma(P + 1/P) \geq 2$ with domain $D_0$ be the generator of $\Gamma(e^{it(P+1/P)})$. We get $P_0 \geq 2N$. On can now use for example the massive representation and it is well known that real Schwartz test function with support in $W_R$ embed densely into $H$ and define a subspace $H_\infty \subset H$ which is analytic for $P + 1/P$. We get bounds from the proof of Lemma 10.3.7 and because the multiplicity is finite that $\| (1 + P_0)^{-1} \phi(f) \| < \infty$ on $D_0$ for $f \in H_\infty$ and similar for the commutator $[P_0, \phi(f)]$ and for $J\phi(f)J$. By the commutator theorem [DF77] one can conclude that $e^{it\hat{f}/2}$ and $e^{it\hat{J}/2}$ commute for all $f, g \in H_\infty$ which by continuity implies that $\mathcal{M}$ and $M_2$ commute.

The property of the modular operators (3) is proved like in [BL04, Prop. 3.1.] and $\Omega$ is the unique translation invariant vector. \hfill $\Box$

**Corollary 10.3.12.** For each Borchers pair (with finite multiplicity in the reducible case) $(H, T)$ and $S \in S_2(H, T)$ exists a 1D Borchers triple, and therefore a half-ray local dilation translation covariant net on $\mathbb{R}$.

Special cases of such models were constructed in [BLM11] and were proposed as scaling limit of 2D models with factorizing $S$-matrix. In the next section we will present a direct relations to massive models in 2D via a class of Longo–Witten unitaries like in Section 10.1.2 in other words the idea of lightfront holography. The idea of construction half-ray local nets on $S$-symmetric Fock space came (independently) before the appearance in [BLM11], when the author studied possible quantizations of nets of standard subspaces, but it took some progress and new ideas coming from [LW11, Tan12a, BT11] to see that this models can be seen as building blocks for massive and massless models in 2D.

### 10.4. Longo–Witten endomorphisms and Lechners models

Let $(H_0, T_0)$ be a standard pair $V \in \mathcal{E}(H_0)$ and $S \in S_2(H, T)$ then $\Gamma(V) \in \mathcal{E}(\mathcal{M}_S(H_0), T)$ with $T = \Gamma(T_0)$.

Let us start with the irreducible standard pair $(H, T)$ and $S = S_2(Q \otimes 1 - 1 \otimes Q)$. We fix some $m > 0$. Then by taking $V_\varepsilon = e^{-im^2s^2/p} \in \mathcal{E}(H)$ we obtain Lechner’s models by the construction of Proposition 10.1.6 namely by Proposition 10.2.1 we know that the one-particle space carries the representation $U_m$ of the Poincaré group. So in our framework this models arise naturally from the $S$-symmetric second quantization of the irreducible standard pair and the (only) Longo–Witten unitary semigroup with negative generator. We remark that the 1D model contains no information about the mass $m$, it can obtained any mass $m > 0$ by rescaling the semigroup.

Let us denote by $\mathcal{S}_b^-$ the scattering functions $\psi$, for which

$$\kappa(\psi) := \inf \left\{ \text{Im} \zeta : \zeta \in \mathbb{R} + i(0, 2) \right\}, \quad \psi(\zeta) = 0 > 0$$

and for which $\psi(0) = -1$ and

$$\|\psi\|_\kappa := \sup \{ |\psi(\zeta)| : \zeta \in \mathbb{R} + i[-\kappa, \pi + \kappa] \}, \quad \kappa \in (0, \kappa(\psi)).$$

We call this scattering functions regular. We note the important result by Lechner [Lec08], which shows that for the class of regular scattering functions the obtained Borchers triples are strictly local.
In the reducible case, i.e. the case of several particles one has to build a Longo–Witten unitary semigroup $V_m(s)$ with negative generator which commutes with $S$ in the sense that $S(V_m \otimes V_m) = (V_m \otimes V_m)S$.

## 10.5. Construction of (wedge-local) massless factorizing models

Construction of massless wedge-local nets were obtained before in [DT11, Tan12a, BT11], but they involve just a $S$-matrix between left and right moving particles. In [ZZ92] scattering models with a left-left, right-right and left-right $S$-matrix are considered. The left-left and right-right $S$-matrix comes just from analogy with the massive models but has no direct physical meaning, because massless particles moving in the same direction cannot be distinguished, but form just one wave. Nevertheless the more general Fock structure gives new examples and leaves the possibility for constructing non-trivial left-right scattering matrices on the Fock space. The construction of the (left-right) $S$-matrix is analogous to the case where the chiral algebras were free Bosonic ([Tan12a]) or Fermionic fields (see Chapter 9) where the left-left and right-right $S$-matrix were $\pm 1$. The mixed Yang–Baxter relation gives restrictions.

### Definition 10.5.1

Let $(M_{\pm}, T_{\pm}, \Omega_{\pm})$ be two 1D Borchers triple on $\mathcal{H}_{\pm}$, respectively. A unitary $S \in B(\mathcal{H}_+ \otimes \mathcal{H}_-)$ is called a scattering matrix for $(M_{\pm}, T_{\pm}, \Omega_{\pm})$ if

1. $S$ commutes with $T_+ \otimes T_-$
2. $S(\xi \otimes \Omega_-) = \xi \otimes \Omega_-$ for $\xi \in \mathcal{H}_+$ and $S(\Omega_+ \otimes \eta) = \Omega_+ \otimes \eta$ for $\eta \in \mathcal{H}_-$
3. $x \otimes 1$ commutes with $\text{Ad}S(x' \otimes 1)$ for $x \in M'_+\text{ and } x' \in M_+$
4. $\text{Ad}S(1 \otimes y)$ commutes with $1 \otimes y'$ for $y \in M_-\text{ and } y' \in M'_-$

### Proposition 10.5.2

Let $(M_{\pm}, T_{\pm}, \Omega_{\pm})$ be two 1D Borchers triple on $\mathcal{H}_{\pm}$, respectively and $S \in S(M_+, T_+, M_-, T_-)$. Then $(M_S, T, \Omega)$ with

$$
\begin{align*}
M_S &= (M'_+ \otimes 1) \lor \text{Ad}S(1 \otimes M_-) \\
T(t, x) &= T_+((t - x)/\sqrt{2}) \otimes T_-((t + x)/\sqrt{2}) \\
\Omega &= \Omega_+ \otimes \Omega_-
\end{align*}
$$

is an asymptotically complete Borchers triple with asymptotic algebra given by the Borchers triple $(M_+ \otimes M_-, T_+ \otimes T_-, \Omega_+ \otimes \Omega_-)$ with $S$-matrix the operator $S$.

**Proof.** The proof can be copied wordly from the version using Möbius covariant local nets. □

Given a Borchers pair $(H_L, T_L)$ on $\mathcal{H}_L$ with two particle scattering operator $S^L$ and $(H_R, T_R)$ on $\mathcal{H}_R$ with two particle scattering operator $S^R$.

Again, we can take as input Longo–Witten unitaries to knit up the $S$-matrix. Let $S^{LR} \in U(\mathcal{H}_L, \mathcal{H}_R)$ which decomposes as

$$
S^{LR} = \sum_{i} \int V_{q,i}^L \otimes dE_{R,j}(q) = \sum_{j} \int dE_{L,i}(q) \otimes V_{q,j}^R
$$

$V_{q,i}^L \in \mathcal{E}(H_L, T_L)$

$V_{q,j}^R \in \mathcal{E}(H_R, T_R)$
so it commutes in particular with \( T_L \otimes 1 \) and \( 1 \otimes T_R \) and let \( S^{LR} \) satisfy the so-called \textit{mixed Yang–Baxter identities}:
\[
S^L_{12} S^{LR} S^L_{13} S^{LR} S^L_{23} = S^R_{23} S^{LR} S^R_{13} S^{LR} S^R_{12}.
\]
Then we can define \( S \in \mathcal{U}(\mathcal{F}_{H_{L,S_L}} \otimes \mathcal{F}_{H_{R,S_R}}) \) first on the full Fock space by
\[
S \uparrow H^m_{L} \otimes \Omega = 1
\]
and \( \Omega+ \otimes H^n_{R} = 1 \)
\[
S \uparrow H^m_{L} \otimes H^n_{R} = S^{LR}_{1,m+1} S^{LR}_{2,m+2} \cdots S^{LR}_{m,m+n}
\]
and we claim it restricts to the space \( \mathcal{F}_{H_{L,S_L}} \otimes \mathcal{F}_{H_{R,S_R}} \), which follows from the Yang–Baxter identity. Namely, we check just the first part, it is sufficient to check that \( S \) commutes with \( \Phi_{L_{i+1}}^{S_L} \) and \( \Phi_{R_{j+1}}^{S_R} \) on the space \( M \) of \( \mathcal{F}_{H_{L,S_L}} \otimes \mathcal{F}_{H_{R,S_R}} \) from the Yang–Baxter identity, and with \( S_{mk} \) for \( m \neq i, j \) it commutes trivially.

Then we claim that \( S \) is a scattering matrix for \( (M_\pm, \Omega_\pm, T_\pm) \), where \( M_+ = R_S(H_L), T_+ = \Gamma(T_L) \) and \( M_- = R_S(H_R), T_- = \Gamma(T_R) \). We just need to check (3) and (4) which follow like before from the spectral decomposition, namely for \( x' \in M_+ \) with
\[
S = \bigoplus_{m} \sum_{1}^{n} \int \bigotimes_{k=1}^{n} V_{q_k,i_k} \otimes dE_{R,i_1i_1}(q_1) \otimes \cdots \otimes dE_{R,i_ni_n}(q_n)
\]
\[
= \bigoplus_{n} \sum_{1}^{m} \int \bigotimes_{k=1}^{n} V_{q_k,i_k} \otimes dE_{R,i_1i_1}(q_1) \otimes \cdots \otimes dE_{R,i_ni_n}(q_n)
\]
we get that
\[
\text{Ad} S(x' \otimes 1) = \bigoplus_{n} \sum_{1}^{n} \int \text{Ad} \left( \bigotimes_{k=1}^{n} \Gamma(V_{q_k,i_k}) \right) (x') \otimes dE_{R,i_1i_1}(q_1) \otimes \cdots \otimes dE_{R,i_ni_n}(q_n)
\]
commutes with \( x \otimes 1 \) for \( x \in M_+ \) and the other condition is checked the same way.

### 10.5.1. Discussion about Möbius covariance of 1D factorizing models

The net obtained from \( S^L, S^R, S^{LR} \) is in general only wedge-local. But let us assume it would be strictly local, then we get several other constructions. Namely, by [Tan12a] we know that the chiral parts \( (M_\pm, T_\pm, \Omega_\pm) \) are actually local Möbius covariant nets, which we denote by \( \mathcal{A}_+ \) (such that \( \mathcal{A}(\mathbb{R}_+, a) = \text{Ad} T(a) \mathcal{A}(\mathbb{R}_+) \)). So the existence of such strictly local massless scattering models would provide 1D factorizing models which are local Möbius covariant nets.

So let us on the other hand now assume that for a given matrix \( S^L \) the model \( (M_+, T_+, \Omega_+) \) gives rise to a local conformal net \( \mathcal{A}_+ \). We note that this does still not imply that massless models are strictly local if \( S_{LR} \neq 1 \).

But we know that \( \Gamma \) maps \( V_L \in \mathcal{E}(H_L, T_L) \) with \( [V_L \otimes V_L, S^L] = 0 \) into \( V_+ = \Gamma(V_L) \in \mathcal{E}(\mathcal{A}_+, T_+) \). This gives time-translation covariant on Minkowski half-plane (boundary net) \( \mathcal{A}_{+,v_+} \), given by \( \mathcal{A}_{+,v_+}(I \times J) = \mathcal{A}_+(I) \vee V_+ \mathcal{A}_+(J) V_+^* \). Thus we can also built one-parameter groups \( V_+ = \Gamma(v_+) \) from \( v_+ = e^{-i \sum m_i^2 / R_i} \) in \( \mathcal{E}(H_L, T_L) \) for \( s \geq 0 \) with a decomposition \( P_{L} = \sum P_{L,s} \) such that the above commutation with \( S^L \) is fulfilled. These have negative generator and \( V_+ \in \mathcal{E}(\mathcal{A}_+) \) for \( s \geq 0 \). Thus we obtain by
the construction a massive net in 2D Minkowski space which is strictly local. So as soon as we have Möbius covariance this would give a new way of showing strict locality.

It is conjectured in [BLM11] that only the cases $S_2 = \pm 1$ leads to non trivial local Möbius covariant subnets. But there seem to be hints in the literature that the constructed 1D Borchers triple can contain non-trivial local Möbius covariant subnets for non constant $S$, especially in the reducible case, i.e. several particles. For example for $(H, T) = (H_0 \oplus H_0, T_0 \oplus T_0)$ and the $S_{SU(2)}$ matrix of the $SU(2)$ invariant Thirring model it is conjectured that the factorizing scattering model is equivalent to $SU(2)_1$ Kac–Moody current algebra, so we conjecture in our case that the 1D Borchers triple associated with this $S$-matrix contains the $SU(2)_1$ loop group model as local Möbius covariant subnet, but we expect that $\mathcal{H}_A$ could be smaller then $\mathcal{F}_{H_{SU(2)}}$.

Strictly local massive theories which fulfill the light strip property give a two parameter subfamily of boundary nets on Minkowski half-plane, which come from the Longo–Witten endomorphisms obtained by the restriction of the massive Borchers triple to a 1D Borchers triple. We draw an example in Figure 2 how a double cone in the Minkowski half-plane corresponds to light strips in the massive model.

![Figure 2. Boundary net from a massive net with light strip property](image-url)
CHAPTER 11

Conclusions and outlook

In this third part we gave new constructions of models in two-dimensional quantum field theory and gave relations between different constructions.

First, in Chapter 9 we combined the Longo–Witten endomorphisms of Chapter 5 to construct interacting wedge-local nets with $\mathcal{A}_R \otimes \mathcal{A}_R$ as the asymptotic algebra and showed that their scattering operators do not preserve the $n$-particle space of its Bosonic Fock space. It is important to note that particle production is a necessary feature of interacting models in higher dimensions [Aks65], thus this result gives some hope for algebraic construction of higher dimensional interacting models.

However, there are at least two shortcomings with the present method. The first is that we proved only wedge-locality of the models. We gave examples of wedge-local nets that are dilation-covariant and at the same time interacting. On the other hand, a *strictly local* dilation-covariant (asymptotically complete) net is necessarily not interacting, as shown in [Tan12b]. Hence, interaction of wedge-local nets could be just a false-positive and strict locality is desired. The second is the fact that the concept of particle in the massless case is not intrinsically defined. Although the Fock space structure is easily understood, its interpretations should be treated with care.

These issues could be overcome by considering massive cases. As for strict locality, it has been shown that the deformation of the massive free field by a suitably regular function is again strictly local [Lec08, Lec11]. On the other hand, in the massless case, even the simplest case $\varphi(p) = -1$ is already not strictly local [Tan12a]. Hence we believe that strict locality should be addressed in massive models. Furthermore, for a massive asymptotically complete model, the notion of particle production is intrinsic.

In Chapter 10 we first gave some relations between half-ray local nets on the real line and massive wedge-local nets on 2D Minkowski space. These two families are related via semi-groups of Longo–Witten unitaries with negative energy. If the net on the real line is Möbius covariant, i.e. a conformal net then the massive net in 2D turns out to be strictly local. This gives a new construction of massive local models in 2D. But as we saw in Part II the construction of the necessary Longo–Witten unitaries seems to be rather complicated, because it involves the inner symmetric function $\varphi_t(p) = e^{-it/p}$. This particular function is too singular for the extension of the free Bosonic models (Chapter 4) and we conjecture that it is also to singular for non-trivial extensions coming from free Fermions (7) using the framed vertex operator construction. The known examples so far are just related to free field theories and the question arises if this construction always leads to free theories. On the other hand we show that we get a conformal net from a massive theory on a possibly smaller Hilbert space. However it can happen the extreme case, that we get the trivial conformal net $\mathbb{C} \cdot 1$ on the Hilbert space $\mathbb{C} \cdot \Omega$. That the Hilbert space has the same size is equivalent to cyclicity for lightlike strips in the massive model; we called this property the light strip property. So the above question could be formulated also as: are massive models with this the light strip property free?

We showed that by starting with half-ray local nets a family of massive models which are interacting and having all asked properties including strict locality could be reached. This is the class of
factorizing S-matrix models construct in [Lec08]. For this purpose we gave a general second quantization of Borchers pairs \((H, T)\) on a \(S\)-symmetric Fock space, which gives half-ray local nets on the real line. The Fock space structure provides Longo–Witten unitaries with negative energy which are used to construct the massive models and massless models.

The massive construction gives back factorizing S-matrix models and as a special case the well established models of Lechner. Also models with several particles can be obtained, but it is not known if strict locality holds. In the same time, such models were also constructed more directly in [LS12].

Finally, it was shown that the half-ray local models obtained by second quantization of Borchers triple on \(S\)-symmetric Fock space can also be used to construct new massless wedge-local models. These are obtained by taking a product of two such 1D theories and construct an \(S\)-matrix using the Fock space structure similar as it was done in the intermediate step in Chapter 9 on the Fermionic Fock space. While these models are again just wedge-local there is some hint in the literature [ZZ92] that these models could be in very special cases turn out to be strictly local. An example would be the massless \(SU(2)\) invariant Thirring model. Such a result as we discussed would also give Möbius covariance for the involved wedge-local models and a construction of certain conformal nets on the \(S\)-symmetric Fock space. To close such a circle this would also give new examples to construct Longo–Witten unitaries and would prove strict locality for massive models obtained from this chiral models.
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Bibliography


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